Dipak Roy

Exercises in Analysis

Creative Incorporate

To understand mathematics means to be able to do mathematics.
And what does it mean doing mathematics? In the first place it
means to be able to solve mathematical problems.
–George Pólya (1887–1985)

In this book I have tried to collect the most attractive problems (elementary and advanced) in Real Analysis accessible to first year students majoring or minoring in Mathematics. The major part of the book contains problems known from books and journals only. I believe that they will be of interest to many readers.

Features of the book:

1. There are more than 3000 problems (from elementary to advanced) including solutions (more than 1700) and exercises (more than 1300).
2. A few additional topics have been included, most notably,
   - Differentials,
   - Differentiation of functions of several variables,
   - Geometric proof of $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$.
   - Concepts of nearness and net convergence, and
   - an introduction to Cantor set.
3. This book also serves as a question bank on Real Analysis.

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Exercises in Analysis

Dedicated to my parents

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Preface to the second edition

In this new edition some misprints and few corrections have been made, and no remarkable changes except a few problems.

Dipak Roy.

Preface to the first edition

This book is an outcome of an experience of teaching in different colleges for more than forty years. For most students, solving problems is a vital part of learning mathematics well. Furthermore, I think students learn the most from challenging problems that demand serious thought and help develop a deeper understanding of important ideas.
It contains a collection of problems in Real Analysis for strong advanced undergraduates or beginning graduate students in mathematics. Some of the problems will be challenging even for very talented students. These problems can be used by students taking real analysis course who want more challenge or some interesting enrichment to their course. They can also be used by more experienced students for review or to solidify their knowledge of the subject. Professors teaching "real analysis" courses may use this book as a source to supplement the problems from their textbook. The assumed background for those undertaking these problems includes familiarity with the basic set-theoretic language of mathematics and the ability to write rigorous mathematical proofs. Keep in mind that the solutions provided represent *one* way of answering a question or solving an exercise. In many cases there are alternatives, so make sure that you don't dismiss your solution just because it does not look like the solution in this book. Solved and unsolved problems are collected from different books, different university exams and mathematical magazines for a period of years.
No solutions are provided for the exercises given here (though there are many hints). The guideline for this is that readers who do not succeed with a first effort at a difficult problem can often progress and learn more by going back to it at a later time. I have provided solutions along with the problem: for solutions in the back of the book offer too much temptation to give up working on a problem too soon.

The presentation of the material in the book follows the pattern below:

- A theorem for which proofs can be found in most textbooks and monographs is stated often without proof and always with at least one reference.

- A result that has not yet been expounded in a textbook or monograph is given with at least one reference and, as space permits, with a proof, an outline of a proof, or with no proof at all.

Acknowledgements: It is always a pleasure to acknowledge the contribution of those who make a project of this nature possible. I wish to acknowledge Professor Himadrisekhar Sarkar and Professor Tapasendra Basu of Jadavpur University, and my students Professor Biswajyoti Saha of University of Hyderabad and Professor Ekata Saha of IIT (Delhi) and Professor Soumen Samanta of Bethune College and Manotosh Mondal of Tamralipta Mahavidyalaya who have read the manuscript, correcting many errors and giving their invaluable suggestions for improvement. I am deeply indebted to all those authors whose books (research papers) on real analysis influenced in learning of the subject and take this opportunity to express my gratitude to them. My largest debt, though, is to the students who read the manuscripts and kept a keen interest in learning throughout the difficult and unpleasant situation. Many examples and solutions to the problems are due to them. While these friends are not in any way responsible for the defects as the book may possess, their encouragement and criticism have been the most helpful in its preparation. The manuscript was cheerfully typed by me. My son Atandrita frequently helped me with my writing. Mr. Subhashis Dey of Creative Incorporate provided the much needed technical assistance. Mr. Dey was the best publisher, this ambitious project could have had. Finally, nothing would be possible without my wife Momi who is always by my side supporting me in every possible way.

I apologize in advance for whatever mistakes the alert reader may be able to detect. None were intentionally included; nevertheless, the detection and rectification of mistakes is a good exercise, and fosters a healthy skepticism about the printed word.

I strongly believe that no 'work' is ever complete till it has had its share of criticism, and hence I will be only too glad to receive comments and suggestions for the improvement of this book. These may please be forwarded to the e-mail address:

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Dipak Roy.

Ramakrishna Mission Residential College,
April, 2020.

Symbols Used

Symbols	Meaning
$\mathbb{N}$	Set of all positive integers.
$\mathbb{N}_0$	Set of non-negative integers.
$\mathbb{Z}$	Set of all integers.
$\mathbb{Q}$	Set of all rational numbers.
$\mathbb{Q}^*$	Set of all non-zero rational numbers.
$\mathbb{Q}^+$	Set of all positive rational numbers.
$\mathbb{R}$	Set of all real numbers.
$\mathbb{R}^*$	Set of all non-zero real numbers.
$\mathbb{R}^+$	Set of all positive real numbers.
$\overline{\mathbb{R}}$	Set of extended real numbers.
$\mathbb{C}$	Set of complex numbers.
$\mathbb{C}^*$	Set of all non-zero complex numbers.
$\vee$	or.
$\wedge$	and.
$\neg$	not.
$\Rightarrow$	implies.
$\Leftarrow$	implied by.
$\Leftrightarrow$	if and only if or iff.
$\forall$	for all.
$\exists$	there exists.
$\exists!$	there exists unique.
$\in$	belong(s) to.
$\notin$	does not belong to.
$A \cup B$	A union B.
$A \cap B$	A intersection B.
$\emptyset$	Empty set.
$A \subset B$	A is a proper subset of B.
$A \subseteq B$	A is a subset of B.
$A \supseteq B$	A is a superset of B.
$A \simeq B$	A is equipotent to B.

$B(a;r)$ or $N(a;r)$	Neighborhood(nbhd.) centered at a with radius r.
$\hat{B}(a;r) = B(a;r) \setminus \{a\}$	Neighborhood(nbhd.) with deleted center with radius r.
$A \triangle B$	A symmetric difference B.
$A \setminus B$	elements of A not in B.
$\omega_f(p)$	oscillation of f at a point p.
$\omega_f(D)$	oscillation of f on a set D.
$\|A\|$	cardinality of A.
$\aleph_0$	cardinality of $\mathbb{N}$.
$\mathfrak{c}$	cardinality of $\mathbb{R}$.
$\delta(A)$	diameter of a subset $A \subseteq \mathbb{R}$.
$f \sim g$	f is asymptotic to g.
$[x]$	integral part of $x \in \mathbb{R}$.

Chapter 1

Basic Set Theory

I see it, but I don't believe it.
...Cantor to Dedekind 29 June 1877.

Georg Cantor (1845-1918):
Georg Cantor was born on March 3, 1845, in St. Petersburg, Russia. He received his doctorate in mathematics from the University of Berlin in 1867, having studied under Weierstrass, Kummer and Kronecker. In 1869 he accepted a teaching position at the University of Halle and became a full professor in 1879. Cantor wanted to obtain a professorship at the University of Bertin, where both pay and prestige were higher, but Kronecker believing that much of Cantor's work (particularly his "transfinite numbers") was unsound, stood firmly in Cantor's path. Others, however, acknowledged Cantor is genius. Cantor was an honorary member of the London Mathematical Society and received honorary doctorates from both Christiania and St. Andrews. Hilbert said Cantor's work was "...the finest product of mathematical genius and one of the supreme achievements of purely intellectual human activity". Known as the founder of set theory, Cantor also made fundamental contributions to classical ana1ysis. Many concepts in modern mathematics bear his name, among which are Cantor series and Cantor sets; he also developed the first usable definition of the continuum. The controversy surrounding his work took a heavy toll on Cantor; beginning in 1884, bouts of deep depression drove him often to a sanitarium. Cantor died in a psychiatric clinic at the University of Halle (where he had remained as a professor) on January 6, 1918.

1.1 Intrduction to Set Theory

The term **"set"** is an undefined concept. In most occasions, we make use of terms that are accepted, understood without definitions. The terms we shall discuss but not attempt to define formally are **object, equals, element, is an element of.** Any attempt to define all the terms used will be an unsuccessful attempt. So it will be understood naïvely(non-axiomatically). In our daily life we frequently deal with notions representing "**collection**" of certain things, which "**belong**" to the collection in quesion, while other things do not belong to it, e.g. a class in a school is a collection of pupils belonging to the class, other things (persons, animals, plane figures etc.) do not belong to the class.
We mean by a "**set**" a notion of the type outlined above, i.e. a set is understood simply by a collection of objects and need not bear any obvious relationship to each other. The words collection,

family are sometimes used as synonyms for set. An object belonging to a set is called **an element or a member** of the set. We write $x \in A$, to mean x is a member of A and $x \notin A$, to mean x is not a member of A.
We shall use the word **"definition"** to represent some concept in a single word or phrase, using undefined terms or previously defined terms.That is, "definition" is a statement that explains the meaning of a term (a word, phrase, or other set of symbols). As for example, in geometry, points and lines are undefined terms but triangles and sqares are defined in terms of points and lines. It is necessary to know something about new terms, and axiom provides the information about the terms and their relationship.

- A **statement** is a sentence which is either true or false.
- An **axiom** is a statement that is assumed to be true.
- A **theorem** is a statement which is true and follows from the axioms, definitions and known results.
- A **lemma** is a derived result whose only real purpose is a tool in the proof of a theorem.
- A **corollary** is a result that follows almost immediately from a theorem.
- **Examples** are the objects that illustrate definitions and other concepts. And we shall study only the concepts that can be described by the examples.

The term defined by "axioms" play an important role in mathematics (and other areas as well) and that certain others are of no interest. In everyday language "axiom" means a self-evident truth. But we are not using everyday language; we are dealing with mathematics. An axiom is not a universal truth–but one of several rules spelling out a given mathematical structure. The axiom is true in the system we are studying because we have forced it to be true by hypothesis. It is a license, in the particular structure, to do certain things.

1.1.1 Note. We should remember one thing that the words lemma, theorem, proposition has no exact meaning and their uses vary from author to author. The common feature is that they represent a derived result.

Let S be a set. A **variable** $x(\in S)$ in a sentence P (often written $P(x)$ if x occurs in P) is said to be **bound variable** if it is preceeded in P by either $\forall\ x$ or $\exists\ x$; otherwise x is **free variable**. A sentence is **closed** if it contains no free variables, otherwise it is **open**. An open sentence which involves x as a free variable is called a **condition** on x. In other words, by an open sentence $P(\cdot)$ on S, we mean $P(\cdot)$ becomes a sentence either true or false, whenever " $\cdot$ " is replaced by the members of S. The symbol $\{x \in S; P(x)\}$ represents the set of members of $x \in S$ for which $P(x)$ is true.
If $P(\cdot)$ is an open sentence on a linearly ordered (see 1.3.10) set S, we consider three types of statements:

1. $\exists\ x \in S; P(x)$.
2. $\forall\ x \in S; P(x)$.
3. $\exists\ x' \in S; P(x)$ such that $\forall\ x > x';\ P(x)$. (is the symbolic definition of ultimately)

This means $P(x)$ is true for all sufficiently large values of x.

1.2 Some Useful Notions:

- **If-then statements**:
  In mathematics, statements A, B to be proved can often be put in the form

  1. if A, then B; $A \Rightarrow B$ (read: A implies B) or $B \Leftarrow A$ (read: B is implied by A). A is the **hypothesis**: "what is given", "what is known"; B is the **conclusion**: "what follows", "what is to be proved".
  2. **Converse**: If we interchange hypothesis and conclusion in $A \Rightarrow B$, we get $B \Rightarrow A$ which is called the **converse** to the statement (1).

  In $A \Rightarrow B$: A is a **sufficient** condition for B (if A is true, B follows);
  In $B \Rightarrow A$: A is a **necessary** condition for B (i.e. B can't be true unless A is also true, since B implies A). The following all mean the same

  1. A implies B.
  2. if A then B.
  3. A is sufficient for B.
  4. A only if B.
  5. B if A.
  6. B whenever A.
  7. B is necessary for A.

- **Equivalent statements**: We can combine the two implication arrows into one double-ended arrow: $A \Leftrightarrow B$ which is a true statement if both $A \Rightarrow B$ and $B \Rightarrow A$ are true. If this is so, we say A and B are **equivalent** statements.
  There are two verbal forms of $A \Leftrightarrow B$ which are in common use. They are A **if and only if** B (abbreviated: A **iff** B) and A is a **necessary and sufficient** condition for B. The "if and only if" is also separated: "A, **if** B": $B \Rightarrow A$; "A, **only if** B": $A \Rightarrow B$.

- **Stronger and weaker**. If $A \Rightarrow B$ is true, but $B \Rightarrow A$ is false, we say: A is a **stronger** statement than B; B is **weaker** than A.
  We turn now to discuss a style of mathematical proof which involves forming the negatives of statements.

- **Negation**: If A is a statement, we will use either not-A or $\neg A$ to denote its negation.

- **Contrapositive** proof: In proving $A \Rightarrow B$, sometimes it is more convenient to use contraposition, i.e. to prove the statement in its contrapositive form: $\neg B \Rightarrow \neg A$ (contrapositive of $A \Rightarrow B$).

- **Indirect** proof: To give an indirect proof that a statement S is true, we assume it is not true and derive a contradiction. To prove $A \Rightarrow B$ indirectly, assume A true but B false, and derive a contradiction: C and $\neg C$ are both true.

- **Without loss of generality**: If the proofs for the cases in a case distinction are very similar, it is customary to assume **without loss of generality** that one of these similar cases is true. This is not a loss of generality, because it is assumed that what is presented enables the reader to fill in the proof(s) for the other case(s).

- **Counterexample**: If a general statement claims something is true for every member of some class of objects, to show it is false we only have to produce a single object in that class for which the general statement fails to hold. Such an object is called a **counterexample** to the general statement.
- Proof by **Mathematical induction**: Let $P(\cdot)$ be an open sentence on $\mathbb{N}$. To prove $P(n), n > n_0$,
  1. prove $P(n_0)$ (the **basis step**);
  2. prove $P(n+1)$: in the proof you are allowed to use $P(n)$, and if necessary, $P(k)$ for any lower values, $n_0 < k < n$, as well (the **induction step**).
- Proof by **Strong Mathematical induction**: When one uses in the proof of $P(n)$ not just the preceding value but lower values of n as well, the proof method is generally referred to as **strong** or **complete induction**; in this style of induction, often more than one value of n is needed for the basis step. i.e. in strong induction, the basis step consists of all $P(n)$ not covered by the argument in the induction step, i.e., for which there are no lower $P(k)$ available to imply $P(n)$.

1.2.1 Definition. A set A is said to be a **subset** of B or B is a **superset** of A iff every member of A is a member of B, and is denoted by $A \subseteq B$ or $B \supseteq A$.

1.2.2 Definition. Two sets A and B are said to be **equal** iff $A \subseteq B$ and $B \subseteq A$, and is denoted by $A = B$. If $A \subseteq B$ and $A \neq B$, then A is said to be a **proper subset** of B and is denoted by $A \subsetneq B$ or $A \subset B$.

1.2.3 Definition. A set which is a subset of any other set is called a **null set** or **empty set** and is denoted by $\emptyset$. Note also that,

1. All empty sets are equal.
2. The empty set has no elements.
3. The only set with no elements is the empty set. For proofs see (1.10.4.1).

1.2.4 Definition. For the sets A and B, the **union (join)** denoted by $A \cup B$ and defined by $A \cup B = \{x; x \in A \text{ or } x \in B\}$.

1.2.5 Definition. For the sets A and B, the **intersection (meet)** denoted by $A \cap B$ and defined by $A \cap B = \{x; x \in A \text{ and } x \in B\}$.

1.2.6 Definition. Two sets A and B are said to be **disjoint** if they have no common elements i.e. iff $A \cap B = \emptyset$ and they **intersect** iff they have common elements iff $A \cap B \neq \emptyset$.

1.2.7 Definition. The family of all subsets of X is called the **power set** of X, and is denoted by $\mathcal{P}(X)$.

According to G. Cantor (1845-1918), who initiated the theory of sets in the last part of the nineteenth century: "A **set** is a collection into a whole of definite, distinct objects of our intuition or our thought. The objects are called the elements (members) of the set." We refer to the "whole of distinct objects" in Cantor's definition as the universal set. In other words,

1.2.8 Definition. If all sets under consideration in a certain discussion are subsets of a set U, then U is called the **Universal set** (for that discussion). Let $A, B \subseteq U$, then we define a set $A \setminus B = \{x; x \in A \text{ and } x \notin B\}$ and call it **complement of** B **relative to** A. And the set A^C defined by $A^C = U \setminus A$ is called the **complement** of A. The sets A, B generating a new set $A\Delta B$ is defined by $A\Delta B = (A \setminus B) \cup (B \setminus A)$ is called the **symmetric difference** of A and B.

1.3 Cartesian product, Relations

A pair of objects x and y, in which their order is relevant is known as **ordered pair** written (x, y) which is different from an unordered pair $\{x, y\}$ and having the characteristic property $(x, y) = (a, b)$ iff $x = a$ and $y = b$. Since the notion of ordered pair is undefined one, so define a set which behaves as an ordered pair satisfying above characteristic property.

1.3.1 Definition.

1. **(Kazimierz-Kuratowski in 1921)** An **ordered pair** (x, y)=$\{\{x\}, \{x, y\}\}$.
2. **(Norbert-Wiener in 1914)** $(x, y) = \{\{\{x\}, \emptyset\}, \{\{y\}\}\}$.
3. $(x, y) = \{\{x, \emptyset\}, \{y, \{\emptyset\}\}\}$.
4. $(x, y) = \{\{x, \emptyset\}, \{y\}\}$.
5. $(x, y) = \{\{x, \emptyset\}, y\}$.

We can prove that these above conditions satisfy the desired properties that two elements x and y, and $(x, y) = (a, b)$ iff $x = a$ and $y = b$. The definition given by Kazimierz-Kuratowski is in general use today.

1.3.2 Definition. If X and Y are sets, then the **cartesian product** of X and Y is the set defined by $X \times Y = \{(x, y); x \in X, y \in Y\}$.

1.3.3 Definition. If X and Y are sets, then a **relation or binary relation** from X to Y is any subset ρ of $X \times Y$. If $(x, y) \in \rho$ we write $x\rho y$ or x is ρ-related to y or y is ρ-relative to x or $y = \rho(x)$. If $\rho \subseteq X \times Y$, then the **domain** of ρ is dom(ρ)=$\{x \in X; (x, y) \in \rho\}$ and the **range** of ρ is ran(ρ)=$\{y \in Y; (x, y) \in \rho\}$. If $\rho \subseteq X \times Y$, then the **converse** relation $\rho^{-1} = \{(y, x); (x, y) \in \rho\}$. If $X = Y$, then we say that ρ is a relation on X.

1.3.4 Definition. If X, Y and Z are sets, and $\alpha \subseteq X \times Y$ and $\beta \subseteq Y \times Z$ then a **composition relation** from X to Z is defined by

$$\beta \circ \alpha = \{(x, z); \text{for some } y \in Y \text{such that} (x, y) \in \alpha, (y, z) \in \beta\}.$$

1.3.5 Definition. Let X be a set and ρ be a relation on X, then we define

1. ρ is **reflexive** iff $\Delta_X \subseteq \rho$ where $\Delta_X = \{(x, x); x \in X\}$,
2. ρ is **irreflexive** iff $\Delta_X \cap \rho = \emptyset$. In other words, $(x, x) \notin \rho \,\forall\, x \in X$.
3. ρ is **symmetric** iff $\rho^{-1} \subseteq \rho$,
4. ρ is **asymmetric** iff $\rho \cap \rho^{-1} = \emptyset$ i.e. if $(x, y) \in \rho$ then $(y, x) \notin \rho$.

5. ρ is **anti-symmetric** iff $\rho \cap \rho^{-1} \subseteq \Delta_X$. Equivalently, if $(x, y) \in \rho$ and $x \neq y$ then $(y, x) \notin \rho$.

6. ρ is **transitive** iff $\rho \circ \rho \subseteq \rho$. Equivalently, if $(x, z) \in \rho$ and $(z, y) \in \rho$ then $(x, y) \in \rho$.

7. ρ is **connected** iff $\rho \cup \rho^{-1} \cup \Delta_X = X \times X$.

8. ρ is an **equivalence** relation iff it is reflexive, symmetric and transitive. In other words, iff $\Delta_X \subseteq \rho, \rho^{-1} \subseteq \rho$ and $\rho \circ \rho \subseteq \rho$.

1.3.6 Definition. A family $\mathcal{P} = \{A_\alpha; \alpha \in \Lambda\}$ of sets $A_\alpha \subseteq X$ is said to be a **partition** of X iff

1. $X = \bigcup_{\alpha \in \Lambda} A_\alpha$

2. $A_\alpha = A_\beta$ if $\alpha = \beta$ and $A_\alpha \cap A_\beta = \emptyset$ if $\alpha \neq \beta$.

1.3.7 Theorem. Let ρ be an equivalence relation on a set X, then the equivalence class $[x]$ determined by $x \in X$ is defined by $[x] = \{y; (x, y) \in \rho\}$, and we can show that

1. $X = \bigcup_{x \in X} [x]$

2. $[x] = [y]$ or $[x] \cap [y] = \emptyset$ for $x, y \in X$.

The collection of equivalence classes is a partition $\mathcal{P}_\rho = \{[x]; x \in X\}$ determined by ρ of X and is also denoted by X/ρ, what we call a **quotient set** of X by ρ and if $\mathcal{P}$ is a partition of X, then the relation $\rho_{\mathcal{P}}$ on X defined by $(a, b) \in \rho_{\mathcal{P}}$ iff $\exists$ an element $A \in \mathcal{P}$ such that $a, b \in A$, then $\rho_{\mathcal{P}}$ is an equivalence relation on X determined by the partition $\mathcal{P}$. Moreover, we have a natural onto map

$$q : X \to X/\rho$$

defined by $q(x) = [x]$ called the **quotient map**. Theorem 1.3.7 basically says that $x\rho y$ if and only if $q(x) = q(y)$.

1.3.8 Definition. If a relation ρ on X is reflexive, and transitive, then ρ is called a **pre-ordering** of X and the pair (X, ρ) is called a **pre-ordered** set. If a relation ρ on X is reflexive, anti-symmetric and transitive, then ρ is called a **partial ordering (ordering)** of X and the pair (X, ρ) is called a **partial ordered** set or simply an **ordered** set. The symbols $<$ or $\leq$ are often used to denote orderings.
In other words, if a relation ρ on X satisfies $\Delta_X \subseteq \rho,\ \rho \cap \rho^{-1} \subseteq \Delta_X$ and $\rho \circ \rho \subseteq \rho$, then ρ is called a **partial ordering (ordering)** of X.

1.3.9 Definition. Let $a, b \in X$ and let $\leq$ an ordering of X, we say that a, b are **comparable** in the ordering $\leq$, if $a \leq b$ or $b \leq a$, and are **incomparable** if they are not comparable.

1.3.10 Definition. An ordering $\leq$ or $<$ of X is called a **linear** or **total** order if any two elements of X are comparable i.e. for $x, y \in X$ either $x \leq y$ or $y \leq x$ and then the pair $(X, \leq)$ is called a **linearly ordered** set. If $S \subseteq X$, where X is ordered by $\leq$, then S is called a **chain**, if any two elements of S are comparable.

1.3.11 Definition. Let $(X, \leq)$ be a partially ordered set, then for a non-empty subset $S \subseteq X$,

1. $b \in S$ is called a **least** element of S in the ordering $\leq$, if $b \leq x, \forall x \in S$.

2. $b \in S$ is called a **minimal** element of S, if $\exists$ no $x \in S$ such that $x \leq b$ and $x \neq b$.

3. $b \in S$ is called a **greatest** element of S, if $x \leq b, \forall\, x \in S$.

4. $b \in S$ is called a **maximal** element of S, if $\exists$ no $x \in S$ such that $b \leq x$ and $x \neq b$.

1.3.12 Example.

- Let $(X, \subseteq)$ be a partially ordered set where $X = \{a, b, c\}$, then consider $\mathcal{A} = \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{c, a\}\}$, here $\{a\}, \{b\}, \{c\}$ are minimal elements of $\mathcal{A}$ and $\{a, b\}, \{b, c\}, \{c, a\}$ are the maximal elements of $\mathcal{A}$. But does not have a least and a greatest elements.

- Let $X = \{2, 3, 4, 6, 8, 12, 24\}$ with the partial order relation " $\leq$ " defined by $x \leq y$ iff x is a divisor of y. Here 2 and 3 are minimal elements of X but does not have the least element and 24 is the greatest element.

- Let $(X, \subseteq)$ be a partially ordered set where $X = \{a, b, c\}$, then consider $\mathcal{A} = \{\{a\}, \{b\}, \{c\}, \{b, c\}\}$ here $\{a\}, \{b\}, \{c\}$ are the minimal elements of $\mathcal{A}$ and $\{a\}, \{b, c\}$ are the maximal elements of $\mathcal{A}$ and does not have the greatest element of $\mathcal{A}$.

1.3.13 Definition. Let $(X, \leq)$ be a partially ordered set, then for a non-empty subset $S \subseteq X$,

1. $a \in S$ is called a **lower bound** of S in the ordering $\leq$, if $a \leq x, \forall x \in S$.

2. $b \in S$ is called the **infimum (inf) or greatest lower bound (glb)** of S, if it is the greatest element of all lower bounds of S.

3. $a \in S$ is called the **upper bound** of S, if $x \leq a, \forall x \in S$.

4. $b \in S$ is called the **supremum (sup) or least upper bound (lub)** of S, if it is the least element of all upper bounds of S.

1.3.14 Definition. Let $(X, \leq)$ be a partially ordered set, then $(X, \leq)$ is said to be

1. a **directed set** if every two-element subset of X is bounded above;

2. a **lattice** if every two-element subset of X has both lub and glb;

3. an **inductively ordered set** if every chain in every subset of X has an upper bound;

4. an **well-ordered set** if every non-empty subset of X has a least element.

1.4 Functions

1.4.1 Definition. Let X, Y be non-empty sets then f is said to be a **mapping** or **function** from X to Y iff

1. $f \subseteq X \times Y$,

2. (x, y) and $(x, z) \in f \Rightarrow y = z$.

In this definition first part says that f is a relation from X to Y, and in the second part, we say that f is **well-defined** or **single-valued.** And $(x, y) \in f$ means $y = f(x)$, we say that y is the **image** of x under f, and x is the **pre-image** of y under f. Thus, we say that f is **well-defined** if $f(x) \neq f(y) \Rightarrow x \neq y$. In other words, $x = y \Rightarrow f(x) = f(y)$. If the domain of f is X, then we write $f : X \to Y$ or $X \xrightarrow{f} Y$ and we say f is **on** X, if range of $f \subseteq Y$ we say that f is **into** Y, Y is called co-comain of f and if range of f is Y we say that f is **onto** or **surjective**. We see that if $f\colon X \to Y$, then $f \subseteq X \times Y$; hence $f \in \mathcal{P}(X \times Y)$. The set $\{(x, f(x)); x \in X\}$ is called the the **graph** of a function f and is denoted by $gr(f)$.

1.4.2 Remark. The function f is identified with the graph of a function in its usual definition (by means of correspondence), i.e. The effect of the function f on an element x of X is denoted by $x \mapsto f(x)$ and thus

$$x \mapsto f(x) \Leftrightarrow (x, f(x)) \in gr(f).$$

1.4.3 Note. We can define an **unary** operation on a set X is a function $f : X \to X$, e.g. let X be any set and $\mathcal{P}(X)$ be the power set of X, then define a mapping $f : \mathcal{P}(X) \to \mathcal{P}(X)$ by

$$f(B) = B^C, \forall\, B \in \mathcal{P}(X),$$

where B^C denotes the complement of B.

1.4.4 Remark. The word "function" was used first by Leibniz in 1673, although not quite in the present-day meaning. In 1698, in a letter to Leibniz, Johann Bernoulli narrowed the meaning closer to what we accept today. During the 18th century, the notion evolved to describe an expression or formula involving variables and constants. It took a joint work of many mathematicians throughout the 19th century to hammer down the definition that we use nowadays.

1.4.5 Definition. Let X, Y be non-empty sets, then $Y^X = \{f; f\colon X \to Y\}$.

1.4.6 Remark. This is not the only reasonable approach to the notion of "function".(In fact, in **category theory** the order is reversed: the notion of "function" is one of the primitive notions, and the "set" is defined in terms of these!) However, the present approach is very convenient in the context of set theory.

1.4.7 Remark. Let Y be any set. Then $\emptyset \subseteq \emptyset \times Y$. Further, because $\emptyset$ has no elements, it follows trivially that to each $x \in \emptyset$ there is a unique $y \in Y$ such that $(x, y) \in \emptyset$. Hence by definition (1.4.1), $\emptyset$ is a function from $\emptyset$ to Y. Further, $\emptyset$ is the only function from $\emptyset$ to Y (since $\emptyset$ is the only relation with domain$(\emptyset) = \emptyset$.)
N.B. We cannot interchange $\emptyset$ and Y since for non-empty Y, domain$(\emptyset) \neq Y$ and by definition a function on Y has domain Y.
In other words, for a nonempty set Y, we have $\emptyset^Y = \emptyset$. This is because no function could have a nonempty domain and an empty range. On the other hand, $Y^\emptyset = \{\emptyset\}$ for any set Y, because $\emptyset : \emptyset \to Y$, but $\emptyset$ is the only function with empty domain. As a special case, we have $\emptyset^\emptyset = \{\emptyset\}$.

1.4.8 Definition. Consider the power set $\mathcal{P}(X)$ of X and let Λ be an arbitrary set, then define a function $f : \Lambda \to \mathcal{P}(X)$ by $f(a) = A_a \subseteq X; a \in \Lambda$. Then the family $\{A_\alpha; \alpha \in \Lambda\}$ is called an **indexed** family of sets with Λ as an **index** set.

Now we can generalize the notion of "cartesian product" which we have defined in **1.3.2** for a finite number of sets only, to an arbitrary family of sets.

1.4.9 Definition. Let $\mathcal{A} = \{X_i; i \in \Lambda\}$ be a family $\mathcal{A}$ of sets. The cartesian product of the family of sets X_i is denoted by $\prod_{i\in\Lambda} X_i$ and is defined by

$$\prod_{i\in\Lambda} X_i = \left\{ x;\ x\colon \Lambda \to \bigcup_{i\in\Lambda} X_i; x_i \in X_i\ \forall\ i \in \Lambda \right\}.$$

1.4.10 Definition. Let $f : X \to Y$, then f is called **injective or 1-1** if the distinct elements have distinct images in Y, i.e. $x \neq y \Rightarrow f(x) \neq f(y)$ which is equivalent to $f(x) = f(y) \Rightarrow x = y$. A function that is both injective and surjective is called **bijective.**

1.4.11 Definition. Let $f : X \to Y$,and $A \subseteq X$. Then the set $f(A) = \{f(x); x \in A\}$ is called the **direct image** of A under f and if $B \subseteq Y$. then the set $f^{-1}(B) = \{x \in X; f(x) \in B\}$ is called the **inverse image** of B under f.

1.4.12 Example. Let $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 4x^2 - 4x$. Then it can easily be verified that

$$\begin{aligned}
&f^{-1}(\mathbb{R}) = \mathbb{R},\ f^{-1}(\emptyset) = \emptyset,\ f^{-1}[-1,\infty) = \mathbb{R},\ f^{-1}(-1,\infty) = \mathbb{R} \setminus \left\{\frac{1}{2}\right\} = \left(-\infty, \frac{1}{2}\right) \bigcup \left(\frac{1}{2}, \infty\right),\\
&f^{-1}[0,\infty) = (-\infty, 0) \cup [1,\infty),\ f^{-1}(0,\infty) = (-\infty, 0) \cup (1,\infty),\ f^{-1}(-\infty,-1] = \left\{\frac{1}{2}\right\},\\
&f^{-1}(-\infty,-1) = \emptyset,\ f^{-1}(-\infty, 0] = [0,1],\ f^{-1}(-\infty,0) = (0,1),\ f^{-1}(0) = \{0,1\},\\
&f^{-1}(24) = \{-2,3\}.\ f^{-1}(8,24) = (-2,-1) \cup (2,3),\ f^{-1}[8,24] = [-2,-1] \cup [2,3].
\end{aligned}$$

We see from this example that the preimage of a set can be a single point, or a set of points, or it may even be empty.

1.4.13 Definition. Let $X, Y$and Z be sets such that $g : X \to Y$ and $f : Y \to Z$, then $f \circ g : X \to Z$ is called the **composition** function defined by $(f \circ g)(x) = f(g(x)), \forall x \in X$.

1.4.14 Note. It can be easily verified that, if $A \subseteq X$ and $B \subseteq Z$ then

$$(f \circ g)(A) = f(g(A)) \text{ and } (f \circ g)^{-1}(B) = g^{-1}(f^{-1}(B))$$

1.4.15 Definition. Let $f : X \to Y$, $g : Y \to X$ and $h : Y \to X$ are functions such that $g \circ f = \iota_X$ and $f \circ h = \iota_Y$, then g is called the **left inverse** of f and h is called the **right inverse** of f, where $\iota_A : A \to A$ is defined by $\iota(a) = a\ \forall\ a \in A$.

1.4.16 Definition. Let $f : X \to Y$, and $A \subseteq X$ the function $\iota : A \to X$ such that $\iota(x) = x, \forall x\ \in A$, then ι is called an **inclusion** function on X. If $\iota : A \to X$ then $g = f \circ \iota : A \to Y$ is called the **restriction** of f on A and is sometimes denoted by $f_{|A} : A \to Y$. and f is called an **extension** of g.

1.4.17 Definition. Let X be a set, and $A \subset X$, the function χ_A (or $\mathbf{1}_A$) from X to $\{0,1\}$ is called the **Characteristic Function (or indicator function)** of A, and is defined by

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A. \end{cases}$$

Given a universal set X, we obviously have $\chi_X = 1$ and $\chi_\emptyset = 0$ where by 1 and 0 we mean the constant functions identically equal to 1 and 0, respectively. Let $A \subseteq X$ be any set, and let

$\Delta = \Delta_A = \{(a, a); a \in A\}$ be the diagonal in $A \times A$: Then **Kronecker's delta**, $\delta = \chi_\Delta$, is the characteristic function of Δ:

$$\delta_{xy} = \delta(x, y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y. \end{cases}$$

1.4.18 Definition. Let ρ be an equivalence relation on X. As X/ρ is a partition of X, then the function $\alpha : X \to X/\rho$ defined by $\alpha(x) = [x]$ is called the **quotient map or canonical or natural mapping** on X onto X/ρ.

1.4.19 Remark (A decomposition of an arbitrary function)**.** Using the above we can write a function $f : X \to Y$ as a composition of a injective and surjective functions. Let ρ be the equivalence relation on X defined by $(x_1, x_2) \in \rho$ iff $f(x_1) = f(x_2)$. Let $\alpha : X \to X/\rho$ and $\beta : X/\rho \to f(X) \subseteq Y$, defined by $\beta([x]) = f(x)$. It clear that α is onto and it is an easy exercise to show that β is injective and $f = \beta \circ \alpha$.

1.4.20 Definition. Let $(X, \leq)$ and $(X', \leq')$ be two ordered sets, then a function f is called **order-preserving (isotone)** relative to " $\leq$ " for X and " $\leq'$ " for X' iff $x \leqslant y$ implies $f(x) \leq' f(y)$. An **isomorphism** between the partially ordered sets $(X, \leq)$ and $(X', \leq')$ is a surjective function $f : X \to X'$ which is order-preserving, and if such a function exists, then it is called an **isomorphic image** of the other or they are **isomorphic.** If f is an isomorphism then it is order-preserving which implies it is injective and hence bijective. Hence f^{-1} exists and it is order-preserving as f is order-preserving.

We will prove now that every partially ordered set $(X, \leq)$ can be represented as a partially ordered set $(\mathcal{Y}, \subseteq)$, where $\mathcal{Y} \subseteq \mathcal{P}(X)$.

1.4.21 Theorem. (**Representation theorem** for partially ordered sets)
Let $(X, \leq)$ be a partially ordered set, then there exists a partially ordered set $(\mathcal{Y}, \subseteq)$, where $\mathcal{Y} \subseteq \mathcal{P}(X)$, such that $(X, \leq)$ is isomorphic to $(\mathcal{Y}, \subseteq)$.

Proof. For $x \in X$, let $X(x) = \{y \in X; y \leq x\}$. Now let $\mathcal{Y} = \{X(x); x \in X\}$ and define $f\colon X \to \mathcal{Y}$ by $f(x) = X(x)$. We can check easily that f is a bijection from X to $\mathcal{Y}$, such that $x_1 \leq x_2 \Leftrightarrow f(x_1) \subseteq f(x_2)$. $\square$

1.4.22 Example. Consider $(X, \leq)$ where $X = \{2, 3, 4, 6, 8, 12\}$ with the partial order relation " $\leq$ " defined by $x \leq y$ iff x is a divisor of y. Here $(X, \leq)$ can be represented by $(\mathcal{Y}, \subseteq)$ where $\mathcal{Y} = \{\{2\}, \{2, 4\}, \{2, 4, 8\}, \{3\}, \{2, 3, 6\}, \{2, 3, 4, 6, 12\}\}$

1.4.23 Definition. A partially ordered set $(X, \leq)$ is **well-odered** if each non-empty subset of X has a least element.

1.4.24 Theorem. Let $(X, \leq)$ be a partially ordered set, then

1. S has at most one least element,
2. the least element of S (if it exists) is the minimal element,
3. if S is a chain, then every minimal element is the least element.

1.4.25 Definition (**PMI-1: Principle of Mathematical Induction**)**.** If $P(n)$ is a statement containing the variable n such that

1. $P(1)$ is true, and
2. for each $k \in \mathbb{N}$, $P(k)$ is true implies $P(k+1)$ is true, then $P(n)$ is true for all $n \in \mathbb{N}$.

1.4.26 Definition (PMI-2: Principle of Mathematical Induction). Suppose that $P(n)$ is a statement containing the variable n. If

1. for $k \in \mathbb{N}, P(k)$ is true, and
2. for each $m \in \mathbb{N}, m \geq k$, $P(m)$ is true implies $P(m+1)$ is true, then $P(n)$ is true for all $n \in \mathbb{N}, n \geq k$.

1.4.27 Remark. It can be shown that PMI-1 and PMI-2 are equivalent.

1.4.28 Theorem.

1. The set of natural numbers $\mathbb{N}$ is well ordered implies PMI-1.
2. PMI-1 implies $\mathbb{N}$ is well ordered.

Proof.

1. (1)$\Rightarrow$ (2)
   Assume (1) is true and $P(n)$ be a statement such that $P(1)$ is true and $P(n)$ is true implies $P(n+1)$ is true. Let $\exists\ m \in \mathbb{N}$ such that $P(m)$ is false. Let $E = \{t \in \mathbb{N}; P(t) \text{ is false}\}$. Since $m \in E$ then $E \neq \emptyset$. By (1) E has a least element, say n, by hypothesis $n \neq 1$, since $n \in E \subseteq \mathbb{N}, n-1 \notin E$ i.e. $P(n-1)$ is true which implies $P(n)$ is true i.e. $n \notin E$, a contradiction.
2. (2)$\Rightarrow$ (1)
   We prove this claim by contradiction. Suppose (2) is true and (1) is false i.e there exists a subset $A \subseteq \mathbb{N}$ which has no least element. Now consider the property $P(n)$ of n such that $P(n)$ is true if n is a lower bound of A. Let $B = \{n \in \mathbb{N}; P(n) \text{ is true}\}$. Clearly $1 \in B$ i.e. $P(1)$ is true. Now suppose $m \in B$ i.e. $P(m)$ is true. Since A has no least element, $m \notin A$, so we have $m < a\ \forall\ a \in A$. This implies $m+1 \leq a\ \forall\ a \in A$, i.e. $m+1$ is a lower bound of A. So $P(m+1)$ is true implies $m+1 \in B$. Thus $B = \mathbb{N}$. But this implies $A = \emptyset$, because if $x \in A$ then $x \in \mathbb{N} = B$ which means x is a lower bound of A, hence x is the least element of A, which is impossible.

$\square$

1.4.29 Theorem. The following are equivalent:

1. The set of natural numbers $\mathbb{N}$ has no upper bound.
2. If $x, y \in \mathbb{R}$ and $x > 0$, then $\exists\ n \in \mathbb{N}$ such that $nx > y$.
3. If $x \in \mathbb{R}$ and $x > 0$, then $\exists\ n \in \mathbb{N}$ such that $0 < 1/n < x$.
4. If $x \in \mathbb{R}$ and $x > 0$, then $\exists\ n \in \mathbb{N}$ such that $n \leq x < n+1$.
5. If $x \in \mathbb{R}$, then $\exists\ n \in \mathbb{N}$ such that $n > x$.

1.5 Finite sets, Infinite sets and Cardinal Numbers.

We ordinarily associate with every finite set a certain abstract identity "the number of elements" in the set. For infinite sets, common sense does not envisage a corresponding use of numbers like 0,1,2...But every set, finite or infinite, we take an attempt to generalize the notion of number so that it will apply to all sets without restriction.
Let $S_n = \{1, 2, .., n\}$ then we have

1.5.1 Theorem. Let $X \neq \emptyset$ be a set and let $X' = X \setminus \{w\}$ be a set obtained by deleting an element w from X. Then there exists a bijection from X to S_{n+1} iff there exists a bijection from X' to S_n.

1.5.2 Theorem. If there exists a bijection from S_m to S_n, then $m = n$.

1.5.3 Definition. A set X is said to be **finite**, iff there exists a bijection from X to S_n for some $n \in \mathbb{N}$.

1.5.4 Definition. A set X is said to be **infinite**, iff it is not finite.

1.5.5 Theorem. Every non-empty subset of natural numbers has a least element.

1.5.6 Corollary. The set $\mathbb{N}$ of all natural numbers is infinite.

1.5.7 Theorem. Let A be a non-empty subset of natural numbers with n elements. Then there exists a unique bijection $f : S_n \to A$ such that $f(p) < f(q)$ whenever $p, q \in S_n$ and $p < q$.

1.5.8 Note. The result of the above theorem says in effect that we can write any finite set of natural numbers in the form $\{n_1, n_2, .., n_k\}$, where $n_1 < n_2 < .. < n_k$. The result can be extended to an infinite set of natural numbers.

1.5.9 Theorem. If X is a finite set with n elements, then every subset of X is finite and has at most n elements.

In mathematics we now and then run into objects or notions, which are *"essentially the same"*. An example of such a case is the field of complex numbers $a + ib$ and the matrix ring $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ where $a, b \in \mathbb{R}$. Although the objects look radically different, they behave identically with respect to the algebraic operations. In algebra the term "isomorphic" and in topology "homeomorphic" are used to define the notions. And in naïve set theory, as the sets have no structure, so all that counts is their size (or cardinality). Hence the proper notion in this case is *equivalence*.

1.6 Cardinals

1.6.1 Definition. A set A is said to be **equivalent or equipotent** to a set B iff there is a bijection $f\colon A \to B$. We denote this by $A \simeq B$.

1.6.2 Definition. A set S is

1. **finite** if it is empty or for some $n \in \mathbb{N}, S \simeq \{1, ..., n\}$.
2. **infinite** if it is not finite.
3. **denumerable** if $S \simeq \mathbb{N}$.

4. **countable** if it is finite or denumerable.

5. **uncountable** if it is not countable.

In the literature we also meet the synonyms **equipotent, equipollent, similar or are of same cardinality** (german: gleichmächtig).
Now for a set X, define a relation " $\simeq$ " on $\mathcal{P}(X)$ by $A \simeq B$ iff there is a bijection $f\colon A \to B$, then " $\simeq$ " is an equivalence relation on $\mathcal{P}(X)$. Now if B belongs to the eqivalence class $[A]$ determined by A, then we say that the two sets A and B have the same **cardinality** i.e. in notation: $|A| = |B|$ or $cardA = cardB$. This is a "safe" imitation of Frege's definition of cardinals: $|\emptyset| = 0,\ |\{\emptyset\}| = 1,\ |\{\emptyset, \{\emptyset\}\}| = 2, \ldots$ and $\aleph_0$ is by definition of the cardinal number of $\mathbb{N}$ and $|\mathbb{R}| = \mathfrak{c}$ or $\aleph_1$.

1.6.3 Note. The word cardinal indicates the number of elements in the set. The cardinal numbers are 0,1,2,.. The first infinite cardinal number is **aleph null** or **aleph naught**, $\aleph_0$. We say that the $\mathbb{N}$ has $\aleph_0$ elements. A mystery of mathematics is the **Continuum Hypothesis** which states that $\mathbb{R}$ has cardinality $\mathfrak{c}$ or $\aleph_1$, the second infinite cardinal. Equivalently, if $\mathbb{N} \subseteq S \subseteq \mathbb{R}$, the Continuum Hypothesis asserts that $S \simeq \mathbb{N}$ or $S \simeq \mathbb{R}$. No intermediate cardinalities exist. It was shown as recently as 1963 by P. J. Cohen that this question is 'unanswerable' in the sense that the hypothesis is independent of the usual axioms of set theory. What this amounts to is that a mathematician may choose to accept or reject the hypothesis depending on the needs of the mathematics he wishes to develop. You can pursue this issue in Paul Cohen's book- Set Theory and the Continuum Hypothesis.

1.6.4 Theorem. For all sets X, $\mathcal{P}(X) \simeq \{0,1\}^X$.

1.6.5 Corollary. $\mathcal{P}(\mathbb{N}) \simeq \{0,1\}^{\mathbb{N}}$.

1.6.6 Definition. A set A is equivalent to a subset of B iff $\exists$ an injection $f\colon A \to B$. We write $A \lesssim B$.

1.6.7 Proposition. $\mathbb{N}$ is not equivalent to $\{0,1\}^{\mathbb{N}}$.

1.6.8 Proposition. $\mathbb{N}$ is not equivalent to $[0,1]$.

1.6.9 Remark (Denumerable sets)**.** There is a curious application of set theory to the field of hotel management, which is attributed to David Hilbert.
In a town X, there is a remarkable hotel, the *Hilbert Hotel*, which is distinguished from the average hotel by its size. The Hilbert Hotel is widely known for the fact that it contains denumerably many rooms, numbered 1,2,3,....
At the day of a big congress in X when a late guest wished to register, the Hilbert hotel was fully booked. Of course, he was kindly, but firmly, shown the door, but because of his persistence (the Hilbert Hotel was the hotel in X!) the desk clerk called for the manager. The manager apologized profusely, quoting the hotel-axiom: full is full. Fortunately the daughter of the manager, who couldn't sleep, came. The clever girl considered the problem for a moment and then gave a solution, which was both brilliant and simple: "Dad, request each guest to move to the room with next number, then this gentleman can take the number 1". This solved the whole problem; the manager, extremely relieved, noticed that in this way he could accomodate another hundred guests.
However, just when everybody was about to retire cheerfully for the night, a bus with the complete delegation drove up. Now, if the delegation had been finite, the manager could have accomodate it easily.

Unfotunately, the delegation was denumerable! It was again the daughter of the manager, who provided the solution: "It is quite simple, Dad, this time you request every guest to move to the room of which the number is twice the number of his present room. Then all rooms with an odd number will be vacant, and the representatives can be put in these rooms". The reader can easily verify that her proposal was correct; everything worked out perfectly.
So far the Hilbert Hotel had overcome all difficulties. The real problem started only, when the next day each guest wanted to accomodate denumerably many friends. How the manager will provide everybody with a room? Even the daughter found that the problem not totally trivial (as she said herself). She retreated in a small room next to reception desk, from which she immerged after a quarter of an hour, with the words: "Dad, it is trivial after all". Indeed, her solution turned out to be not so complicated. (Hint: Denumerable union of denumerable sets is denumerable.) The reader is invited to give a solution himself.
The Hilbert Hotel is still flourishing. What has become of the clever daughter is not known to us; some say she took up the study of Mathematics, other say she has married and lives comfortably and contently in a modest cottage at the edge of the forest.......

1.7 A Note on Axiom of Choice

1.7.1 Note. (Axiom of Choice)
Let $\mathcal{C}$ be a collection of non-empty sets. Then we can **choose** a member from each set in that collection. In other words, there exists a function f defined on $\mathcal{C}$ with th the property that, for each set $S \in \mathcal{C}, f(S) \in S$. The function f is then called a **choice function.** To understand this axiom better, let us consider a few examples:

- if $\mathcal{C}$ be the collection of non-empty subsets of $\mathbb{N} = \{1, 2, 3, ...\}$, then we can define f quite easily: just let $f(S)$ be the smallest member of S.
- if $\mathcal{C}$ be the collection of non-empty bounded intervals in $\mathbb{R}$, the set of real numbers, then we can define $f(S)$ to be the midpoint of the interval S.
- if $\mathcal{C}$ be some more general collection of subsets of $\mathbb{R}$, we may be able to define f by using a more complicated rule.

However, if $\mathcal{C}$ be any arbitrary collection of non-empty subsets of $\mathbb{R}$, it is not clear how to find a suitable function f, such that for $S \in \mathcal{C}, f(S) \in S$. In fact, no one has ever found a suitable function f for any collection of sets.

1.7.2 Definition. (Axiom of Choice:) Consider a set X and $\mathcal{C}$ be a family of non-empty pairwise disjoint subsets in $\mathcal{P}(X)$, then there **exists** a set D consisting of exactly one point from each member of $\mathcal{C}$.

If we want to obtain a set by application of this existential axiom, that cannot be uniquely determined. We can overcome the controversy, if we can interpret the words "choose" and "exists" in the axiom. If we follow the constructivists, and "exists" means "find", then the axiom is false, since we cannot find a choice function for the non-empty subsets of the reals. However, most people give "exists" a much weaker meaning, and they consider the Axiom to be true: To define $f(S)$, just arbitrarily "pick any member" of S.

In effect, when we accept the Axiom of Choice, this means we agree the convention that we shall permit ourselves to use a choice function f in proofs, as though it exists in some sense, even though we cannot give an explicit example of it or an explicit algorithm for it.

1.7.3 Note.

- A question: "**Do we need the axiom of choice?**" It purely depends on what kind of mathematics we are engaged in.
  Illustration of the use of axiom of choice: (By M.E.Rudin.)
  Let us consider some banality like the following.
  We know that countable union of countable sets is countable. In other words, we want to show that:
  If $A_i \neq \emptyset$ is countable for $i \in \mathbb{N}$, then $A = \bigcup_{i=1}^{\infty} A_i$ is countable.
  This is used in real analysis all over and most of us do not even realize that the proof uses the axiom of choice. The proof goes like this. Since each A_i is countable, so the enumeration A_i for each i as follows. Consider

$$
\begin{aligned}
&A_1 = \{a_{11}, a_{12}, a_{13}, ..., a_{1n}, ..\} \\
&A_2 = \{a_{21}, a_{22}, a_{23}, ..., a_{2n}, ..\} \\
&A_3 = \{a_{31}, a_{32}, a_{33}, ..., a_{3n}, ..\} \\
&... \\
&A_i = \{a_{i1}, a_{i2}, a_{i3}, ..., a_{in}, ..\} \\
&...
\end{aligned} \tag{1.1}
$$

  Thus $A_i = \bigcup_{j=1}^{\infty} \{a_{ij}\}$ and

$$A = \bigcup_{i=1}^{\infty} A_i = \bigcup_{j=1}^{\infty} \bigcup_{i=1}^{\infty} \{a_{ij}\}.$$

  Hence we can arrange the elements of the union into a countable sequence using well known counting method.

$$a_{11}, a_{12}, a_{22}, a_{21}, a_{13}, a_{23}, a_{33}, a_{32}, a_{31}, ... \tag{1.2}$$

  So where is the axiom of choice? Let us go over the proof once more, we are given a countable collection of sets $\mathcal{A} = \{A_1, A_2, .., A_i, ..\}$. Each A_i is a countable set and so for each i, there exists an enumeration of $A_i = \{a_{i1}, a_{i2}, ..., a_{ij}, ..\}$. However, there exists more than one enumeration of A_i. If for each i, E_i denotes the set of all enumerations of the form (1.2), then we are confronted with the following problem:
  If we want to apply the diagram (1.1), we have to choose one specific enumeration of A_i, for each i. In other words, we have to choose one element from each E_i. And here we are; we need a choice function on the family $\mathcal{A} = \{A_1, A_2, ..., A_i, ..\}$. Naturally the argument above shows that axiom of choice is used in the particular proof of the theorem, that countable union of countable sets is countable. Also note that it does not rule out the possibilty of finding an alternative proof that would make no reference to the axiom of choice.

- Another application: For any mapping $\alpha : S \to T$, there exists a surjection β and an injection γ such that $\alpha = \gamma \circ \beta$. (Compare 1.10.65)

1.7.4 Theorem. THE FOLLOWING ARE EQUIVALENT:

1.(WOP) (The **well-ordering principle**, also called **Zermelo's Theorem**) Every nonempty set can be well ordered. (E.F.F.Zermelo was a German mathematician).

2.(ZL) (**The Zorn's Lemma**): Let $(S, \leq)$ be a preordered set. Assume that every chain in S has an upper bound in S. Then S has at least a maximal element. (The lemma is named after M. Zorn.)

3.(AC) (**The Axiom of Choice**): Let $\mathcal{C}$ be a collection of non-empty sets. Then there exists a function f defined on $\mathcal{C}$ with th the property that, for each set $S \in \mathcal{C}, f(S) \in S$. (The function f is then called a **choice function.**)

1.7.5 Remark. There is something misleading about the names of the three statements (WOP), (ZL), and (AC) above. The first one is called a "Principle" (or a "Theorem"), the second a "Lemma," and the third an "Axiom." This is just for historical reasons. Specially the use of "Theorem" and "Lemma" is unappropriate. It suggests that they can be proved from more basic results. The theorem above dismisses this suggestion as incorrect; if one admits one of the three statements he must admit the two others. Their logic value is the same. As a matter of fact, the assumption of the validity of one of them allows to prove the rest. The negation of one of them forces the negation of the other two.
A long list of equivalent statements has been added by several authors to the above theorem. For example, and just mentioning concepts used in this book, the statement called Tychonoff's Theorem: An arbitrary product of compact sets is itself compact (Kelley:[33]), or the statement: In the product topology, the closure of a product of subsets is equal to the product of their closures. Another comes from the realm of Linear Algebra: Every nonempty vector space has an algebraic basis (Hamel basis)[11].

1.8 Axiomatic Set Theory

(**Russell's Paradox**). A set is either a member of itself, or it is not. Let R denote the set of all sets which are not members of themselves. Then if $R \in R$ it follows that $R \notin R$. If $R \notin R$, it follows that $R \in R$. Hence it cannot be that $R \in R$ or that $R \notin R$. It is clear from this and other paradoxes, that there is a need for the axiomatization of intuitive set theory. These paradoxes are avoided in axiomatic set theory by the elimination of "sets" that are "too large." To develop the theory of sets from the axiomatic point of view is a long and difficult process, far removed from Real Analysis. For this reason we have made no effort to be rigorous in dealing with sets, but have rather appealed to intuition.

1.9 Axioms and Choice

We shall give a basic treatment of set theory, namely the framework in which most of our mathematics will take place. Our undefined terms are **class** and the relation "$\in$" called the **belonging relation**. For any two classes A and $B, A \in B$ is either true or false.
A class also consists of elements and is characterized by its elements. All sets are classes. But a class is a set iff it can be an element of some class. This implies that it is forbidden to make use of certain classes as elements for the construction of a new set. An example of such a class is the class

of all sets. On the other hand, for each set A there is canonically defined a new set $\mathcal{P}(A)$ which, in particular, contains A as an element: $\mathcal{P}(A)$ is the set of all subsets of A. The collection of all conceivable sets into a class not forming a set permits one to avoid the well-known paradoxes that arose in "naïve" set theory.

1.9.1 Definition. We write $A \subseteq B$ if $x \in A \Rightarrow x \in B$. We say A and B are equal and write $A = B$ if $A \subseteq B$ and $B \subseteq A$. For a given class A if there exists a class B such that $A \in B$ then we call A a **set**. Otherwise it is a proper class.

Axiom 1: **The Axiom of Existence**: There exists a set which has no elements.

Axiom 2: **The Axiom of Extensionality**: If every element of X is an element of Y and every element of Y is an element of X then $X = Y$.

Axiom 3: **The Axiom Schema of Comprehension**: For each formula p in which only set variables are quantified and in which the class variable A does not appear, there is a class A whose members are just those sets having property p.

Axiom 4: **The Axiom of Pair**: For any sets A and B, there is a set C such that $x \in C$ if and only if $x = A$ or $x = B$. We write $\{A, B\}$.

Axiom 5: **The Axiom of Union**: For any set S, there is a set U such that $x \in U$ if and only if $x \in A$ for some $A \in S$.

Axiom 6: **The Axiom of Power Set**: For any set A, there is a set $P(A)$ called the power set of A such that $X \in P(A)$ if and only if $X \subseteq A$.

Axiom 7: **The Axiom of Replacement**: Let ϕ be a formula such that for every set x there is a unique set y for which $\phi(x, y)$ is true. For every set A there is a set B such that for all $x \in A$ there is a $y \in B$ for which $\phi(x, y)$ holds.

Axiom 7′: **The Axiom of Replacement**: If A is a set and $f : A \to B$ is a a function, then the range of f is a set.

Axiom 8: **The Axiom of Infinity**: There exists a set A with the properties: (i) $\emptyset \in A$ and (ii) if $a \in A$, then $a \cap \{a\} \in A$.

Axiom 9: **The Axiom of Foundation**: For every nonempty set A there is a $x \in A$ such that $x \cap A = \emptyset$.

Axiom 10: **The Axiom of Choice**: Given any nonempty family $\{A_i; i \in \Lambda\}$ where each A_i and Λ are sets, then there exists a set S consisting of exactly one element from each A_i.

It can be shown that there is a unique set satisfying Axiom 1. It is called the empty set and written as $\emptyset$. Axiom 4 and 7 give us that we may take Cartesian products and hence define functions. Thus, the reason why there are two Axiom of Replacement listed. Axioms 1-9 are called Zermelo-Frankel set theory or often times **ZF**. When one includes Axiom 10, we write it as **ZFC**. Choice is indeed independent of the other axioms.

1.9.2 Lemma (ZF). There does not exist a set x satisfying $x \in x$.

Proof. Otherwise, by the Axiom of Power Set we could form the set $A = \{x\}$. Now, this set must satisfy foundation but it only has one element, namely x and so $x \cap A = \emptyset$. $\square$

1.9.3 Theorem (Schróder-Bernstein). If $A \leq B$ and $B \leq A$, then $A \simeq B$.

The proof of the Schróder-Bernstein theorem is long and technical. As it does not illustrate anything new we shall leave it to the interested reader to find a suitable reference. At this point there are one of two ways to define cardinal numbers. One is to simply define the **cardinality** of a set A to be the equivalence class under $\leq$. But through this way you obtain that cardinalities are not sets but proper classes. The other way is to innocuously define cardinals as a collection of sets with the property that each arbitrary set A is assigned a unique cardinal $|A|$ and that $A \simeq B$ if and only if $|A| = |B|$. You then can induce the order onto the class of cardinals. Now, again the last two have the feature that if you simply want to discuss cardinalities and obtain known results such as

1.9.4 Theorem (Cantor's Theorem). For any set A, we have $|A| \leq |P(A)|$.

1.10 Problems and Solutions on Chapter 1

Warning! *Never Never Ever read this solution unless you tried problems quite long time and gave up. Doing so may impair your ability to think and solve problems.*

1.10.1 Problem. Show that $\{x; x \neq x\} = \emptyset$.

1.10.1.1 Solution. If $\{x; x \neq x\} \neq \emptyset$, then there exists x such that $x \neq x$, a contradiction.

1.10.2 Problem. Let $\{A_n; n \in \mathbb{N}\}$ be a collection of sets.

1. Consider $B_n = \bigcup_{i=1}^n A_i, n \in \mathbb{N}$. Show that $\bigcup_{i=1}^\infty B_i = \bigcup_{i=1}^\infty A_i$.
2. Consider $B_n = \bigcap_{i=1}^n A_i, n \in \mathbb{N}$. Show that $\bigcap_{i=1}^\infty B_i = \bigcap_{i=1}^\infty A_i$.

1.10.2.1 Solution.

1. Let

$$\begin{aligned} x \in \bigcup_{i=1}^{\infty} B_i \Leftrightarrow x \in B_i = \bigcup_{k=1}^{i} A_k \text{ for some } i \\ \Leftrightarrow x \in A_k \text{ for some } k \Leftrightarrow x \in \bigcup_{i=1}^{\infty} A_i. \end{aligned}$$

2. Similar to above. $\square$

1.10.3 Problem. Prove that the implication is left distributive with respect to the disjunction.

1.10.3.1 Solution. We have to prove

$$A \Rightarrow (B \vee C) = (A \Rightarrow B) \vee (A \Rightarrow C).$$

By the basic properties of the $\vee$ operation (idempotency, commutativity, associativity) and the identity $(X \Rightarrow Y) = \neg X \vee Y$, thus

$$\begin{aligned} A \Rightarrow (B \vee C) &= \neg A \vee (B \vee C) = (\neg A \vee \neg A) \vee (B \vee C) \\ &= (\neg A \vee B) \vee (\neg A \vee C) \\ &= (A \Rightarrow B) \vee (A \Rightarrow C). \quad \square \end{aligned}$$

1.10.4 Problem.

1. All empty sets are equal.
2. The empty set has no elements.
3. The only set with no elements is the empty set.

1.10.4.1 Solution.

1. Suppose $\emptyset_1$ and $\emptyset_2$ are any two empty sets. Since an empty set is a subset of any other set, we must have both $\emptyset_1 \subseteq \emptyset_2$ and $\emptyset_2 \subseteq \emptyset_1$. By the definition of equality of two sets, we get $\emptyset_1 = \emptyset_2$. This proves (1).
2. Suppose $x \in \emptyset$. Since for any set A we have $\emptyset \subseteq A$ and $\emptyset \subseteq A^C$, we must have both $x \in A$ and $x \in A^C$, which contradicts the existence of x. This proves (2).
3. Suppose X is a set with no elements and $X \neq \emptyset$. Since $\emptyset \subseteq X$, then $X \nsubseteq \emptyset$. Then there must be an element $x \in X$ such that $x \notin \emptyset$. But X has no elements which is a contradiction. Hence the proof.

1.10.5 Problem. Show that, for any subsets A, B and C of X,

1. $A\Delta B = \emptyset$ iff $A = B$.
2. $A\Delta B = X$ iff $A = B^C$.
3. $A\Delta\emptyset = A$ and $A\Delta X = A^C$.
4. $(A\Delta B) \cap C = (A \cap C)\Delta(B \cap C)$.
5. $(A \cup B) \cap (B \cup C) \cap (C \cup A) = (A \cap B) \cup (B \cap C) \cup (C \cap A)$
   $=(A \cup B \cup C) \cap (A^C \cup B \cup C) \cap (A \cup B^C \cup C) \cap (A \cup B \cup C^C)$
   $= (A \cap B \cap C) \cup (A^C \cap B \cap C) \cup (A \cap B^C \cap C) \cup (A \cap B \cap C^C)$.
6. $(A \setminus B) \cup (B \setminus C) \cup (C \setminus A) = (A \cup B \cup C) \setminus (A \cap B \cap C)$
   $= (A\Delta B) \cup (B\Delta C) \cup (C\Delta A) = (B \setminus A) \cup (C \setminus B) \cup (A \setminus C)$.

1.10.5.1 Solution. To solve some of these problems, we adopt the following short-hand algebraic notations so that the calculations can be done quickly:

$+$ for $\cup$, $\cdot$ for $\cap$, $'$ for complementation ^{C}, 0 for $\emptyset$, 1 for X and we write $A \cdot B = AB$; $A \cdot 1 = A; A + A' = 1; A \cdot A' = 0$
Then $A \cup B = A + B$, $A \cap B = A.B$ and $A \setminus B = A \cap B^C = AB'$, $A\Delta B = (A \setminus B) \cup (B \setminus A) = AB' + A'B$ and we can write $A \cdot B = AB, A + A' = 1, AA' = 0, 1' = 0, 0' = 1$.

1.

$$\begin{aligned}
&A\Delta B=\emptyset\\
\Leftrightarrow&(A\setminus B)\cup(B\setminus A)=\emptyset\\
\Leftrightarrow&A\setminus B=\emptyset \text{ and } B\setminus A=\emptyset\\
\Leftrightarrow&A\subseteq B \text{ and } B\subseteq A\\
\Leftrightarrow&A=B.
\end{aligned}$$

2.

$$\begin{aligned}
&A\Delta B=X\\
\Leftrightarrow&AB'+A'B=1\\
\Rightarrow&A(AB'+A'B)=A1=A \text{ and}\\
&B'(AB'+A'B)=B'1=B'\\
\Rightarrow&AB'=A \text{ and } AB'=B'\\
\Rightarrow&A=B'.
\end{aligned}$$

Hence $A=B^C$. Converse part is left to the reader.

3. Left to the reader.

4. Now, RHS is

$$\begin{aligned}
(A\cap C)\Delta(B\cap C)&=AC(BC)'+(AC)'BC\\
&=AC(B'+C')+(A'+C')BC\\
&=ACB'+A'BC=(AB'+A'B)C=(A\Delta B)\cap C
\end{aligned}$$

Hence $(A\Delta B)\cap C=(A\cap C)\Delta(B\cap C)$.

5. We have

$$\begin{aligned}
&(A\cup B)\cap(B\cup C)\cap(C\cup A)\\
=&(A+B)(B+C)(C+A)\\
=&(AB+AC+BB+BC)(C+A)\\
=&ABC+ACC+BBC+BCC+ABA+ACA+BBA+BCA\\
=&ABC+AC+BC+BC+AB+AC+BA+BCA\\
=&ABC+BC+AB+AC\\
=&BC+AB+AC=AB+BC+CA.
\end{aligned}$$

Thus $(A\cup B)\cap(B\cup C)\cap(C\cup A)=(A\cap B)\cup(B\cap C)\cup(C\cap A)$.
Again,

$$\begin{aligned}
BC+AB+AC&=BC(A+A')+AB(C+C')+AC(B+B')\\
&=BCA+BCA'+ABC+ABC'+ACB+ACB'\\
&=BCA+BCA'+ABC'+ACB'.
\end{aligned}$$

Thus $(B \cup C) \cap (A \cup B) \cap (A \cup C)$
$= (A \cap B \cap C) \cup (A' \cap B \cap C) \cup (A \cap B' \cap C) \cup (A \cap B \cap C')$. In a similar way, we get the next result.

6. Again,

$$\begin{aligned}
&(A \cup B \cup C) \setminus (A \cap B \cap C) \\
=&(A + B + C)(ABC)' = (A + B + C)(A' + B' + C') \\
=&AB' + AC' + BA' + BC' + CA' + CB' \\
=&(AB' + BA') + (BC' + CB') + (CA' + C'A) \\
=&A\Delta B \cup B\Delta C \cup C\Delta A.
\end{aligned}$$

and

$$\begin{aligned}
&(AB' + BA') + (BC' + CB') + (CA' + C'A) \\
=&(AB' + BA')(C + C') + (BC' + CB')(A + A') + (CA' + C'A)(B + B') \\
=&(AB'C + A'BC + AB'C' + A'BC') + (ABC' + AB'C + A'BC' + A'B'C) \\
&+ (A'BC + ABC' + A'B'C + AB'C') \\
=&AB'C + A'BC + AB'C' + ABC' + A'BC' + A'B'C \\
=&AB'(C + C') + BC'(A + A') + CA'(B + B') \\
\text{or } =&A'B(C + C') + B'C(A + A') + C'A(B + B') \\
=&AB' + BC' + CA' \text{ or } = A'B + B'C + C'A. \quad \square
\end{aligned}$$

1.10.6 Problem. Show that, for any sets A and B, there is a set C such that $A\Delta C = B$: Is C unique?

1.10.6.1 Solution. Suppose D be a set such that $A\Delta D = B$ then

$$\begin{aligned}
&A\Delta C = A\Delta D \\
\Rightarrow&A\Delta(A\Delta C) = A\Delta(A\Delta D) \\
\Rightarrow&(A\Delta A)\Delta C = (A\Delta A)\Delta D \\
\Rightarrow&C = D.
\end{aligned}$$

Hence C is unique. $\square$

1.10.7 Problem. Given two sets A and B, show that

$$A \cup B = A\Delta B\Delta(A \cap B) \text{ and } A \setminus B = A\Delta(A \cap B) :$$

Deduce that, if $A \cap B = \emptyset$, then $A\Delta B = A \cup B$.

1.10.7.1 Solution. Now,

$$
\begin{aligned}
&A\Delta B\Delta(A\cap B)\\
=&(AB'+A'B)\Delta(AB)\\
=&(AB'+A'B)(A.B)'+(AB'+A'B)'(AB)\\
=&(AB'+A'B)(A'+B')+(A'+B)(A+B')(AB)\\
=&(AB'+A'B)+(AB+A'B')(AB)\\
=&(AB'+A'B)+AB\\
=&(AB'+A'B)+AB+AB\\
=&(AB'+AB)+(A'B+AB)\\
=&A(B'+B)+(A'+A)B\\
=&A+B.
\end{aligned}
$$

Thus $A\Delta B\Delta(A\cap B)=A\cup B$. Again, if $A\cap B=\emptyset$ then $A\setminus B=A$ and $B\setminus A=B$. Hence $A\Delta B=(A\setminus B)\cup(B\setminus A)=A\cup B$. □

1.10.8 Problem. Is it true for all subsets X,Y,Z of U that

1. $X\cap(Y\Delta Z)=(X\cap Y)\Delta(X\cap Z)$.
2. $X\cup(Y\Delta Z)=(X\cup Y)\Delta(X\cup Z)$?

1.10.8.1 Solution.

1. True. We have

$$
\begin{aligned}
(X\cap Y)\Delta(X\cap Z)&=(XY)(XZ)'+(XY)'(XZ)\\
&=(XY)(X'+Z')+(X'+Y')(XZ)\\
&=(XY)X'+(XY)Z'+X'(XZ)+Y'(XZ)\\
&=(XY)Z'+Y'(XZ)=X(YZ'+Y'Z)\\
&=X\cap(Y\Delta Z).
\end{aligned}
$$

2. False. For, if $X=\{1,2,3\},Y=\{2,3,4\},Z=\{3,4,5\}$ then $(X\cup Y)\Delta(X\cup Z)=\{5\}$ and $X\cup(Y\Delta Z)=\{1,2,3,5\}$. □

1.10.9 Problem. Show that, given two sets A and B, we have $A=\emptyset\Leftrightarrow B=A\Delta B$.

1.10.9.1 Solution. We have

$$
\begin{aligned}
&B=A\Delta B\\
\Rightarrow&B\Delta B=(A\Delta B)\Delta B\\
\Rightarrow&0=A\Delta(B\Delta B)\\
\Rightarrow&\emptyset=A\Delta\emptyset=A.
\end{aligned}
$$

and other part is clear. □

1.10.10 Problem. Suppose $\{A_\alpha;\alpha\in\Lambda\}$ be a family of subsets of X, then

1. $\bigcup_{\alpha\in\Lambda} A_\alpha = \emptyset$, if $\Lambda = \emptyset$.
2. $\bigcup_{\alpha\in\Lambda} A_\alpha = \emptyset$ iff either $\Lambda = \emptyset$ or $A_\alpha = \emptyset$.
3. $\bigcap_{\alpha\in\Lambda} A_\alpha = X$, if $\Lambda = \emptyset$.

1.10.10.1 Solution.

1. Suppose $x \in \bigcup_{\alpha\in\Lambda} A_\alpha \Rightarrow \exists\, \alpha \in \Lambda$ such that $x \in A_\alpha \Rightarrow \Lambda \neq \emptyset$.
2. Left to the reader.
3. $\bigcap_{\alpha\in\Lambda} A_\alpha = X \Rightarrow \bigcup_{\alpha\in\Lambda} A_\alpha^C = \emptyset$ and apply 1. □

1.10.11 Problem. If (E_n) be a sequence of subsets of a set X, we define

$$\lim_{n\to\infty} \sup E_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k = \{x \in X; x \in E_n \text{ frequently}\}$$

and

$$\lim_{n\to\infty} \inf E_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} E_k = \{x \in X; x \in E_n \text{ ultimately}\}.$$

Show that

1. $\lim_{n\to\infty} \inf\ E_n \subseteq \lim_{n\to\infty} \sup E_n$.
2. $\lim_{n\to\infty} \sup E_n = \{x \in X; x \in E_n, \text{for infinitely many } n\}$.
3. $\lim_{n\to\infty} \inf E_n = \{x \in X; x \in E_n, \text{for all but finitely many } n\}$.
4. if (a) $E_1 \supseteq E_2 \supseteq ...$ and if (b) $E_1 \subseteq E_2 \subseteq ..$, then compute $\lim_{n\to\infty} \inf E_n$ and $\lim_{n\to\infty} \sup E_n$ in both cases.
5. $\lim_{n\to\infty} \inf\ E_n^C = (\lim_{n\to\infty} \sup E_n)^C$.
6. $(\lim_{n\to\infty} \inf\ E_n \cup \lim_{n\to\infty} \inf F_n) \subseteq \lim_{n\to\infty} \inf (E_n \cup F_n)$, and equality holds if $\cup$ is everywhere replaced by $\cap$.
7. $\lim_{n\to\infty} \sup (E_n \cap F_n) \subseteq (\lim_{n\to\infty} \sup E_n \cap \lim_{n\to\infty} \sup F_n)$, and equality holds if $\cap$ is everywhere replaced by $\cup$.

1.10.11.1 Solution.

1. Let $P_n = \bigcap_{k=n}^{\infty} E_k$, then
$$P_1 \subseteq P_2 \subseteq ... \subseteq P_n \subseteq P_{n+1} \subseteq ...$$
and let $Q_n = \bigcup_{k=n}^{\infty} E_k$, then
$$Q_1 \supseteq Q_2 \supseteq ... \supseteq Q_n \supseteq Q_{n+1} \supseteq ....$$
Since
$$P_1 \subseteq P_2 \subseteq ... \subseteq P_n \subseteq P_{n+1} \subseteq ..... \subseteq Q_{n+1} \subseteq Q_n \subseteq ... \subseteq Q_2 \subseteq Q_1,$$

hence

$$\begin{aligned}
&\sup_n P_n \subseteq \inf_n Q_n \\
&\Rightarrow \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} E_k \subseteq \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k \\
&\Rightarrow \lim_{n\to\infty} \inf \ E_n \subseteq \lim_{n\to\infty} \sup E_n.
\end{aligned}$$

2. Let $x \in \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k$, then $x \in \bigcap_{n=1}^{\infty} Q_n$. So, $x \in Q_n \ \forall \ n = 1, 2, \ldots.$ Suppose $x \in Q_n$ implies $x \in E_{k_n} \ \forall \ n = 1, 2, \ldots$ Then $x \in E_k$ for infinitely many k.
3. Let $x \in \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} E_k$, then $x \in \bigcup_{n=1}^{\infty} P_n$ implies $x \in P_k$ for some k. Suppose $x \notin P_1, \ldots, P_{k-1}$ implies $x \in E_i \ \forall \ i \geq k$. Then $x \in E_k$ for all but finitely many k.
4. (a) If $E_1 \supseteq E_2 \supseteq \ldots$, then $\bigcup_{k=n}^{\infty} E_k = E_n$, so, $\lim_{n\to\infty} \sup E_n = \bigcap_{n=1}^{\infty} E_n$ and $\bigcap_{k=n}^{\infty} E_k = \bigcap_{k=1}^{\infty} E_k$, so, $\lim_{n\to\infty} \inf E_n = \bigcap_{n=1}^{\infty} E_n$
   (b) If $E_1 \subseteq E_2 \subseteq \ldots$, then $\bigcap_{k=n}^{\infty} E_k = E_n$, so, $\lim_{n\to\infty} \inf E_n = \bigcup_{n=1}^{\infty} E_n = \lim_{n\to\infty} \sup E_n$.
5. Left to the reader.
6. Left to the reader.
7. Left to the reader. □

1.10.12 Problem. For each $n \in \mathbb{N}$, let $M_n = \left\{\frac{k}{n}; k \in \mathbb{Z}\right\}$. Then show that $\limsup_n M_n = \mathbb{Q}$ and $\liminf_n M_n = \mathbb{Z}$.

1.10.12.1 Solution. Here, we can write $M_n = \left\{\frac{k}{n}; k \in \mathbb{Z}\right\} = \frac{1}{n}\mathbb{Z}$. Now, let $m \in \mathbb{Z}$, then $m = \frac{mn}{n} \in M_n \Rightarrow \mathbb{Z} \subseteq M_n \ \forall \ n \in \mathbb{N}$. We show that $M_p \cap M_q = \mathbb{Z}$, $\forall \ p, q \in \mathbb{N}$, with $(p, q) = 1$. Suppose that $\frac{m}{p} = \frac{n}{q}$. Then, $q = \frac{pn}{m} \Rightarrow m|n$ similarly $n|m \Rightarrow m = n \Rightarrow p = q$, a contradiction. Now, if a, b are integers with no common factor and $b > 1$, the number $\frac{a}{b}$ lies in no M_n, where n is a natural number such that b does not divide n. Thus, $\lim_{n\to\infty} \sup M_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} M_k = \mathbb{Q}$ and $\lim_{n\to\infty} \inf M_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} M_k = \mathbb{Z}$. □

1.10.13 Problem. Every sequence (M_n) of countable sets has a convergent subsequence.

1.10.13.1 Solution. We need to show that a subsequence can be extracted so that, if an element lies in an infinite number of sets in this subsequence, then it belongs to all but finitely many sets if this subsequence. This can be achieved by a Cantor diagonal procedure. More precisely, put $M_n = \{x_{n,k}; k \in \mathbb{N}\}$ for $n \in \mathbb{N}$ and order the set $M = \bigcup_{n=1}^{\infty} M_n$ as $x_{1,1} \leq x_{2,1} \leq x_{1,2} \leq x_{3,1} \leq x_{2,2} \leq x_{1,3} \leq x_{4,1} \leq ..$ If for every $x \in M$ there exists $n \in \mathbb{N}$ such that $x \notin \bigcup_{k=n}^{\infty} M_k$, then $\limsup M_n = \emptyset$, and there is nothing to prove. Otherwise, select the first element in M that belongs to infinitely many M_n's and consider the subsequence (M_{n_k}) of those M_n's. Now, if no other element in $\bigcup_{k=1}^{\infty} M_{n_k}$, belongs to infinitely many elements in the subsequence, we are done. Otherwise, pick the next element in $\bigcup_{k=1}^{\infty} M_{n_k}$, with this property, and choose the corresponding subsequence of M_{n_k}. Proceed recursively. Either we stop at some step (and then we obtain a convergent subsequence) or, on the contrary, the process continues forever. In this case, choose a subsequence consisting of the "diagonal" of the successive subsequences so obtained. This concludes the proof. □

1.10.14 Problem. Prove or disprove the following assertion.

1. If ρ is a transitive relation on X, then so is ρ^{-1}.
2. Let ρ be a reflexive and transitive relation on X. Then $\rho \cap \rho^{-1}$ is an equivalence relation.
3. If ρ_1, ρ_2 are transitive relations on X, then so is $\rho_1 \circ \rho_2$.
4. Let ρ be a relation X such that ρ is symmetric and transitive. Then ρ is relexive iff $dom(\rho) = X$. Moreover, ρ is an equivalence relation.

1.10.14.1 Solution.

1. Since ρ is a transitive relation on X, so $\rho \circ \rho \subseteq \rho$, then $\rho^{-1} \circ \rho^{-1} = (\rho \circ \rho)^{-1} \subseteq \rho^{-1}$ implies that ρ^{-1} is transitive.
2. Suppose that ρ is reflexive and transitive relation on X, let

$$\begin{aligned}
&(x, y) \in \rho \cap \rho^{-1} \\
\Rightarrow &(x, y) \in \rho \text{ and } (x, y) \in \rho^{-1} \\
\Rightarrow &(y, x) \in \rho^{-1} \text{ and } (y, x) \in \rho \\
\Rightarrow &(y, x) \in \rho \cap \rho^{-1}
\end{aligned}$$

shows that $\rho \cap \rho^{-1}$ is symmetric. Again, let

$$\begin{aligned}
&(x, y), (y, z) \in \rho \cap \rho^{-1} \\
\Rightarrow &(x, y), (y, z) \in \rho \text{ and } (x, y), (y, z) \in \rho^{-1} \\
\Rightarrow &(x, z) \in \rho \text{ and } (y, x), (z, y) \in \rho \\
\text{Now } &(y, x), (z, y) \in \rho \Rightarrow (z, x) \in \rho \Rightarrow (x, z) \in \rho^{-1}. \\
\text{Thus } &(x, z) \in \rho \cap \rho^{-1}
\end{aligned}$$

shows that $\rho \cap \rho^{-1}$ is transitive. Finally, $\Delta_X \in \rho \cap \rho^{-1}$ shows $\rho \cap \rho^{-1}$ is reflexive and hence an equivalence relation.
3. Let $X = \{1, 2, 3, 4\}$, take $\rho_1 = \{(1, 3), (2, 4)\}$ and $\rho_2 = \{(1, 2), (4, 1), (4, 2)\}$ then $\rho_1 \circ \rho_2 = \{(1, 4), (4, 3), (4, 4)\}$ is not transitive.
4. Suppose that ρ is relexive, then $(x, x) \in \rho \;\forall\, x \in X$ implies $dom(\rho) = X$.
Conversely, let $x \in X$ then for some $y \in X, (x, y) \in \rho$, so by symmetricity $(y, x) \in \rho$ and by transitivity $(x, y), (y, x) \in \rho$ implies $(x, x) \in \rho$. Thus ρ is reflexive and hence ρ is an equivalence relation. □

1.10.15 Problem. In this problem we deal with (binary) relations in the set of real numbers $\mathbb{R}$, i.e. the subsets ρ of $\mathbb{R}^2$.

1. What is the natural interpretation of such relations?
2. What is the geometric meaning of the sets dom(ρ) and ran(ρ)?
3. What is the geometric meaning of the property of reflexivity?
4. What is the geometric meaning of the property of symmetry?

5. What is the geometric meaning of the property of connectedness?
6. What is the geometric meaning of the property of antireflexivity?
7. What is the geometric meaning of the property of asymmetry?
8. What is the geometric meaning of the property of antisymmetry?

1.10.15.1 Solution.

1. The relations are interpreted as the subsets of the Cartesian plane.
2. $\mathrm{dom}(\rho)$ is the projection of ρ onto the axis Ox and $\mathrm{ran}(\rho)$ the projection of ρ onto the axis Oy.
3. The diagonal (i.e. the straight line described by the equation $y = x$) is included in ρ.
4. The diagonal is an axis of symmetry of ρ.
5. The set-theoretical union of ρ, the diagonal, and the symmetric image of ρ with respect to the diagonal is the whole plane.
6. The diagonal is disjoint from ρ.
7. ρ does not contain a pair of points symmetric with respect to the diagonal.
8. The only points of $\mathbb{R}$ which have in ρ images which are symmetric (with respect to the diagonal) belong to the diagonal.

1.10.16 Problem. If ρ and μ be the the relations on a set X, then show that $(\rho\circ\mu)^{-1} = \mu^{-1}\circ\rho^{-1}$.

1.10.16.1 Solution.

$$\begin{aligned}
&(a,b) \in (\rho\circ\mu)^{-1}\\
&\Leftrightarrow (b,a) \in \rho\circ\mu\\
&\Leftrightarrow \exists\, c \in X \text{ such that } (b,c) \in \mu \text{ and } (c,a) \in \rho\\
&\Leftrightarrow \exists\, c \in X \text{ such that } (c,b) \in \mu^{-1} \text{ and } (a,c) \in \rho^{-1}\\
&\Leftrightarrow (a,b) \in \mu^{-1}\circ\rho^{-1}.
\end{aligned}$$

Thus $(\rho\circ\mu)^{-1} = \mu^{-1}\circ\rho^{-1}$. □

1.10.17 Problem. Let ρ and μ be two equivalence relations on a set X. Show that $\rho\circ\mu$ is an equivalence relation on X if and only if $\rho\circ\mu = \mu\circ\rho$ and that, in this case, $\rho\circ\mu$ is the intersection of all the equivalence relations on X that contain both ρ and μ.

1.10.17.1 Solution. Suppose that $\rho\circ\mu$ is an equivalence relation on X. Here

$$(a,b) \in \rho\circ\mu \text{ implies } (b,a) \in \rho\circ\mu \text{ (by symmetricity)}$$

$$\begin{aligned}
\text{Now } &(b,a) \in \rho\circ\mu\\
&\Rightarrow \exists\, c \in X \text{ such that } (a,c) \in \mu \text{ and } (c,b) \in \rho\\
&\Rightarrow (b,c) \in \rho \text{ and } (c,a) \in \mu\\
&\Rightarrow (b,a) \in \mu\circ\rho\\
&\Rightarrow \rho\circ\mu \subseteq \mu\circ\rho.
\end{aligned}$$

Similarly, $\mu \circ \rho \subseteq \rho \circ \mu$. Thus $\rho \circ \mu = \mu \circ \rho$.
Conversely, let $\rho \circ \mu = \mu \circ \rho$.
then, $\forall\ a \in X,\ (a, a) \in \rho$ and $(a, a) \in \mu$ implies $(a, a) \in \rho$ implies $(a, a) \in \rho \circ \mu$ implies $\rho \circ \mu$ is reflexive.
Now, $(\rho \circ \mu)^{-1} = \mu^{-1} \circ \rho^{-1} \subseteq \mu \circ \rho = \rho \circ \mu \Rightarrow\ \rho \circ \mu$ is symmetric.
And $(\rho \circ \mu) \circ (\rho \circ \mu) = \rho \circ (\mu \circ \rho) \circ \mu = \rho \circ (\rho \circ \mu) \circ \mu = \rho^2 \circ \mu^2 \subseteq \rho \circ \mu$ implies $\rho \circ \mu$ is transitive.
Hence $\rho \circ \mu$ is an equivalence relation.
Let $\mathcal{R} = \{\alpha; \alpha$ is an equivalence relation on $X, \rho, \mu \subseteq \alpha\}$. Let $(x, y) \in \rho \circ \mu$, then $\exists\ z \in X$ such that $(x, z) \in \mu \subseteq \alpha, (z, y) \in \rho \subseteq \alpha$ and this implies $(x, y) \in \alpha$. Thus $\rho \circ \mu \subseteq \alpha$. Hence the result follows. □

1.10.18 Problem. Give an example of each of the following :

1. A relation which is not reflexive, but symmetric and transitive,
2. A relation which is reflexive, symmetric and transitive,
3. A relation which is not reflexive, not symmetric but transitive,
4. A relation which is not reflexive, not symmetric and not transitive,
5. A relation which is neither reflexive nor transitive but symmetric,
6. A relation which is reflexive, but not symmetric and not transitive,
7. A relation which is reflexive and symmetric but not transitive,
8. A relation which is reflexive and transitive but not symmetric.
9. A relation which is reflexive and transitive, but not antisymmetric.
10. A relation which is transitive and antisymmetric but not reflexive.
11. A relation which is reflexive and antisymmetric but not transitive.

1.10.18.1 Solution.
Let $X = \{a, b, c\}, X \times X = \{(a, a), (b, b), (c, c), (b, a), (a, b), (b, c), (c, b), (a, c), (c, a)\}$.

1. $\rho = \{(b, b), (c, c), (b, c), (c, b)\}$.
2. $\rho = \{(a, a), (b, b), (c, c), (b, c), (c, b), (a, c), (c, a)\}$.
3. $\rho = \{(b, b), (c, c), (b, c), (c, a), (b, a)\}$.
4. $\rho = \{(b, b), (c, c), (b, c), (c, a)\}$.
5. $\rho = \{(c, c), (b, c), (c, b), (a, c), (c, a)\}$.
6. $\rho = \{(a, a), (c, c), (b, c), (a, c), (c, a)\}$.
7. $\rho = \{(a, a), (b, b), (c, c), (b, c), (c, b), (a, b), (b, a)\}$.
8. $\rho = \{(a, a), (b, b), (c, c), (b, c)\}$.

9. $\rho = \{(a,a),(b,b),(c,c),(a,b),(b,a)\}$.
10. $\rho = \{(a,b),(a,a)\}$.
11. $\rho = \{(a,a),(b,b),(c,c),(a,b),(b,c)\}$.

1.10.19 Problem. If S be a set, and $|S| = n$ is finite, then

1. $|S \times S| = n^2$.
2. there are 2^{n^2} relations on S,
3. there are 2^{n^2-n} reflexive relations on S,
4. there are $2^{(n^2+n)/2}$ symmetric relations on S,
5. there are $2^{(n^2-n)/2}$ reflexive and symmetric relations on S.

1.10.19.1 Solution.

1. Let $S = \{a_1, a_2, .., a_n\}$ then
$$|S \times S| = \left|\bigcup_{i=1}^{n}(\{a_i\} \times S)\right| = \sum_{i=1}^{n} |\{a_i\} \times S| = n^2.$$
2. Since any subset of $S \times S$ is a relation and there are $|\mathcal{P}(S \times S)| = 2^{n^2}$ subsets of $S \times S$, hence there are 2^{n^2} binary operations on S.
3. Let $\triangle = \{(a,a); a \in S\}$, then for any $\rho \subset (S \times S) \setminus \triangle$, $\rho \cup \triangle$ is a reflexive relation, as $|(S \times S) \setminus \triangle| = n^2 - n$, so $|\mathcal{P}((S \times S) \setminus \triangle)| = 2^{n^2-n}$ is the number of reflexive relations on S.
4. For symmetric and reflexive relations $\rho, \triangle \subseteq \rho, \rho^{-1} \subseteq \rho$, so let
$$T = \{\{(a,b),(b,a)\}; a, b \in S, a \neq b\}.$$
Thus, any subset of $T \cup \triangle$ gives us symmetric relations, so the total number of subsets is $|\mathcal{P}(T \cup \triangle)| = 2^{|T \cup \triangle|} = 2^{(n^2+n)/2}$.
5. For symmetric and reflexive relations $\rho, \triangle \subseteq \rho, \rho^{-1} \subseteq \rho$, so let
$$T = \{\{(a,b),(b,a)\}; a, b \in S, a \neq b\}.$$
Thus, any subset of T gives us symmetric relations, so the total number of subsets is $|\mathcal{P}(T)| = 2^{|T|} = 2^{(n^2-n)/2}$ □

1.10.20 Problem. Let $A = \{1, 2, ..., m\}$ and $B = \{1, ..., n\}$.

1. How many different functions $f : A \to B$ do exist?
2. Suppose $n \geq m$. How many different injective functions $f : A \to B$ do exist?
3. How many different functions $f : A_0 \to B$ do exist, where $A_0 \subseteq A$ is arbitrary?
4. Suppose $m \geq n$. How many different surjective functions $f : A \to B$ do exist?

1.10.20.1 Solution.

1. $|B|^{|A|}$. The number of all functions $f : A \to B$ is equal to the number of all variations with repetition of n elements of the class m, i.e., it is $n^m = |B|^{|A|}$.

2. The number of injective maps from a set A with m elements to a set B with n elements is easy to find; one need only notice that an injective map corresponds to the choice of m distinct elements of B, in some order.
i.e., $\binom{n}{m} \cdot m! = n(n-1)(n-2).......(n-m+1)$.

3. $(|B|+1)^{|A|} = (n+1)^m$.

4. The total number of maps from X to Y is n^m. We shall find the number of maps that are not surjective and then take the difference to obtain the number of surjective maps.
Let $Y = \{y_1, ..., y_n\}$. A map $f : X \to Y$ is not surjective if there is at least one element $y_i \in Y$ that does not belong to its image. In other words, the set of non-surjective maps $f : X \to Y$ is the union of the sets $F_1, .., F_n$, where F_i is the set of maps whose image does not contain y_i. The cardinality of each set F_i (there are n such sets, one for each value of i) is the number of maps from X to the set $Y \setminus \{y_i\}$ and therefore equal to $(n-1)^m$. The cardinality of each of the $\binom{n}{2}$ sets $F_i \cap F_j$ is the number of maps from X to the set $Y \setminus \{y_i, y_j\}$, that is, $(n-2)^m$, and so on. In conclusion, we have by the inclusion-exclusion principle

$$\begin{aligned}
&|A_1 \cup .... \cup A_n| \\
=&\sum_{1\le i\le n} |A_i| - \sum_{1\le i<j\le n} |A_i \cap A_j| + \sum_{1\le i<j<k\le n} |A_i \cap A_j \cap A_k| - ... \\
=&\binom{n}{1}(n-1)^m - \binom{n}{2}(n-2)^m + \binom{n}{3}(n-3)^m - ....
\end{aligned}$$

The number of surjections from A to B is $n^m - \binom{n}{1}(n-1)^m + \binom{n}{2}(n-2)^m ... + (-1)^{n-1}\binom{n}{n-1}1^m$.

□

1.10.21 Problem. Determine the largest set $A \subseteq \mathbb{R}$ such that the following functions are defined. (We call then A the natural domain of the function given with that formula.)

1. $f(x) = x\sqrt{\cos\sqrt{x}}$.
2. $f(x) = \sqrt{\sin x^2}$.
3. $f(x) = \ln^3\left(\sin\left(\frac{\pi}{x}\right)\right)$ where $\ln^3 x = \ln(\ln(\ln x))$.

1.10.21.1 Solution.

1. The natural domain of the function $g(t) = \sqrt{t}$ is the set $\{t \in \mathbb{R}; t \ge 0\}$, which means that $A \subseteq [0, \infty)$. Again,

$$\cos u \ge 0 \Leftrightarrow u \in \bigcup_{k\in\mathbb{Z}} \left[(4k-1)\frac{\pi}{2}, (4k+1)\frac{\pi}{2}\right]$$

or, equivalently,

$$(4k-1)\frac{\pi}{2} \le u \le (4k+1)\frac{\pi}{2}, \quad k \in \mathbb{Z}$$

Using the condition $A \subseteq [0,\infty)$. and putting in the last relation $u=\sqrt{x}, x\geq 0$, we obtain that if $x \geq 0$, then $\cos(\sqrt{x}) \geq 0$ holds if and only if

$$0 \leq x \leq \pi^2/4 \text{ or } (4k-1)^2\frac{\pi^2}{4} \leq u \leq (4k+1)^2\frac{\pi^2}{4},\quad k\in\mathbb{Z}$$

for some natural number k. This means that the natural domain of f is the set

$$D=\left[0,\frac{\pi^2}{4}\right]\bigcup\left(\bigcup_{k\in\mathbb{N}}\left[(4k-1)^2\frac{\pi^2}{4},(4k+1)^2\frac{\pi^2}{4}\right]\right).$$

2. The function $g(t)=\sin t$ is non-negative iff $2k\pi \leq t \leq (2k+1)\pi, k\in\mathbb{Z}$, hence

$$\sin x^2 \geq 0 \Leftrightarrow 2k\pi \leq x^2 \leq (2k+1)\pi \text{ for } k \in\in \mathbb{N}_0.$$

Since for every $x\in\mathbb{R}$, $\sqrt{x^2}=|x|$, we obtain finally

$$\begin{aligned} D&=\left\{x\in\mathbb{R}; \sqrt{2k\pi}\leq |x| \leq \sqrt{(2k+1)\pi} \text{ for some } k\in\mathbb{N}_0\right\}\\ &=\left(\bigcup_{k\in\mathbb{N}_0}\left[\sqrt{2k\pi},\sqrt{(2k+1)\pi}\right]\right)\bigcup\left(\bigcup_{k\in\mathbb{N}_0}\left[-\sqrt{(2k+1)\pi},-\sqrt{2k\pi}\right]\right).\end{aligned}$$

3. The natural domain of the logarithmic function $g(t)=\ln t$ is the open interval $(0,\infty)$, hence the natural domain D of the function f will be the set of all $x\in\mathbb{R}$ such that $\sin\left(\frac{\pi}{x}\right)>0$ The last inequality is true iff there exists an integer k such that

$$2k\pi \leq \frac{\pi}{x} \leq (2k+1)\pi, k\in\mathbb{Z}$$

Solving these inequalities for x gives three cases.

(a) If $k=0$, then $0<\frac{1}{x}<1$, hence $x\in(1,\infty)$;

(b) If $k\in\mathbb{Z}, k>0$ then $x\in\left(\frac{1}{2k+1},\frac{1}{2k}\right)$

(c) If $k\in\mathbb{Z}, k<0$ then $x\in\left(\frac{1}{2k+1},\frac{1}{2k}\right)$.

So, we obtained that

$$D=\left(\bigcup_{k\in\mathbb{Z},k<0}\left(\frac{1}{2k+1},\frac{1}{2k}\right)\right)\bigcup\left(\bigcup_{k\in\mathbb{Z},k>0}\left(\frac{1}{2k+1},\frac{1}{2k}\right)\right)\bigcup(1,\infty). \quad \square$$

1.10.22 Problem. Let the function $f:\mathbb{R}\setminus\{-1\}\to\mathbb{R}$ be given with the formula $f(x)=\frac{x}{1+x}$. Find the function $f_n, n\in\mathbb{N}$, where

$$f_1=f, f_2=f\circ f_1, ..., f_n=f\circ f_{n-1}, ...$$

Determine also the natural domains of these composite functions.

1.10.22.1 Solution. The domain of the function f is the set $\mathbb{R} \setminus \{-1\}$. Then

$$f_2(x) = (f \circ f_1)(x) = f(f(x)) = f\left(\frac{x}{1+x}\right) = \frac{\frac{x}{1+x}}{1+\frac{x}{1+x}} = \frac{x}{1+2x}$$

Clearly, the natural domain of f_2 is the set $\mathbb{R} \setminus \{-1/2\}$, the definition of f_2 reduces its domain to the set $\mathbb{R} \setminus \{-1, -1/2\}$. Let us prove by mathematical induction that for $n = 2, 3, ...$ we have

$$f_n(x) = \frac{x}{1+nx}, \text{ for } x \in \mathbb{R} \setminus \left\{-1, -\frac{1}{2}, .., -\frac{1}{n}\right\}$$

We proved already the formula for $n = 2$. Suppose it holds for $n = k$, for some natural number $k > 1$. Then

$$\begin{aligned} f_{k+1}(x) &= (f \circ f_k)(x) = f(f_k(x)) = f\left(\frac{x}{1+kx}\right) \\ &= \frac{\frac{x}{1+kx}}{1+\frac{x}{1+kx}} = \frac{x}{1+(k+1)x} \text{ for } x \in \mathbb{R} \setminus \left\{-1, -\frac{1}{2}, .., -\frac{1}{n}\right\}. \quad \square \end{aligned}$$

1.10.23 Problem. Give an example of a partially ordered set which has a unique minimal element but no smallest element.

1.10.23.1 Solution. Consider the set $(0, 1] \cup [2, 3)$ with the ordering $\prec$ given by $x \prec y$ if and only if either $x, y \in (0, 1]$ and $x < y$, or $x, y \in [2, 3)$ and $x < y$. Then "$\prec$" is a partial ordering on the set, 2 is the unique minimal element and there is no smallest element. $\square$

1.10.23.2 Solution. If S and T are subspaces of V, we must find the largest subspace W of V contained in the subspaces S and T and the smallest subspace U of V containing the subspaces S and T. i.e. $S \cap T$ and $S \oplus T$.

1.10.24 Problem. Let A, B be two sets such that $A \subseteq B$ a relation ρ is defined on $\mathcal{P}(B)$ by $(X, Y) \in \rho$ iff $X \cap A = Y \cap A$. Then show that

1. ρ is an equivalence relation on $\mathcal{P}(B)$;
2. $\langle \emptyset \rangle$ is the equivalence class determined by $\emptyset$ is $\mathcal{P}(B \setminus A)$.

1.10.24.1 Solution.

1. A routine verification is left to the reader.
2. Now,

$$\begin{aligned} \langle \emptyset \rangle &= \{X \in \mathcal{P}(B); (X, \emptyset) \in \rho\} \\ &= \{X \in \mathcal{P}(B); X \cap A = \emptyset \cap A = \emptyset\} \\ &= \{X \in \mathcal{P}(B); X \subseteq A^C\} \\ &= \{X; X \subseteq B \cap A^C\} \\ &= \{X; X \subseteq B \setminus A\} \\ &= \mathcal{P}(B \setminus A). \quad \square \end{aligned}$$

1.10.25 Problem. Let $\mathcal{P}$ be a given partition of a non-empty set X. Define a relation ρ_P(depending on P) on X by $(x, y) \in \rho_P$ iff there exists $A \in P$ such that $x, y \in A$ (i.e., iff y belongs to the same class as x). Then ρ_P is an equivalence relation, induced by $\mathcal{P}$. Moreover,

1. $\mathcal{P}(\rho_{\mathcal{P}}) = \mathcal{P}$ (partition induced by $\rho_{\mathcal{P}}$);

2. $\rho(\mathcal{P}_\rho) = \rho_{\mathcal{P}}$ (equivalence relation induced by partition $\mathcal{P}$).

1.10.25.1 Solution. Let $x \in X$. Since $\mathcal{P}$ is a partition, there exists $A \in \mathcal{P}$, such that $x \in A$. So, $(x, x) \in \rho$. This implies ρ is reflexive. If $(x, y) \in \rho$, then $(y, x) \in \rho$ from the definition of ρ. So, ρ is symmetric. Let $(x, y) \in \rho$ and $(y, z) \in \rho$. Then there exist $A, B \in \mathcal{P}$ such that $x, y \in A$ and $y, z \in B$. Consequently, $y \in A \cap B$. Since $\mathcal{P}$ is a partition, $A = B$. Consequently, $x, z \in A \in \mathcal{P}$; therefore, $(x, z) \in \rho$. So, ρ is transitive. As a result ρ is an equivalence relation.

1. We now show that $\mathcal{P}(\rho_{\mathcal{P}}) = \mathcal{P}$. Let $A \in \mathcal{P}$ and $x \in A$. Then for every $y \in A, (x, y) \in \rho_{\mathcal{P}}$. Consequently, $y \in (x) \Rightarrow A \subseteq (x)$. Next let $z \in (x)$. Then there exists some $B \in \mathcal{P}$ such that $x, z \in B$. But $x \in A \Rightarrow x\ A \cap B \Rightarrow A = B$ (by the property of a partition). Consequently, $z \in A \Rightarrow (x) \subseteq A$. As a result,
$$(x) = A, \text{ but } (x) \in \mathcal{P}(\rho_{\mathcal{P}}).$$
Consequently, $\mathcal{P} \subseteq \mathcal{P}(\rho_{\mathcal{P}})$. Moreover, both $\mathcal{P}$ and $\mathcal{P}(\rho_{\mathcal{P}})$ are partitions of the same set X. Hence, $\mathcal{P}(\rho_{\mathcal{P}}) = \mathcal{P}$.

2. To prove $\rho(\mathcal{P}_\rho) = \rho_{\mathcal{P}}$, let $(x, y) \in \rho$. Then $y \in (x) \in \mathcal{P}_\rho \Rightarrow (x, y)\rho(\mathcal{P}_\rho) \Rightarrow$. Again $(x, y)\rho(\mathcal{P}_\rho) \Rightarrow$ there is an equivalence class (z) such that $x, y \in (z) \Rightarrow (z, x)\rho$ and $(z, y) \in \rho(x, z)$ and $(z, y) \in \rho \Rightarrow (x, y) \in \rho$ (by transitive property of ρ) $\Rightarrow \rho(\mathcal{P}_\rho) \subseteq \rho$. Hence $\rho(\mathcal{P}_\rho) = \rho_{\mathcal{P}}$. □

1.10.26 Problem. In $\mathcal{P}(X)$, the power set of X, define a relation ρ as follows $A\rho B$ iff $A\Delta C = B\Delta C$ for some $C \in P(X)$. Show that ρ is an equivalence relation.

1.10.26.1 Solution. We observe that
$$\begin{aligned}
& A\Delta C = B\Delta C \\
\Rightarrow & (A\Delta C)\Delta C = (B\Delta C)\Delta C \\
\Rightarrow & A\Delta(C\Delta C) = B\Delta(C\Delta C) \\
\Rightarrow & A\Delta\emptyset = B\Delta\emptyset \\
\Rightarrow & A = B.
\end{aligned}$$

Thus $A\rho B$ iff $A = B$. Thus ρ is an equivalence relation. □

1.10.27 Problem.

1. $f^{-1}(\emptyset) = \emptyset$.

2. If $A, B \subseteq X$, then
   (a) If $A \subseteq B \subseteq X$, then $f(A) \subseteq f(B)$.
   (b) $f(A \cup B) = f(A) \cup f(B)$.
   (c) $f(A \cap B) \subseteq f(A) \cap f(B)$.

   (d) $f(A \setminus B) \supseteq f(A) \setminus f(B)$.

3. If $P, Q \subseteq Y$, then

   (a) If $P \subseteq Q \subseteq Y$, then $f^{-1}(P) \subseteq f^{-1}(Q)$.
   (b) $f^{-1}(P \cup Q) = f^{-1}(P) \cup f^{-1}(Q)$.
   (c) $f^{-1}(P \cap Q) = f^{-1}(P) \cap f^{-1}(Q)$
   (d) $f^{-1}(P \setminus Q) = f^{-1}(P) \setminus f^{-1}(Q)$

4. If $A \subseteq X$ and $B \subseteq Y$, then

$$A \cap f^{-1}(B) \subseteq f^{-1}(f(A) \cap B) \text{ and } f(A) \cap B = f(A \cap f^{-1}(B)).$$

5. If $A \subseteq X$ then $A \subseteq f^{-1}(f(A))$ and $A = f^{-1}(f(A))$ iff f is injective.

6. If $B \subseteq Y$ then $f(f^{-1}(B)) \subseteq B$ and $f(f^{-1}(B)) = B$ iff f is surjective.

1.10.27.1 Solution.

1. If $x \in f^{-1}(\emptyset)$ then $f(x) \in \emptyset$, which is absurd.

2. (a) $y \in f(A) \Rightarrow \exists\, x \in A \subseteq B \Rightarrow y = f(x) \in f(B)$.
   (b) $y \in f(A \cup B) \Rightarrow \exists\, x \in A \cup B$ such that $y = f(x) \in f(A)$ or $f(B) \Rightarrow y \in f(A) \cup (B)$.
   (c) $A \cap B \subseteq A \Rightarrow f(A \cap B) \subseteq f(A)$ and $A \cap B \subseteq B \Rightarrow f(A \cap B) \subseteq f(B)$, thus by combining these, we get the result.
   (d) Let $y \in f(A) \setminus f(B) \Rightarrow \exists\, x \in A$ such that $y = f(x) \in f(A)$ and $y = f(x) \notin f(B) \Rightarrow x \notin B$. Thus $x \in A \setminus B \Rightarrow y = f(x) \in f(A \setminus B)$, and the result follows.

3. (a) $x \in f^{-1}(P) \Rightarrow f(x) \in P \subseteq Q \Rightarrow x \in f^{-1}(Q)$.
   (b) $x \in f^{-1}(P \cup Q) \Rightarrow f(x) \in P \cup Q \Rightarrow f(x) \in P$ or $f(x) \in Q \Rightarrow x \in f^{-1}(P)$ or $x \in f^{-1}(Q) \Rightarrow x \in f^{-1}(P) \cup f^{-1}(Q)$. Thus $f^{-1}(P \cup Q) \subseteq f^{-1}(P) \cup f^{-1}(Q)$. Again, $P \subseteq P \cup Q$ and $Q \subseteq P \cup Q$ implies $f^{-1}(P) \subseteq f^{-1}(P \cup Q)$ and $f^{-1}(Q) \subseteq f^{-1}(P \cup Q)$, thus by combining these, we get the result.
   (c) Now, $x \in f^{-1}(P \cap Q) \Leftrightarrow f(x) \in P \cap Q \Leftrightarrow f(x) \in P$ and $f(x) \in Q \Leftrightarrow x \in f^{-1}(P) \cap f^{-1}(Q)$.
   (d) Again, $x \in f^{-1}(P \setminus Q) \Leftrightarrow f(x) \in P$ and $f(x) \notin Q \Leftrightarrow x \in f^{-1}(P)$ and $x \notin f^{-1}(Q) \Leftrightarrow x \in f^{-1}(P) \setminus f^{-1}(Q)$.

4. $x \in A \cap f^{-1}(B) \Rightarrow \exists\, y \in B$ such that $y = f(x) \Rightarrow f(x) \in f(A) \cap B \Rightarrow x \in f^{-1}(f(A) \cap B)$. Again, $y \in f(A) \cap B \Rightarrow \exists\, x \in A$ such that $y = f(x) \in B$ thus $x \in A \cap f^{-1}(B) \Rightarrow y = f(x) \in f(A \cap f^{-1}(B))$, other part is similar.

5. $x \in A \Rightarrow f(x) \in f(A) \Rightarrow x \in f^{-1}(f(A)) \Rightarrow A \subseteq f^{-1}(f(A))$. Again, $x \in f^{-1}(f(A)) \Rightarrow f(x) \in f(A) \Rightarrow \exists\, y \in A$ such that $f(x) = f(y)$, by injectivity $x = y \in A$, which shows $f^{-1}(f(A)) \subseteq A$. Hence f is injective implies $A = f^{-1}(f(A))$. Now, let $f(x) = f(y)$ then taking $A = \{x\}$, we get $y \in f^{-1}(f(\{x\})) = \{x\}$ which shows $y = x$. Thus f is injective.
   Alternatively, Let $x \neq y$ then $x \notin \{y\} = f^{-1}(f(y))$ implies $f(x) \notin \{f(y)\} \Rightarrow f(x) \neq f(y)$.

6. Let $y \in f(f^{-1}(B))$ then $\exists\, x \in f^{-1}(B)$ such that $y = f(x) \in B$. Hence $f(f^{-1}(B)) \subseteq B$. Again, let $y \in B$ then, by surjectivity $\exists\, x \in A$ such that $y = f(x) \in B \Rightarrow x \in f^{-1}(B) \Rightarrow f(x) \in f(f^{-1}(B))$ i.e. $y = f(x) \in f(f^{-1}(B))$. Thus $B \subseteq f(f^{-1}(B))$ hence $B = f(f^{-1}(B))$.
   Now, let $y \in B = f(f^{-1}(B))$ then $\exists\, x \in f^{-1}(B) \subseteq A$ such that $y = f(x)$. Hence f is surjective. $\square$

1.10.28 Problem. Show that, for a function $f\colon X \to Y$,

1. f is injective iff $f(A \cap B) = f(A) \cap f(B)\ \forall\, A, B \subseteq X$.
2. f is surjective iff $Y \setminus f(A) \subseteq f(X \setminus A)\ \forall\, A \subseteq X$.
3. f is bijective iff $Y \setminus f(A) = f(X \setminus A)\ \forall\, A \subseteq X$.

1.10.28.1 Solution.

1. Suppose that f is injective. Now, $A \cap B \subseteq A$ and $A \cap B \subseteq B \Rightarrow f(A \cap B) \subseteq f(A) \cap f(B)$ and $x \in f(A) \cap f(B)$ implies $\exists\, a \in A, b \in B$ such that $f(a) = x = f(b) \Rightarrow a = b \in A \cap B$ (by injectivity), thus $x \in f(A \cap B)$ implies $f(A \cap B) \subseteq f(A) \cap f(B)$. Hence $f(A \cap B) = f(A) \cap f(B)\ \forall\, A, B \subseteq X$.
   On the otherhand, let f be not injective. Then $\exists\, x, y \in X$ such that $f(x) = f(y)$ but $x \neq y$, so, considering $A = \{x\}$ and $B = \{y\}$, we get $f(A \cap B) = f(A) \cap f(B) \Rightarrow \emptyset = \{f(x)\} \neq \emptyset$, a contradiction.

2. Suppose f is surjective. Then, let $A \subseteq X$, and

$$\begin{aligned}
&y \in Y \setminus f(A) \\
\Rightarrow &\,\exists\, x \in X \text{ such that } y = f(x) \notin f(A) \\
\Rightarrow &x \notin A \Rightarrow f(x) \in f(X \setminus A) \\
\Rightarrow &y = f(x) \in f(X \setminus A).
\end{aligned}$$

   Conversely, if $A = \emptyset$ then $Y \setminus f(\emptyset) \subseteq f(X)$ implies $Y \subseteq f(X)$ and $A = X$ implies $f(X) \subseteq Y$. Hence $Y = f(X)$.

3. Note that the condition says that $(f(A))^C = f(A^C)$. Suppose that $x, y \in X$ and $f(x) = f(y)$ but $x \neq y$. Let $A = \{x\}$, so $y \in A^C$ i.e. $f(y) \in f(A^C)$, then by the condition,

$$f(y) \in f(A^C) = \{f(x)\}^C = \{f(y)\}^C,$$

   a contradiction. Thus f injective.
   Again, suppose that f is surjective, so $Y = f(X)$ and then we take $A = X$ and $B = A^C$, hence by injectivity, we have

$$\begin{aligned}
&f(X \cap A^C) = f(X) \cap f(A^C) \\
\Rightarrow &f(X \setminus A) = f(X) \cap (f(A))^C = Y \setminus f(A). \quad \square
\end{aligned}$$

1.10.29 Problem. Let $f : X \to Y$ be a function and $\{A_i; i \in I\}$ be an indexed family of subsets of Y, prove the following assertions:

1. $\left(\bigcap_{i \in I} A_i\right)^C = \bigcup_{i \in I} A_i^C$.

2. Suppose that $B \subseteq Y$, show that $B \setminus \left(\bigcup_{i\in I} A_i\right) = \bigcap_{i\in I} (B \setminus A_i)$.
3. $f^{-1}\left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I} f^{-1}(A_i)$
4. $f^{-1}\left(\bigcap_{i\in I} A_i\right) = \bigcap_{i\in I} f^{-1}(A_i)$

1.10.29.1 Solution.

1. Let
$$\begin{aligned} & x \in \left(\bigcap_{i\in I} A_i\right)^C \\ \Leftrightarrow & x \notin \bigcap_{i\in I} A_i \\ \Leftrightarrow & x \notin A_i, \text{ for some } i \in I \\ \Leftrightarrow & x \in A_i^C, \text{ for some } i \in I \\ \Leftrightarrow & x \in \bigcup_{i\in I} A_i^C. \end{aligned}$$
Thus, $\left(\bigcap_{i\in I} A_i\right)^C = \bigcup_{i\in I} A_i^C$.
2. Similar to (1).
3. Let
$$\begin{aligned} & x \in f^{-1}\left(\bigcup_{i\in I} A_i\right) \\ \Leftrightarrow & f(x) \in \bigcup_{i\in I} A_i \\ \Leftrightarrow & f(x) \in A_i \text{ for some } i \in I \\ \Leftrightarrow & x \in f^{-1}(A_i) \text{ for some } i \in I \\ \Leftrightarrow & x \in \bigcup_{i\in I} f^{-1}(A_i). \end{aligned}$$
Thus, $f^{-1}\left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I} f^{-1}(A_i)$.
4. Let
$$\begin{aligned} & x \in f^{-1}\left(\bigcap_{i\in I} A_i\right) \\ \Leftrightarrow & f(x) \in \bigcap_{i\in I} A_i \\ \Leftrightarrow & f(x) \in A_i \text{ for all } i \in I \\ \Leftrightarrow & x \in f^{-1}(A_i) \text{ for all } i \in I \\ \Leftrightarrow & x \in \bigcap_{i\in I} f^{-1}(A_i). \end{aligned}$$
Thus, $f^{-1}\left(\bigcap_{i\in I} A_i\right) = \bigcap_{i\in I} f^{-1}(A_i)$. □

1.10.30 Problem. Let S denote the collection of all subsets of a given set T. Let $f : S \to \mathbb{R}$ be a real-valued function defined on S. The function f is called **additive** if $f(A \cup B) = f(A) + f(B)$ whenever A and B are disjoint subsets of T. If f is additive, prove that for any two subsets A and B we have

1. $f(A \cup B) = f(A) + f(B \setminus A)$ and $f(A \cup B) = f(A) + f(B) - f(A \cap B)$.
2. Assume f is additive and assume also that the following relations hold for two particular subsets A and B of T :
   (a) $f(A \cup B) = f(A^C) + f(B^C) - f(A^C)f(B^C)$
   (b) $f(A \cap B) = f(A)f(B), f(A) + f(B) \neq f(T)$,

   where $A^C = T \setminus A, B^C = T \setminus B$. Prove that these relations determine $f(T)$, and compute the value of $f(T)$.

1.10.30.1 Solution.

1. Since $A \cap (B \setminus A) = \emptyset$ and $A \cup B = A \cup (B \setminus A)$, we have
$$f(A \cup B) = f(A \cup (B \setminus A)) = f(A) + f(B \setminus A). \tag{1.3}$$
   In addition, since $(B \setminus A) \cap (A \cap B) = \emptyset$ and $B = (B \setminus A) \cup (A \cap B)$, we have
$$f(B) = f((B \setminus A) \cup (A \cap B)) = f(B \setminus A) + f(A \cap B)$$
   which implies that
$$f(B \setminus A) = f(B) - f(A \cap B) \tag{1.4}$$
   By (1.3) and (1.4), we have $f(A \cup B) = f(A) + f(B) - f(A \cap B)$.
2. Write
$$f(T) = f(A) + f(A^C) = f(B) + f(B^C),$$
   then
$$\begin{aligned}
[f(T)]^2 &= \left(f(A) + f(A^C)\right)\left(f(B) + f(B^C)\right) \\
&= f(A)f(B) + f(A^C)f(B) + f(A)f(B^C) + f(A^C)f(B^C) \\
&= f(A)f(B) + [f(T) - f(A)]\, f(B) + f(A)\, [f(T) - f(B)] + f(A^C)f(B^C) \\
&= [f(A) + f(B)]f(T) - f(A)f(B) + f(A^C)f(B^C) \\
&= [f(A) + f(B)]f(T) - f(A)f(B) + f(A^C) + f(B^C) - f(A \cup B) \\
&= [f(A) + f(B)]f(T) - f(A)f(B) + [f(T) - f(A)] + [f(T) - f(B)] \\
&\quad - [f(A) + f(B) - f(A \cap B)] \\
&= [f(A) + f(B) + 2]f(T) - f(A)f(B) - 2[f(A) + f(B)] + f(A \cap B) \\
&= [f(A) + f(B) + 2]f(T) - 2[f(A) + f(B)]
\end{aligned}$$
   which implies that $[f(T)]^2 - [f(A) + f(B) + 2]f(T) + 2[f(A) + f(B)] = 0$. Let $f(A) + f(B) = a$ and $f(T) = x$. Then we have
$$x^2 - (a + 2)x + 2a = 0 \Rightarrow (x - a)(x - 2) = 0$$
   So, $x = 2$ since $x \neq a$ by $f(A) + f(B) \neq f(T)$. □

1.10.31 Problem.

1. Prove that, for a set X, $f\colon \mathcal{P}(X) \to \mathcal{P}(X)$ defined by $f(A) = A^C$ is a bijection.
2. Let X be a set, $f\colon \mathcal{P}(X) \to \mathcal{P}(X)$ defined by $f(A) = A \cap B$ for $B \in \mathcal{P}$. Under what conditions is f one-to-one and onto?
3. Let X be a set, $f\colon \mathcal{P}(X) \to \mathcal{P}(X)$ defined by $f(A) = A\Delta B$ for $B \in \mathcal{P}$. Under what conditions is f one-to-one and onto?
4. Let $f : X \to Y$ be a mapping. Let $F : 2^X \to 2^Y$ be the mapping defined by $F(A) = f(A)\ \forall\ A \in \mathcal{P}(X)$. Show that F is one-to-one (onto) iff f is one-to-one (onto).

1.10.31.1 Solution.

1. Let $f(A) = f(B)$, then $A^C = B^C \Rightarrow A = B$ shows that f is injective and for $A \in \mathcal{P}(X), (A^C)^C = A \Rightarrow f(A^C) = A$ implies f is surjective. Thus f is bijective.
2. We see that $B\cap B = B\cap X \Rightarrow f(B) = f(X)$, so f is one-to-one implies $f(B) = f(X) \Rightarrow B = X$. So, for any $A \in \mathcal{P}(X)$, we get $f(A) = A\cap X = A \Rightarrow f$ is the identity mapping which is bijective. Hence the required condition is $B = X$.
3. We have that, for $P, Q \in \mathcal{P}(X)$

$$\begin{aligned}
&f(P) = f(Q)\\
\Rightarrow &P\Delta B = Q\Delta B\\
\Rightarrow &(P\Delta B)\Delta B = (Q\Delta B)\Delta B\\
\Rightarrow &P = Q \Rightarrow\ f \text{ is one-to-one.}
\end{aligned}$$

   And for any $A \in \mathcal{P}(X)$, we get $f(A\Delta B) = (A\Delta B)\Delta B = A \Rightarrow f$ is onto. Hence f is bijective for any B.
4. Let F be one-one. Then taking $A = \{x\}$ and $B = \{y\}$, we get

$$f(x) = f(y) \Rightarrow f(\{x\}) = f(\{y\}) \Rightarrow \{x\} = \{y\} \Rightarrow x = y$$

   shows that f is one-one.
   Let f be one-one, then for $A, B \in \mathcal{P}(X)$. Now $F(A) = F(B) \Rightarrow f(A) = f(B)$ and $a \in A$ implies $f(a) \in f(A) = f(B)$ implies $f(a) \in f(B)$ implies $f(a) = f(b)$ for some $b \in B$, since f is one-one, so $a = b \in B$. Thus $A \subseteq B$, similarly $B \subseteq A$. Hence $A = B$, and F is one-one.
   Again, suppose that F is onto. Let $b \in B$ then $\exists\ A \in \mathcal{P}(X)$ such that $F(A) = \{b\}$ i.e. $\exists\ a \in A$ such that $f(a) = b$ implies f is onto.
   Suppose that f is onto. Let $B \in \mathcal{P}(Y)$ then, consider the set

$$A = \bigcup\{f^{-1}(b); b \in B\} \subseteq X,$$

   shows that $F(A) = B$. Hence F is onto. □

1.10.32 Problem. Let $f : X \to Y$ be any function. For a subset B of Y, prove that: $f^{-1}(B) = \emptyset \Leftrightarrow B \cap f(X) = \emptyset$.

1.10.32.1 Solution. $B \cap f(X) \neq \emptyset \Leftrightarrow \exists\ y \in B$ and $\exists\ x \in X$ such that $y = f(x) \in B \Leftrightarrow x \in f^{-1}(B) \neq \emptyset$, a contradiction. □

1.10.33 Problem. Let there exist a function $f : X \to Y$ which is not an injection. Prove that $X \neq \emptyset$ and $Y \neq \emptyset$.

1.10.33.1 Solution. Note that $f \subseteq X \times Y$, so if $X = \emptyset$ then $f = \emptyset$ which is injective because if $(x_1, y) \in \emptyset$ and $(x_2, y) \in \emptyset$ then $x_1 = x_2$. If $Y = \emptyset$ and $f : X \to Y$ then $X = \emptyset$ also and $f = \emptyset$ which is again an injection. □

1.10.34 Problem. Let there exist a function $f : X \to Y$ which is not a surjection. Prove that $Y \neq \emptyset$.

1.10.34.1 Solution. If $Y = \emptyset$ and f is a function then f is a surjection. □

1.10.35 Problem. $A^B = \emptyset$ iff $A = \emptyset$ and $B \neq \emptyset$.

1.10.35.1 Solution. Suppose that $A = \emptyset$ and $B \neq \emptyset$, so $f \in A^B \Rightarrow f \subseteq B \times \emptyset = \emptyset$. But $\emptyset$ is not a function from $B \neq \emptyset$ to $A = \emptyset$. Therefore $A^B = \emptyset$. Conversely, suppose that $A \neq \emptyset$. Let $a \in A$ then $\{(x, a); x \in B\}$ is a function $B \to A$ which produces $A^B \neq \emptyset$. On the otherhand $B = \emptyset$, then $\emptyset$ is a function from $B \to A$ and again $A^B \neq \emptyset$. □

1.10.36 Problem. Prove that $A^B = B^A \Rightarrow A = B$.

1.10.36.1 Solution. If $A^B \neq \emptyset$, let $f \in A^B = B^A$. Then $\mathrm{dom} f = B$ since $f \in A^B$. But $\mathrm{dom} f = A$, since $f \in B^A$ thus $A = B$. Again, if $A^B = \emptyset$, then by the previous problem $A = \emptyset$. But $B^A = \emptyset$ also giving $B = \emptyset$. Hence $A = B$. □

1.10.37 Problem. Prove that $A \subseteq B \Rightarrow A^X \subseteq B^X$.

1.10.37.1 Solution. Let $f \in A^X$, then observe,

1. $f \subseteq X \times A \subseteq X \times B$
2. $\forall\ x \in X\ \exists\ y \in A$ (and therefore $y \in B$) such that $(x, y) \in f$

Again, $(x, y_1) \in f$ and $(x, y_2) \in f \Rightarrow y_1 = y_2$. f is therefore a function $X \to B$ and $f \in B^X$. □

1.10.38 Problem. Prove that $A \simeq B \Rightarrow A^X \simeq B^X$.

1.10.38.1 Solution. Let $\phi : A \to B$ be a bijection. Let $f \in A^X$. Define $\Phi : A^X \to B^X$ by $\Phi(f) = \phi \circ f$. We show that Φ is injective. Since $\phi \circ f = \phi \circ g \Rightarrow f = g$ so Φ is injective. Again, for $g \in B^X$, $\Phi(\phi^{-1} \circ g) = \phi \circ (\phi^{-1} \circ g) = g$ shows that Φ is surjective. Hence Φ is a bijection. □

1.10.39 Problem. If $f : X \to X$, then $f \subseteq \iota_X \Rightarrow f = \iota_X$.

1.10.39.1 Solution. To prove $f = \iota_X$, it remains to prove $\iota_X \subseteq f$. Let $x \in X$. Since f is a function, $(x, y) \in f$ for some $y \in X$ and then $f \subseteq \iota_X \Rightarrow (x, y) \in \iota_X \Rightarrow y = x \Rightarrow (x, x) \in f \Rightarrow \iota_X \subseteq f$. □

1.10.40 Problem. If $f : X \to X$, then $\iota_X \subseteq f \Rightarrow f = \iota_X$.

1.10.40.1 Solution. Similar to the above. □

1.10.41 Problem. Let $f, g \in Y^X$ and let $A = \{x \in X;\ f(x) = g(x)\}$. Prove that $f \circ \iota_A = g \circ \iota_A$. Let $B \subseteq X$ such that $f \circ \iota_B = g \circ \iota_B$. Prove that $B \subseteq A$.

1.10.41.1 Solution. Note that,

$$(f \circ \iota_A)(x) = f(\iota_A(x)) = f(x) = g(x) = g(\iota_A(x)) \; \forall \; x \in A$$

implies $f \circ \iota_A = g \circ \iota_A$. Let $x \in B$ then $f(\iota_B(x)) = g(\iota_B(x)) \Rightarrow f(x) = g(x)$. So $x \in A$, thus $B \subseteq A$. □

1.10.42 Problem. Let $f, g \in Y^X$. Prove that there exists an equivalence relation on ρ on Y such that if $\Phi_\rho : Y \to Y/\rho$ be the quotient map then $\Phi_\rho \circ f = \Phi_\rho \circ g$, and moreover, if μ is any other equivalence relation on Y such that $\Phi_\mu \circ f = \Phi_\mu \circ g$, then $\rho \subseteq \mu$, and Y/ρ is finer than Y/μ. Hence there exists $\pi : Y/\rho \to Y/\mu$ such that $\pi \circ \Phi_\rho = \Phi_\mu$.

1.10.42.1 Solution. Let $\sum = \{\lambda \in \mathcal{P}(Y \times Y); \lambda \text{ is an equivalence relation on } Y\}$ such that $\Phi_\lambda \circ g = \Phi_\lambda \circ f$, which means

$$\begin{aligned}
&(\Phi_\lambda \circ f)(x) = (\Phi_\lambda \circ g)(x), \; \forall \; x \in X \\
\Leftrightarrow &[f(x)]/\lambda = [g(x)]/\lambda \; \forall \; x \in X \\
\Leftrightarrow &[f(x)]/\lambda = [g(x)]/\lambda \; \forall \; x \in X \\
\Leftrightarrow &\, (f(x), g(x)) \in \lambda \; \forall \; x \in X.
\end{aligned}$$

Since $Y \times Y \in \sum$, so $\sum \neq \emptyset$. Now, $\rho = \bigcap_\lambda \{\lambda \in \sum\}$ is an equivalence relation on Y. We show that $\Phi_\rho \circ g = \Phi_\rho \circ f$. We show that $(f(x), g(x)) \in \rho \; \forall \; x \in X$. Since $(f(x), g(x)) \in \lambda \; \forall \; \lambda \in \sum$, therefore $(f(x), g(x)) \in \bigcap_\lambda \{\lambda \in \sum\} = \rho \; \forall \; x \in X$. Since $\rho = \bigcap_\lambda \{\lambda \in \sum\}$, so $\rho \subseteq \lambda \; \forall \; \lambda \in \sum$. Now, if μ is any other equivalence relation on Y such that $\Phi_\mu \circ g = \Phi_\mu \circ f$, then $\rho \subseteq \mu$. Now, we define $\pi : Y/\rho \to Y/\mu$ by $\pi([y]_\rho) = [y]_\mu$. We show that π is well-defined, let $[y]_\rho = [z]_\rho \Rightarrow (y, z) \in \rho \Rightarrow (y, z) \in \mu \Rightarrow [y]_\mu = [z]_\mu \Rightarrow \pi([y]_\rho) = \pi([z]_\rho)$. Thus π is well-defined. Again, if $[y]_\mu \in Y/\mu$ then $\pi([y]_\rho) = [y]_\mu \Rightarrow \pi$ is surjective and $\pi \circ \Phi_\rho = \Phi_\mu$. □

1.10.43 Problem. Let A, B, C be sets such that $B \cap C = \emptyset$. Prove that

$$A^{B \cup C} \approx A^B \times A^C.$$

1.10.43.1 Solution. Let $f \in A^{B \cup C}$, then $f = u \cup v$ where $u = f \cap (B \times A)$ and $v = f \cap (C \times A)$. We see that $u \cap v = \emptyset$ for $B \cap C = \emptyset$ now $u : B \to A$ and $v : C \to A$. We define $\Phi : A^{B \cup C} \to A^B \times A^C$ by $\Phi(f) = \Phi(u \cup v) = (u, v)$. We now verify that Φ is well-defined, an injection and a surjection. □

1.10.44 Problem. Let A, B, C be sets. Prove that $A^C \times B^C \approx (A \times B)^C$.

1.10.44.1 Solution. Define $\Phi : A^C \times B^C \to (A \times B)^C$ by

$$(\Phi(f, g))\,(x) = (f(x), g(x)) \; \forall \; x \in C.$$

For injectivity: let

$$\begin{aligned}
&\Phi(f_1, g_1) = \Phi(f_2, g_2) \\
\Rightarrow &\Phi(f_1, g_1)(x) = \Phi(f_2, g_2)(x) \; \forall \; x \in C \\
\Rightarrow &(f_1(x), g_1(x)) = (f_2(x), g_2(x)) \; \forall \; x \in C \\
\Rightarrow &f_1(x) = f_2(x), \; g_1(x) = g_2(x) \; \forall \; x \in C \\
\Rightarrow &f_1 = f_2, \; g_1 = g_2 \\
\Rightarrow &(f_1, g_1) = (f_2, g_2).
\end{aligned}$$

To prove surjectivity; let

$$\begin{aligned}
&h \in (A \times B)^C \text{ then } h(x) \in A \times B \;\forall\, x \in C.\\
&\text{Let } h(x) = (h_1(x), h_2(x))\\
&\text{then } h_1 \in A^C \text{ and } h_2 \in B^C\\
&\Rightarrow (h_1, h_2) \in A^C \times B^C\\
&\Rightarrow \Phi(h_1, h_2) = h \text{ since } \Phi(h_1, h_2)(x) = (h_1(x), h_2(x)) = h(x) \;\forall\, x \in C. \quad \square
\end{aligned}$$

1.10.45 Problem. Let A, B, C be sets. Prove that $A^{B\times C} \approx \left(A^B\right)^C$

1.10.45.1 Solution. Let $f \in A^{B\times C}$. Define $\Phi : A^{B\times C} \to \left(A^B\right)^C$ by

$$\Phi(f) = \left\{(c, g) \in C \times A^B \text{ and } g(b) = f(b, c)\right\}.$$

We show that Φ is bijective. Injective: let

$$\begin{aligned}
&\Phi(f_1) = \Phi(f_2)\\
\Rightarrow &\left\{(c, g) \in C \times A^B \text{ and } g(b) = f_1(b, c)\right\}\\
&\quad = \left\{(c, g) \in C \times A^B \text{ and } g(b) = f_2(b, c)\right\}\\
\Rightarrow &f_1(b, c) = f_2(b, c) \;\forall\, b \in B \text{ and } c \in C\\
\Rightarrow &f_1 = f_2.
\end{aligned}$$

For surjectivity, let $h \in \left(A^B\right)^C$, then $h(c) : B \to A$ and $h(c)(b) \in A$. Let $f : B \times C \to A$ such that $f(b, c) = h(c)(b)$. Then

$$\begin{aligned}
\Phi(f) &= \left\{(c, g) \in C \times A^B \text{ and } g(b) = f(b, c)\right\}\\
&= \left\{(c, g) \in C \times A^B \text{ and } g(b) = h(c)(b)\right\}\\
&= \left\{(c, g) \in C \times A^B \text{ and } g = h(c)\right\} = h. \quad \square
\end{aligned}$$

1.10.46 Problem. If A, B, C are sets, then

1. $A \times B = A \times C \Leftrightarrow$ either $B = C$ or A is empty.
2. $(A \times B) \times C = A \times (B \times C) \Leftrightarrow$ at least one of the sets is empty.
3. $A \times B = B \times A \Leftrightarrow$ either $A = B$ or at least one of the sets is empty.

1.10.46.1 Solution.

1. If $A \neq \emptyset$ then $(x, y) \in A \times B \Rightarrow (x, y) \in A \times C \Rightarrow x \in A$ and $y \in B \Rightarrow y \in C \Rightarrow B \subseteq C$ similarly $C \subseteq B$, thus $B = C$.
2. If one of the sets A, B, C is empty, then by the above $(A \times B) \times C = \emptyset = A \times (B \times C)$. Conversely, suppose $(A \times B) \times C = A \times (B \times C)$. If none of the sets A, B, C is empty, then there exists at least one element $(a, c) \in (A \times B) \times C$ where $a \in A \times B$ and $c \in C$. By hypothesis $(a, c) \in A \times (B \times C) \Rightarrow a \in A$, a contradiction.

3. Suppose that $A \neq \emptyset$ $B \neq \emptyset$ then $\forall\ (x, y) \in A \times B \Leftrightarrow (x, y) \in B \times A$, thus $A \subseteq B$ and $B \subseteq A$. Hence $A = B$. Again, let $A \times B = B \times A$ and $A \neq B$ then without loss of generality, we assume that $\exists\ x \in A$ and $x \notin B$. So for any $y \in B$

$$(x, y) \notin B \times A \Rightarrow (x, y) \in (B \times A)^C = (A \times B)^C = A^C \times B^C \Rightarrow y \in B^C.$$
$$\text{Thus } B \subseteq B^C \Rightarrow B = \emptyset. \quad \square$$

1.10.47 Problem. If X, Y be two sets and $f : X \to Y$ is a surjection, then show that

1. f induces an equivalence relation ρ on X;
2. there is a surjection $g : X \to X/\rho$ and
3. there is an injection $h : Y \to X/\rho$ such that $h \circ f = g$.

1.10.47.1 Solution. Hint:

1. Define a relation ρ on X defined by $(x, y) \in \rho$ iff $f(x) = f(y)$, then, it is easy to see that ρ is an equivalence relation on X. Thus f induces an equivalence relation ρ on X and the equivalence class determined by x is denoted by $[x]$ and A/ρ is the family of equivalence classes $\{[x]; x \in X\}$ is called the quotient set determined by ρ.
2. Define a mapping $g : X \to X/\rho$ by $g(x) = [x]$ the equivalence class determined by x. By definition we see that g is a surjection.
3. Hint. $g : X \xrightarrow{f} Y \xrightarrow{h} X/\rho$. $\square$

1.10.48 Problem. Determine the number of ordered triples (A, B, C) of sets which have the property that

1. $A \cup B \cup C = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and
2. $A \cap B \cap C = \emptyset$.

1.10.48.1 Solution. There is a bijection between triples of subsets of $\{1, 2, 3, 4, ., , 9, 10\}$ and 10×3 matrices with 0,1 entries, sending (A, B, C) to the matrix $B = (b_{ij})$ with $b_{ij} = 1$ if $i \in A_i$ and $b_{ij} = 0$ otherwise. Under this bijection the set of triples satisfying $A \cup B \cup C = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and $A \cap B \cap C = \emptyset$ maps onto the set T of 10×3 matrices with 0,1 entries such that no row is (0 0 0) or (1 1 1). The number of possibilities for each row of such matrices is $2^3 - 2 = 6$, so $|T| = |S| = 6^{10}$. $\square$

1.10.49 Problem. Let A be a set and $f : A \to A$ be a 1-1 function. Then $f^n : A \to A$ is an 1-1 function for all integers $n \geq 1$. (f^n is a standard abbreviation for $f \circ f \circ .. \circ f$ with n occurence of f.)

1.10.49.1 Solution. If possible, let $\exists\ n > 1$ such that f^n is not 1-1. Let $m > 1$ be the smallest positive integer such that f^m is not 1-1. Then $\exists\ a, b \in A$ such that $a \neq b$ and $f^m(a) = f^m(b)$. But then $f(f^{m-1}(a)) = f(f^{m-1}(b))$ and hence $f^{m-1}(a) = f^{m-1}(b)$ since f is 1-1. Again, since m is the smallest positive integer such that f^m is not 1-1, f^{m-1} is 1-1. Hence $a = b$, a contradiction. Therefore, f^n is 1-1 for all $n \geq 1$. $\square$

1.10.50 Problem. Let A be a finite set and $f : A \to A$ be a 1-1 function. Then f is onto.

1.10.50.1 Solution. Let $a \in A$. Then $f^n(a) \in A \;\forall\; n \geq 1$. Hence the set

$$\{a, f(a), f^2(a), ...\} \subseteq A.$$

Since A is finite, so A cannot contain distinct elements and so, there exist m and n such that $m > n$ with $f^m(a) = f^n(a)$. Then $f^n(f^{m-n}(a)) = f^n(a)$. Hence by the previous problem f^n is 1-1. Let $b = f^{m-n-1}(a) \in A$. Then $a = f(b)$. Hence f is onto. □

1.10.51 Problem.

1. For a function $f : X \to X, f^n$ is a standard abbreviation for $f \circ f \circ .. \circ f$ with n occurence of f. Suppose that $f^n = I_X$. Show that f is bijective.
2. If S is a finite set and f is an injection $S \to S$, then show that for some integer $n > 0, f^n = I_S$.
3. If S has m elements, find an $n > 0$ (in terms m) that works simultaneously for all injections $S \to S$.

1.10.51.1 Solution.

1. If $n = 1$, then $f = \iota_X$ which is a bijection. Let $n > 1$ and suppose that f is not an injection, then $\exists\; x_1, x_2 \in X$ such that $x_1 \neq x_2$ and $f(x_1) = f(x_2)$. Thus $f^{n-1}f(x_1) = f^{n-1}f(x_2) \Rightarrow f^n(x_1) = f^n(x_2) \Rightarrow x_1 = x_2$, a contradiction. Again, suppose that f is not a surjection, this means that $\exists\; y \in X$ such that $y \neq f(x) \;\forall\; x \in X$. Thus, we get $f^{n-1}f(z) \neq y \;\forall\; z \in X$. Hence $f^n = \iota_X$ is not a surjection, a contradiction. □
2. Use the previous problem.
3. $n = m!$.

1.10.52 Problem. If X is a non-empty set, then the mapping $\rho \to X/\rho$ defines a bijection from the set $E(X)$ of all equivalence relations on X onto the set $\mathcal{P}(X)$ of all partitions of X.

1.10.52.1 Solution. If ρ is an equivalence relation on X, the set X/ρ of equivalence classes is the partition $\mathcal{P}(X)$ of X so that $\rho \to X/\rho = \mathcal{P}_\rho$ defines a function

$$f : E(X) \to P(X).$$

Define a function $g : P(X) \to E(X)$ as follows: If $P = \{X_i : i \in I\}$ is a partition of X, let $g(P)$ be the equivalence relation ρ_P on X given by:

$$(a, b) \in \rho_P \Leftrightarrow a \in X_i \text{ and } b \in X_i \text{ for some (unique) } i \in I.$$

Then ρ_P is an equivalence relation on X. Hence g is well defined. It is clear that $g \circ f = I_{E(X)}$ and $f \circ g = I_{P(X)}$. Because $g(f(\rho)) = g(P_\rho) = \rho(P_\rho) = \rho$ for all $\rho \in E(X) \Rightarrow g \circ f = I_{E(X)}$. Again $(f \circ g)(P) = f(\rho_P) = P(\rho_P) = P$ for all $P \in \mathcal{P}(X) \Rightarrow f \circ g = I_{P(X)}$. Hence the result.

1.10.53 Problem. Let $f : X \to Y$ be a mapping onto Y. Then there is a mapping $g : Y \to X$ such that $f \circ g$ is the identity map on Y.

1.10.53.1 Solution. For each $y \in Y$, let $A_y = f^{-1}(\{y\})$. Consider the collection $\mathcal{A} = \{A_y; y \in Y\}$. Since f is onto, $A_y \neq \emptyset \;\forall\; y$. By the axiom of choice, there is a function F on $\mathcal{A}$ such that $F(A_y) \in A_y \;\forall\; y \in Y$. i.e. $F(A_y) \in f^{-1}(\{y\})$, so $f(F(A_y)) = y$. Define $g : Y \to X; y \mapsto F(A_y)$. Then $f \circ g = \iota_Y$. □

1.10.54 Problem. A mapping $f : \mathbb{R} \to \mathbb{R}$ is onto iff each line parallel to the x-axis intersects the graph of f at least once.

1. Formulate a similar condition for $f : \mathbb{R} \to \mathbb{R}$ to be one-one.
2. Formulate a similar condition for $f : \mathbb{R} \to \mathbb{R}$ to be both one-one and onto.

1.10.54.1 Solution.

1. A mapping $f : \mathbb{R} \to \mathbb{R}$ is one-one iff each line parallel to the x-axis intersects the graph of f exactly once.
2. A mapping $f : \mathbb{R} \to \mathbb{R}$ is one-one and onto iff each line parallel to the x-axis and each line parallel to the y-axis intersects the graph of f exactly once.

1.10.55 Problem. Define a mapping $f : \mathbb{N} \to \mathbb{N}$ by

$$f(n) = \begin{cases} \dfrac{n+1}{2} & \text{if } n \text{ is odd} \\ \dfrac{n}{2} & \text{if } n \text{ is even.} \end{cases}$$

Here, for each $n \in \mathbb{N}$, the equation $f(x) = n$ has exactly two solutions.
(For example $f(x) = 2$ has the solutions $x = 3$ and $x = 4$.).

1. Define a mapping $f : \mathbb{N} \to \mathbb{N}$ such that for each $n \in \mathbb{N}$, the equation $f(x) = n$ has exactly three solutions.
2. Define a mapping $f : \mathbb{N} \to \mathbb{N}$ such that for each $n \in \mathbb{N}$, the equation $f(x) = n$ has exactly n solutions.
3. Define a mapping $f : \mathbb{N} \to \mathbb{N}$ such that for each $n \in \mathbb{N}$, the equation $f(x) = n$ has infinitely many solutions.

1.10.55.1 Solution.

1. $$f(n) = \begin{cases} \frac{n}{3} & \text{if } n = 3m,\ m \in \mathbb{N} \\ \frac{n+1}{3} & \text{if } n = 3m-1,\ m \in \mathbb{N} \\ \frac{n+2}{3} & \text{if } n = 3m-2,\ m \in \mathbb{N} \end{cases}$$

2. $$f(k) = \begin{cases} \frac{k}{n} & \text{if } k = nm,\ m \in \mathbb{N} \\ \frac{k+1}{n} & \text{if } k = nm-1,\ m \in \mathbb{N} \\ \frac{k+2}{n} & \text{if } k = nm-2,\ m \in \mathbb{N} \\ \dots\dots\dots\dots & \\ \frac{k+n-1}{n} & \text{if } k = nm-n+1,\ m \in \mathbb{N} \end{cases}$$

3. Define $f : \mathbb{N} \to \mathbb{N}$ in the following way

$$\begin{array}{l} 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, ... \\ 1; 1, 2; 1, 2, 3; 1, 2, 3, 4; ... \quad \square \end{array}$$

1.10.56 Problem. Let $X = \{1, 2, 3\}$. Find an example of a function $f : X \to X$ and a set $A \subset X$ so that $f^{-1}(A) \neq \emptyset$ but $f^{-1}(f^{-1}(A)) = \emptyset$.

1.10.56.1 Solution. Define $f : X \to X$ by $f(1) = 2, f(2) = f(3) = 3$ and $A = \{1, 2\}$ then $f^{-1}(A) = \{1\}$ and $f^{-1}\left(f^{-1}(A)\right) = \emptyset$.

1.10.57 Problem.

1. Let $f\colon X \to Y$ be a function. Define $F : \mathcal{P}(X) \to \mathcal{P}(Y)$ by $F(S) = f(S)\ \forall\ S \in \mathcal{P}(X)$.
   (a) What conditions on f ensure that F is 1-1?
   (b) What conditions on f ensure that F is onto?
   (c) Show that if f is a bijection, then so is F.
2. Let $f : X \to Y$ be a function. Define $G : \mathcal{P}(Y) \to \mathcal{P}(X)$ by $G(S) = f^{-1}(S)\ \forall\ S \in \mathcal{P}(Y)$.
   (a) What conditions on f ensure that G is 1-1?
   (b) What conditions on f ensure that G is onto?
   (c) Show that if f is a bijection, then so is G.

1.10.57.1 Solution.

1. (a) Suppose that f is not 1-1, so $\exists\ x, y \in X$ such that $f(x) = f(y)$ but $x \neq y$ then consider the sets $A = \{x, y\}, B = \{x\}$ and then $F(A) = \{f(x), f(y)\} = \{f(x)\}$ and $f(B) = \{f(x)\} \Rightarrow F(A) = F(B)$, if F is 1-1 $F(A) = F(B) \Rightarrow A = B$ i.e. $\{x, y\} = \{x\} \Rightarrow x = y$ which is impossible and thus F is not 1-1. Hence f is 1-1 ensures that F is 1-1.
   (b) Suppose that f is not onto, so $\exists\ y \in Y$ such that $y \notin F(X)$, then the set $T = F(X) \cup \{y\}$ in $\mathcal{P}(Y)$ has no preimage in $\mathcal{P}(X)$. Thus F is not onto. Hence f is onto ensures that F is onto.
   (c) Combining the above results, we can conclude that f is bijective implies F is bijective.
2. (a) We claim that f is onto. If f is not onto, then $\exists\ y \in Y$ such that $y \neq f(x)\ \forall\ x \in X$. In that case $G(\{y\}) = f^{-1}(y)$ does not exist and then dom$G \neq \mathcal{P}(Y)$. Hence f is onto. If f is 1-1, then $G(\{y_1\}) = G(\{y_2\}) \Rightarrow f^{-1}(y_1) = f^{-1}(y_2) \Rightarrow y_1 = y_2 \Rightarrow \{y_1\} = \{y_2\}$ shows that G is 1-1.
   (b) Left to the reader.
   (c) Left to the reader. □

1.10.58 Problem.

1. Show that there exist sets X, Y, Z and functions $f : X \to Y;\ g, h : Z \to X$ such that $f \circ g = f \circ h$ but $g \neq h$. What property is necessary on f such that $f \circ g = f \circ h \Rightarrow g = h$?
2. Show that there exist sets X, Y, Z and functions $f : X \to Y;\ g, h : Y \to Z$ such that $g \circ f = h \circ f$ but $g \neq h$. What property is necessary on f such that $g \circ f = h \circ f \Rightarrow g = h$?

1.10.58.1 Solution.

1. Let $f : X \to Y$ be not injective, then $\exists\ x_1, x_2 \in X$ such that $x_1 \neq x_2 \Rightarrow f(x_1) = f(x_2)$ and if $g \neq h$, then $\exists\ z \in Z$ such that $g(z) \neq h(z)$. Now assign $g(z) = x_1,\ h(z) = x_2$, hence $(f \circ g)(z) = f(g(z)) = f(x_1) = f(x_2) = f(h(z)) = (f \circ h)(z)$ for some $z \in Z$, Thus $f \circ g = f \circ h$ but $g \neq h$. The property on f is that it is injective.

2. Let $f : X \to Y$ be not surjective, then $f(X) \subsetneq Y \Rightarrow \exists\, y \in Y \setminus f(X)$ and $g \neq h$, then $\exists\, y \in Y$ such that $g(y) \neq h(y)$. Now, let $g(y) = z_1,\ h(y) = z_2$, then define g by $g(y) = z_1$ for some $y \in Y$ and
$$h(y) = \begin{cases} z_1 & \text{if } y \in f(X) \\ z_2 & \text{if } y \in Y \setminus f(X). \end{cases}$$
Now, for $y \in Y \setminus f(X),\ h(y) = z_2$ and $g(y) = z_1$, Thus $g \neq h$. Again, $(g \circ f)(x) = z_1 = (h \circ f)(x)\ \forall\, x \in X$. The property on f is that it is surjective.

1.10.59 Problem. Let X, Y be non-empty sets, let $f : X \to Y$. Prove the following:

1. f is injective iff there is a map $g : Y \to X$ such that $g \circ f = \iota_X$.
2. f is surjective iff there is a map $h : Y \to X$ such that $f \circ h = \iota_Y$.

1.10.59.1 Solution.

1. Let f be injective. Then for each $y \in f(X)$ there is a unique $x \in X$ such that $f(x) = y$. Choose a fixed element $x_0 \in X$. Define $g : Y \to X$ by
$$g(y) = \begin{cases} x & \text{if } y \in f(A) \text{ and } f(x) = y \\ x_0 & \text{if } y \notin f(X). \end{cases}$$
Then $g \circ f = \iota_X$.
Conversely, let there be a map $g : Y \to X$ such that $g \circ f = \iota_X$. Since ι_X is bijective, it follows that f is injective.

2. Suppose f is surjective. Then $f^{-1}(y) \subseteq X$ is a non-empty set for every $y \in Y$. For each $y \in Y$, choose $x_y \in f^{-1}(y)$. Then the map $h : Y \to X, y \mapsto x_y$ is such that $f \circ h = \iota_Y$. Conversely, let there be a map $h : Y \to X$ such that $f \circ h = \iota_Y$. Since ι_Y is bijective, $f \circ h$ is bijective and hence it follows that f is surjective. □

1.10.60 Problem. Let $f : A \to B$ be a mapping with $A, B \neq \emptyset$. Then the following are equivalent:

1. f is injective;
2. $\exists$ a mapping $g : B \to A$ such that $g \circ f = \iota_A$;
3. $\forall$ subsets $X \subseteq A$ and $\forall$ mappings $h_1, h_2 : X \to A$ such that $f \circ h_1 = f \circ h_2 \Rightarrow h_1 = h_2$.

1.10.60.1 Solution. We prove by showing $(1) \Rightarrow (2),\ (2) \Rightarrow (3)$ and $\neg(1) \Rightarrow \neg(3)$.

1. $(1) \Rightarrow (2)$ is done in the previous problem.
2. $(2) \Rightarrow (3)$: Let $x \in X$ and $h_1, h_2 : X \to A$ such that $f \circ h_1 = f \circ h_2$, then
$$\begin{aligned} & g \circ (f \circ h_1) = g \circ (f \circ h_2) \\ \Rightarrow & (g \circ f) \circ h_1 = (g \circ f) \circ h_2 \text{ (by the associativity of compositions)} \\ \Rightarrow & (g \circ f) \circ h_1 = (g \circ f) \circ h_2 \\ \Rightarrow & \iota_A \circ h_1 = \iota_A \circ h_2 \\ \Rightarrow & (\iota_A \circ h_1)(x) = (\iota_A \circ h_2)(x)\ \forall\, x \in X \\ \Rightarrow & \iota_A(h_1(x)) = \iota_A(h_2(x))\ \forall\, x \in X \\ \Rightarrow & h_1(x) = h_2(x)\ \forall\, x \in X \\ \Rightarrow & h_1 = h_2. \end{aligned}$$

3. $\neg(1) \Rightarrow \neg(3)$: If f is not injective then there are elements $a_1, a_2 \in A$ such that $a_1 \neq a_2$ but $f(a_1) = f(a_2)$. Now we construct mappings h_1, h_2 such that $h_1 \neq h_2$ but $f \circ h_1 = f \circ h_2$. Since $h_1 \neq h_2$, so $\exists\, a \in A$ such that $h_1(a) = a_1, h_2(a) = a_2$. Now

$$\begin{aligned}
&(f \circ h_1)(a) = f(h_1(a)) = f(a_1) = f(a_2) = f(h_2(a)) = (f \circ h_2)(a) \\
\Rightarrow &(f \circ h_2)(a) = (f \circ h_2)(a) \\
\Rightarrow &f \circ h_2 = f \circ h_2 \text{ for some } a \in A.
\end{aligned}$$

Hence the result follows. □

1.10.61 Problem. Let $f : A \to B$ be a mapping with $A, B \neq \emptyset$. Then the following are equivalent:

1. f is surjective,
2. $\exists$ a mapping $g : B \to A$ such that $f \circ g = \iota_B$,
3. $\forall$ set Y and $\forall$ mappings $h_1, h_2 : B \to Y$ such that $h_1 \circ f = h_2 \circ f \Rightarrow h_1 = h_2$.

1.10.61.1 Solution. We prove by showing $(1) \Rightarrow (2)$, $(2) \Rightarrow (3)$ and $\neg(1) \Rightarrow \neg(3)$

1. $(1) \Rightarrow (2)$: Let $\mathcal{A} = \{A_b = f^{-1}(b); b \in B\}$. Since f is onto, so $f^{-1}(b) \neq \emptyset$. Thus, $\mathcal{A}$ is the family of disjoint sets and $\bigcup_{b \in B} A_b = A$, so we can apply the axiom of choice on $\mathcal{A}$ to get a set S such that for $b \in B$ choose $a_b \in A$ such that $f(a_b) = b$ and $S \cap A_b = \{a_b\}$ is singleton. Now, consider the functions $g : B \to A$ defined by $g(b) = A_b \cap S = \{a_b\}$ and $(f \circ g)(b) = f(g(b)) = f(a_b) = b = \iota_B(b)\ \forall\, b \in B$. Thus $f \circ g = \iota_B$.

2. $(2) \Rightarrow (3)$: Let $y \in Y$ and $h_1, h_2 : B \to Y$ such that $h_1 \circ h_1 = f \circ h_2$, then

$$\begin{aligned}
&g \circ (f \circ h_1) = g \circ (f \circ h_2) \\
\Rightarrow &(g \circ f) \circ h_1 = (g \circ f) \circ h_2 \text{ (by the associativity of compositions)} \\
\Rightarrow &(g \circ f) \circ h_1 = (g \circ f) \circ h_2 \\
\Rightarrow &\iota_A \circ h_1 = \iota_A \circ h_2 \\
\Rightarrow &(\iota_A \circ h_1)(x) = (\iota_A \circ h_2)(x)\ \forall\, x \in X \\
\Rightarrow &\iota_A(h_1(x)) = \iota_A(h_2(x))\ \forall\, x \in X \\
\Rightarrow &h_1(x) = h_2(x)\ \forall\, x \in X \\
\Rightarrow &h_1 = h_2.
\end{aligned}$$

3. $\neg(1) \Rightarrow \neg(3)$ Let $f : A \to B$ be not surjective, then $f(A) \subsetneq B \Rightarrow B \setminus f(A) \neq \emptyset$ and $h_1 \neq h_2$, then $\exists\, b \in B$ such that $h_1(b) \neq h_2(b)$. Now, let $h_1(b) = b_1,\ h_2(b) = b_2$, then define h_1 by $h_1(b) = b_1$ for some $b \in B$ and

$$h_2(b) = \begin{cases} b_1 & \text{if } b \in f(A) \\ b_2 & \text{if } b \in B \setminus f(A). \end{cases}$$

Now, for $b \in B \setminus f(A),\ h_2(b) = b_2$ and $h_1(b) = b_1$, Thus $h_1 \neq h_2$. Again,

$$\begin{aligned}
&(h_1 \circ f)(a) = b_1 = (h_2 \circ f)(a)\ \forall\, a \in A \\
\Rightarrow &h_1 \circ f = h_2 \circ f.
\end{aligned}$$

Hence the result follows. □

1.10.62 Problem (The characterization of injection:)**.** Let $f : X \to Y$. Then f is one-to-one iff for all functions g and h such that $g : Z \to X$ and $h : Z \to X$, $f \circ g = f \circ h$ implies that $g = h$.

1.10.62.1 Solution. Suppose that f is one-to-one and that g and h are mappings on $Z \to X$ for which $f \circ g = f \circ h$. Then $f(g(z)) = f(h(z))\ \forall\ z \in Z$. Now, since f is oneto-one it follows that $g(z) = h(z)\ \forall\ z \in Z$. Hence, $g = h$. Converse part is done in the previous problem. □

1.10.63 Problem (The characterization of surjection:)**.** Let $f : X \to Y$. Then f is onto Y iff for all functions g and h such that $g : Y \to Z$ and $g : Y \to Z$, $g \circ f = h \circ f$ implies $g = h$.

1.10.63.1 Solution. Suppose f is onto Y then let $y \in Y$, since f is onto, so $\exists\ x \in X$ such that $y = f(x)$ so $\forall\ y \in Y$ $g(y) = g(f(x)) = (g \circ f)(x) = h \circ f(x) = h(f(x)) = h(y)$. Thus $g = h$. Converse part is done in the previous problem (1.10.61). □

1.10.64 Problem. Let $A = \{0, 1\}$ and B, C be non-empty sets and $f : B \to C$ be a non-surjective map. Then there exist distinct maps $g, h : C \to A$ such that $g \circ f = h \circ f$.

1.10.64.1 Solution. Define $g, h : C \to A$ by $g(x) = 0$ for all $x \in C$ and

$$h(x) = \begin{cases} 0 & \text{if } x \in f(B) \subsetneq C \\ 1 & \text{if } x \in C \setminus f(B) (\neq \emptyset). \end{cases}$$

Now, for $x \in C \setminus f(B), h(x) = 1$ and $g(x) = 0$, thus $g \neq h$. Again, $(g \circ f)(x) = 0 = (h \circ f)(x)\ \forall\ x \in B$. Hence $g \circ f = h \circ f$. □

1.10.65 Problem (A decomposition of an arbitrary function)**.** Prove that, if $\alpha : S \to T$ is a mapping, then there exists a surjection β and an injection γ such that $\alpha = \gamma \circ \beta$.

1.10.65.1 Solution. Since $\alpha : S \to T$ is a mapping, we get the family $\tau = \{A_t = \alpha^{-1}(t); \forall\ t \in \alpha(S)\}$ of disjoint sets and $\bigcup_t A_t = S$. So by **axiom of choice** there exists a set $C \subset S$, such that $C \cap A_t$ is a singleton set $\forall\ t \in \alpha(S)$. Now Define a function $\beta : S \to C$ by $\beta(s) = C \cap \alpha^{-1}(\alpha(s))$. and $\gamma : C \to T$ by $\gamma(c) = \alpha(c)$, for $s \in S, c \in C$ and we want to show that β is surjective and γ is injective.
Let $c \in C$, then $C \subset S$ and $\{c\} = C \cap A_t$ for some $t \in \alpha(S)$ i.e.$\{c\} = C \cap \alpha^{-1}(t)$ and if $t = \alpha(s)$ then $c = C \cap \alpha^{-1}(\alpha(s)) = \beta(s)$, thus β is onto. Now let $p, q \in C$ and $\gamma(p) = \gamma(q) = \alpha(c) \Rightarrow \{p\} = C \cap \alpha^{-1}(\alpha(c))$ and $\{q\} = C \cap \alpha^{-1}(\alpha(c))$ so $\{p\} = \{q\} \Rightarrow p = q$. Hence γ is injective. □

1.10.66 Problem. Let the function $f : \mathbb{R} \setminus \{-1\} \to \mathbb{R}$ be given with the formula $f(x) = \frac{x}{1+x}$. Find the function f_n, $n \in \mathbb{N}$, where $f_1 = f, f_2 = f \circ f_1$ and $f_n = f \circ f_{n-1}$ for $n = 2, 3, ...$ Determine also the natural domains of these composite functions.

1.10.66.1 Solution. The range of the function f is the set $\mathbb{R} \setminus \{-1\}$. Then

$$f_2(x) = f \circ f_1(x) = f(f(x)) = f\left(\frac{x}{1+x}\right) = \frac{x}{1+2x}.$$

Clearly, the domain is the set $\mathbb{R} \setminus \{-1/2\}$, the domain of f_2 is the set $\mathbb{R} \setminus \{-1, -1/2\}$. Then we can prove by mathematical induction that for $n = 2, 3, ...$, domain of

$$f_n(x) = \frac{x}{1+nx} \text{ is } \mathbb{R} \setminus \{-1, -1/2, ..., -1/n\}. \quad \square$$

1.10.67 Problem. Let A be an uncountable set and let B be a countable subset of A. Show that A is equivalent to $A \setminus B$.

1.10.67.1 Solution.

1. If $A \cap B = \emptyset$ then $A \setminus B = A$ implies the identity function is the bijection from $A \setminus B$ to A. Thus A is equivalent to $A \setminus B$.

2. Let $A \cap B \neq \emptyset$. Now, as $A \setminus B = A \setminus A \cap B$, so let $A \cap B = \{b_1, b_2, ...\} \subseteq B$. Let $C = \{c_1, c_2, ...\}$ be a countable subset of $A \setminus B$. Now define $f : A \setminus B \to A$ by

$$f(x) = \begin{cases} x, & \text{if } x \notin C \\ b_n, & \text{if } x = c_n (n = 1, 2, ...). \end{cases}$$

   Then f is one-to-one and onto, proving that $A \setminus B \simeq A$. □

1.10.68 Problem. Let X be a set, and $f : X \to X$. Let $A_0 = X$ and $A_{n+1} = f(A_n)$.

1. $A_{n+1} \subseteq A_n \ \forall \ n \in \mathbb{N}$.
2. Let $A = \bigcap_{n=1}^{\infty} A_n$. Prove that $f(A) \subseteq A$.
3. Show that it is not necessarily true that $A \subseteq f(A)$.

1.10.68.1 Solution.

1. Now, we have

$$\begin{aligned} & A_0 = X \\ \Rightarrow & f(A_0) = f(X) \subseteq X = A_0 \\ \Rightarrow & A_1 \subseteq A_0 \\ \Rightarrow & f(A_1) \subseteq f(A_0) \\ \Rightarrow & A_2 \subseteq f(A_0) = A_1. \end{aligned}$$

   So, by induction, the result follows.

2. Since $A = \bigcap_{n=1}^{\infty} A_n$, so

$$f(A) = f\left(\bigcap_{n=1}^{\infty} A_n\right) \subseteq \left(\bigcap_{n=1}^{\infty} f(A_n)\right) = \bigcap_{n=1}^{\infty} A_{n+1} \subseteq A.$$

3. Example: Let $X = \{(i, j); i, j \in \mathbb{N}, j \geq i\} \cup \{a, b\}$ (where a,b be any two "extra" elements.) Define $f : X \to X$ by

$$f(i, j) = \begin{cases} a, & \text{if } i = j \\ (i + 1, j), & \text{if } i \neq j \end{cases}$$

   and $f(a) = f(b) = b$. Show that $A_n = \{(i, j); i \geq n+1, j \geq i\} \cup \{a, b\}$ and $\{a, b\} = A = \bigcap_n A_n$. but $a \notin f(A)$. □

1.10.69 Problem. Give an example of a collection $\mathcal{C} = \{A_n; n \in \mathbb{N}\}$ of distinct non-epmty sets A_n such that for any finite set $F \subset \mathbb{N}$ we have $\bigcap_{n \in F} A_n \neq \emptyset$ but $\bigcap_{n=1}^{\infty} A_n = \emptyset$.

1.10.69.1 Solution. Consider $A_n = \{n, n+1, n+2, ...\}$. $\square$

1.10.70 Problem. For any subsets A, B, A_i of X, then prove the following:

1. $\chi_A = 1 \Leftrightarrow A = X$ and $\chi_A = 0 \Leftrightarrow A = \emptyset$,
2. $A \subseteq B \Leftrightarrow \chi_A \leq \chi_B$
3. $\chi_{X \setminus A} = 1 - \chi_A$,
4. $\chi_{A \cap B} = \chi_A \chi_B = \min\{\chi_A, \chi_B\}$,
5. $\chi_{\cap A_i} = \inf_i \chi_{A_i} = \prod_i \chi_{A_i}$,
6. $\chi_{A \cup B} = 1 - (1 - \chi_A)(1 - \chi_B) = \max\{\chi_A, \chi_B\} = \chi_A + \chi_B - \chi_{A \cap B}$,
7. $\chi_{\cup A_i} = \sup_i \chi_{A_i} = 1 - \prod_i (1 - \chi_{A_i})$,
8. $\chi_A = \chi_B$ iff $A = B$,
9. $\chi_{A \triangle B} = \chi_A + \chi_B \pmod 2$,
10. $\chi_{A \triangle B} = |\chi_A - \chi_B|$, where $A \triangle B = (A \setminus B) \cup (B \setminus A)$,
11. if A is finite, then $|A| = \sum_{x \in X} \chi_A(x)$.

1.10.70.1 Solution.

1. By the problem,

$$\begin{aligned}
&\chi_A = 1 \\
&\Rightarrow \forall\ x \in X \text{ implies } \chi_A(x) = 1 \\
&\Rightarrow X \subseteq A \text{ implies } X = A. \\
\text{and } &\chi_A = 0 \\
&\Rightarrow \forall\ x \in X\ \chi_A(x) = 0 \text{ implies } X \subseteq A^C \Rightarrow A = \emptyset.
\end{aligned}$$

   Converse parts are obvious.

2. Suppose that $A \subseteq B$ and $\chi_A > \chi_B$. Then

$$\begin{aligned}
&\chi_A(x) > \chi_B(x) \Rightarrow 1 > 0. \\
\Rightarrow &\chi_A(x) = 1 \text{ and } \chi_B(x) = 0 \\
\Rightarrow &x \in A \text{ and } x \in B^C \\
\Rightarrow &x \in A \cap B^C
\end{aligned}$$

   which contradicts that $A \subseteq B$. Hence $\chi_A \leq \chi_B$.

3. Now, $\chi_{X \setminus A}(x) = 1$ implies $x \in A^C \Rightarrow \chi_A(x) = 0$ implies $\chi_{X \setminus A}(x) + \chi_A(x) = 1$. Again, $\chi_{X \setminus A}(x) = 0$ implies $x \in A \Rightarrow \chi_A(x) = 1$ implies $\chi_{X \setminus A}(x) + \chi_A(x) = 1$. Thus $\chi_{X \setminus A} = 1 - \chi_A$.

4. Here

$$\begin{aligned}
&\chi_{A\cap B}(x) = 1\\
\Rightarrow &x \in A\cap B \Rightarrow x \in A \text{ and } x \in B\\
\Rightarrow &\chi_A = 1 \text{ and } \chi_B = 1\\
\Rightarrow &\chi_A(x)\chi_B(x) = 1\\
\Rightarrow &\left(\chi_A\chi_B\right)(x) = 1.
\end{aligned}$$

Thus $\chi_{A\cap B} = \chi_A\chi_B$.
Again,

$$\begin{aligned}
\min\{\chi_A,\chi_B\}(x) &= \frac{\chi_A(x)+\chi_B(x)-|\chi_A(x)-\chi_B(x)|}{2}\\
&= 1 \text{ (as } x\in A\cap B),
\end{aligned}$$

shows that $\chi_{A\cap B} = \chi_A\chi_B = \min\{\chi_A,\chi_B\}$.

5. Similar to 4.

6. Since $A\cup B = \left(A^C \cap B^C\right)^C$ so,

$$\begin{aligned}
&\chi_{A\cup B}\\
&= \chi_{(A^C\cap B^C)^C}\\
&= 1-\chi_{A^C\cap B^C}\\
&= 1-\chi_{A^C}\chi_{B^C}\\
&= 1-\left(1-\chi_B\right)\left(1-\chi_C\right).
\end{aligned}$$

7. Similar to 5.

8. Suppose $\chi_A = \chi_B$ then $\chi_A(x) = \chi_B(x)\ \forall\ x \in X$. Let $\chi_A(x) = 1$, then $\chi_B(x) = 1$, so, $x\in A \Rightarrow x \in B$, hence $A\subseteq B$. Let $\chi_A(x) = 0$, then $\chi_B(x) = 0$ implies $x \in A^C \Rightarrow x \in B^C$, and $A^C \subseteq B^C \Rightarrow B \subseteq A$. Thus $A = B$.

9. Now, we have

$$\chi_{A\triangle B}(x) = \begin{cases} 1 & \text{if } x\in A\setminus B\\ 1 & \text{if } x \in B\setminus A\\ 0 & \text{otherwise.}\end{cases} \quad \text{and } (\chi_A+\chi_B)(x) = \begin{cases} 1 & \text{if } x\in A\setminus B\\ 1 & \text{if } x \in B\setminus A\\ 2 & \text{if } x\in A\cap B\\ 0 & \text{otherwise.}\end{cases}$$

Thus $\chi_{A\triangle B} = \chi_A + \chi_B \pmod 2$.

10.

$$\left(\chi_A-\chi_B\right)(x) = \chi_A(x)-\chi_B(x) = \begin{cases} 1 & \text{if } x\in A\setminus B\\ -1 & \text{if } x\in B\setminus A\end{cases}$$

Thus $|\chi_A - \chi_B|(x) = 1$, if $x \in A\backslash B$ or $x\in A\backslash B$ implies $\chi_{A\triangle B}(x) = 1$. Hence $\chi_{A\triangle B} = |\chi_A - \chi_B|$,

11. Since A is finite, then $A = \cup_{x\in X}\{x\}$ and $|A| = \sum_{x\in X} \chi_A(x)$. $\square$

1.10.1 Remark. The set of characteristic functions of the subsets of X, is the set $\{0,1\}^X$, and it is sometimes denoted by 2^X, as $card\{0,1\} = 2$.

1.10.71 Problem.

1. Let $\mathcal{F} = \{F \in 2^{\mathbb{N}}; F \text{ is an infinite set}\}$. What is $card\,(\mathcal{F})$?
2. Prove that, for every $n \in \mathbb{N}, \mathbb{N}^n = \mathbb{N}\times\mathbb{N}\times .. \times\mathbb{N}$ is a countable set. So, $\mathcal{A} = \cup_{n\in\mathbb{N}}\mathbb{N}^n$ is countable.

1.10.71.1 Solution.

1. Let p and q be two prime numbers, $(p \neq q)$, then $\forall\, n, m \in \mathbb{N}\setminus\{0\}, p^n \neq q^m$.....(*)
   Let $p_0, p_1, p_2, ..., p_k, ...$ be distinct prime numbers.
   Let for each $k = 0, 1, 2, ...F_k = \{p_k^n; n \in \mathbb{N}\}$. So $p_k = 2 \Rightarrow F_k = \{2, 2^2, 2^3, ...\}$. Hence (*) shows that for $i \neq j, F_i \cap F_j = \emptyset$. Moreover, each F_i is an infinite set.
   Let
   $$\mathcal{F}_0 = \{A; A \in 2^{\mathbb{N}}, A \text{ is finite}\}.$$
   Then $\mathcal{F}_0 \cap \mathcal{F} = \emptyset$, $\mathcal{F}_0 \cup \mathcal{F} = 2^{\mathbb{N}}$. Hence $card\mathcal{F}$ is $\mathfrak{c}$.
2. We define a mapping $f : \mathcal{F}_0 \to \mathcal{A}$ as follows: Let $A \in \mathcal{F}_0$. So A is of the form: $A = \{n_1, n_2, n_3, .., n_k, ..\}$ and $f(A) = (n_1, n_2, ..., n_k) \in \mathbb{N}^k$. Clearly f is 1-1. As $\mathcal{A}$ is countable, so is $\mathcal{F}_0$. $\square$
   Conclusion: The set $\mathcal{F}$ is uncountable. Thus in $\mathbb{N}$ there are uncountably many infinite subsets.

1.10.72 Problem. A mapping $f : \mathbb{N} \to \mathbb{N}$ is said to be strictly increasing if $n < m \Rightarrow f(n) < f(m)$. How many strictly increasing mappings $f : \mathbb{N} \to \mathbb{N}$ do we have?

1.10.72.1 Solution. Hint: If $f : \mathbb{N} \to \mathbb{N}$ is strictly increasing, then the set $M = f(\mathbb{N})$ is an infinite set. Now, let $M \subseteq \mathbb{N}$ be an infinite set. So M is of the form $M = \{n_1, n_2, n_3, ...\}$ with $n_1 < n_2 < n_3 < ....$ To M, we associate the mapping $f : \mathbb{N} \to \mathbb{N}, f(k) = n_k$ so that $f(\mathbb{N}) = M$. Hence there are uncountably many strictly increasing mappings. $\square$

1.10.73 Problem. Let (A_n) be a sequence of sets. Show that $\bigcup_{k=1}^{\infty} A_k$ can written as a disjoint union family of sets. Deduce from this another proof of the fact that the countable union of countably many sets is at most countable.

1.10.73.1 Solution. Hint: Let $B_0 = A_0, B_1 = A_1 \setminus A_0, ..., B_n = A_n \setminus \cup_{k=1}^{n-1} A_k$. Show that the sets B_n are pairwise disjoint, and

$$\bigcup_{k=1}^{n} A_k = \bigcup_{k=1}^{n} B_k \text{ and } \bigcup_{k=1}^{\infty} A_k = \bigcup_{k=1}^{\infty} B_k. \quad \square$$

1.10.74 Problem. Let X be a set. Show that the set $S = \{f; f : X \to \{0,1\}\}$ is equipotent to $\mathcal{P}(X)$. [S is denoted by 2^X and $\mathcal{P}(X)$ is the power set of X]. Furthermore $|X| < |\mathcal{P}(X)| = |2^X|$.

1.10.74.1 Solution. Define a function $\phi : X \to \mathcal{P}(X)$ by $\phi(x) = \{x\}$ for $x \in X$, then ϕ is an injection, which shows that $|X| \leq |\mathcal{P}(X)|$. Now, we show that there cannot be a surjection from $X \to \mathcal{P}(X)$. Suppose that $\pi : X \to \mathcal{P}(X)$ is surjective. Then consider the set $T = \{t \in X; t \notin \pi(t)\}$.

We show that T has no preimage in X. Clearly $T \neq \emptyset$, for $\emptyset \in \mathcal{P}(X) \Rightarrow \exists\, a \in X$ such that $\pi(a) = \emptyset$, thus $a \notin \emptyset = \pi(a)$. So $a \in T \subseteq X$. Again, let $s \in X$ such that $\pi(s) = T$. We observe that

$$\begin{aligned} s \in T \Rightarrow&\ s \notin T \\ \text{and } s \notin T \Rightarrow&\ s \in T, \end{aligned}$$

a contradiction, shows that π is not a surjection. Thus $|X| < |\mathcal{P}(X)|$.
Now, we show that $\mathcal{P}(X)$ is equipotent to 2^X. Define a function $\psi : \mathcal{P}(X) \to 2^X$ by $\psi(P) = \chi_P$, where χ_P is the characteristic function defined on $X \to \{0,1\}$, by $P \in \mathcal{P}(X)$, $\chi_P(x) = \begin{cases} 1, & \text{if } x \in P \\ 0, & \text{if } x \in P^C. \end{cases}$

1. Injectivity: We observe that

$$\begin{aligned} &\psi(A) = \psi(B) \\ \Rightarrow &\chi_A(x) = \chi_B(x)\ \forall\, x \in X. \end{aligned}$$

   Let $\chi_A(x) = 1$, then $\chi_B(x) = 1$, so, $x \in A \Rightarrow x \in B$, hence $A \subseteq B$. Let $\chi_A(x) = 0$, then $\chi_B(x) = 0$ implies $x \in A^C \Rightarrow x \in B^C$, and $A^C \subseteq B^C \Rightarrow B \subseteq A$. Thus $A = B$ shows that ψ is injective.

2. Surjectivity: Let $\phi \in 2^X$, then $\phi(x) = 1$ or $\phi(x) = 0$. Suppose that $\phi(x) = 1$, then consider the set $P = \{x \in X; \phi(x) = 1\}$ then $P^C = \{x \in X; \phi(x) = 0\}$. Hence $\phi = \chi_P$. Thus ψ is surjective. Hence $\mathcal{P}(X)$ is equipotent to 2^X and $|\mathcal{P}(X)| = |2^X|$. □

1.10.75 Problem. Let S be a set. Prove that S is infinite if and only if $|A| = |S|$ for some proper subset A of S.

1.10.75.1 Solution. Assume $|A| = |S|$ where A is a proper subset of S. If S is finite, say $|S| = |\{1, 2, ..., n\}|$ then $|A| = |\{1, 2, ..., n-k\}| < |\{1, 2, ..., n\}|$ where k is the number of elements in $S \setminus A$, contradiction. So S is infinite as desired. Conversely, assume S is infinite. Choose $x_1 \in S$, then $x_2 \in S \setminus \{x_1\}$, then $x_3 \in S \setminus \{x_1, x_2\}$,,..... in this way we have (by induction) distinct points $x_n \in S$ for all $n \in \mathbb{N}$. Let $T = \{x_n; n \in \mathbb{N}\}$ and define $f : T \to T$ by $f(x_n) = x_{2n}$. Since the x_n's are distinct f is injective, so $f : T \to f(T)$ is bijective. Now define $g : S \to S$ by $g(x) = f(x)$ if $x \in T$, and $g(x) = x$ if $x \in S \setminus T$. Since f is injective so is g, so $g : S \to g(S)$ is bijective. Now $A \equiv g(S)$ has the same cardinality as S, yet since $f(T)$ is a proper subset of T, $A = (S \setminus T) \cup f(T)$ is a proper subset of S, as desired. □

1.10.76 Problem.

1. Let $\mathcal{F}(\mathbb{N}) = \{A \subseteq \mathbb{N}; |A| < \infty\}$, then show that $\mathcal{F}(\mathbb{N})$ is countable.

2. There exists a function $\phi : \mathcal{F}(\mathbb{N}) \to \mathbb{N}$ such that $\phi(A) \leq \phi(B)$ if $A \subseteq B$ for$A, B \in \mathcal{F}(\mathbb{N})$.

1.10.76.1 Solution.

1. Hint: Note that $\mathcal{F}(\mathbb{N})$ is equipotent to a subset of

$$\mathbb{N} \cup (\mathbb{N} \times \mathbb{N}) \cup ... \cup (\mathbb{N} \times \mathbb{N} \times .. \times \mathbb{N}) \cup ...$$

2. Left to the reader. □

1.10.77 Problem. Exhibit a bijection between the following sets:

1. $[a, b]$ and $[c, d]$,
2. (a, b) and (c, d),
3. $[a, b)$ and $[c, d)$,
4. $[a, b)$ and $(c, d]$,
5. (a) (a, b) and $[c, d]$
   (b) $[-1, 1]$ and $(-1, 1)$,
6. $(a, b]$ and (c, d),
7. $[a, \infty)$ and $\mathbb{R}$,
8. (a, ∞) and $\mathbb{R}$,
9. $(0, 1)$ and $\mathbb{R}^+$,
10. $(0, 1) \cap \mathbb{Q}^C$ and $\mathbb{R}$,

1.10.77.1 Solution.

1. $[a, b]$ and $[c, d]$, Define $f : [a, b] \to [c, d]$ by
$$f(x) = (x - b)\frac{c - d}{a - b} + d.$$
2. (a, b) and (c, d), Same as before.
3. $[a, b)$ and $[c, d)$, Same as before.
4. $[a, b)$ and $(c, d]$, Define $f : [a, b) \to (c, d]$ by
$$f(x) = (x - a)\frac{c - d}{b - a} + d.$$
5. (a) (a, b) and $[c, d]$. We define a function $f : [c, d] \to [a, b]$ by
$$f(x) = (x - c)\frac{a - b}{c - d} + a.$$
Now, we construct a bijective mapping $g : [a, b] \to (a, b)$ by considering a sequence (x_n) in (a, b) and define g by
$$g(x) = \begin{cases} x_1 & \text{if } x = a \\ x_2 & \text{if } x = b \\ x_{n+2} & \text{if } x = x_n, n \geq 1 \\ x & \text{otherwise.} \end{cases}$$
Then $[c, d] \xrightarrow{f} [a, b] \xrightarrow{g} (a, b)$ defines a composition mapping $g \circ f$ and its inverse the required bijection.

(b) Let $f : [-1, 1] \to (-1, 1)$ defined by

$$f(x) = \begin{cases} \frac{x}{2}, & \text{if } x = \pm\frac{1}{2^{n-1}}\ n = 1, 2, ... \\ x, & \text{otherwise.} \end{cases}$$

It is readily verified that f is a bijection.

6. $(a, b]$ and (c, d), construction is similar to above.

7. $[a, \infty)$ and $\mathbb{R}$. Consider a sequence (x_n) with $x_1 = a$ in $[a, \infty)$ and define $f : [a, \infty) \to \mathbb{R}$ by $f(x) = \begin{cases} x_1 & \text{if } x = a \\ x_{n+1} & \text{if } x = x_n\ n \geq 1 \\ \ln(x - a) & \text{otherwise.} \end{cases}$

8. (a, ∞) and $\mathbb{R}$, $x \mapsto \ln(x - a)$.

9. $(0, 1)$ and $\mathbb{R}^+$, $x \mapsto \tan \dfrac{\pi x}{2}$.

10. $(0, 1) \cap \mathbb{Q}^C$ and $\mathbb{R}$: Consider a sequence (x_n) in $(0, 1) \cap \mathbb{Q}^C$, and enumerate the rationals in $\mathbb{R}$ by $r_1, r_2, r_3...$ define f by

$$f(x) = \begin{cases} r_n & \text{if } x = x_n,\ n = 1, 2, .. \\ \frac{2x-1}{x(1-x)} & \text{otherwise.} \end{cases}$$

1.10.78 Problem. Any complex number that is a root of a (nonzero) polynomial with integer coefficients is called an **algebraic number**. Show that the set of all algebraic numbers are countably infinite.

1.10.78.1 Solution. Let $\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$. Fix $n > 1$. Since every polynomial $p(x) = a_0 + a_1x + .. + a_nx^n$ is determined uniquely by $(a_0, a_1, ..., a_n)$, it is easy to see that the non-zero polynomials of degree $< n$ with integer coefficients are in one-to-one correspondence with the countable set $\mathbb{Z}^{n+1} \setminus \{(0, 0, ..., 0)\}$. Let $(a_0, a_1, ..., a_n)$, be an enumeration of all these polynomials. By the Fundamental Theorem of Algebra, the set $A_k = \{x \in \mathbb{C}; p_k(x) = 0\}$ is a finite set. Thus, the set of all zeros of the polynomials $\{p_1, p_2, ...\}$ of degree $< n$ is precisely the set $R_n = \cup_{k=1}^{\infty} A_k$, which is a countable set. Now, note that the set of all algebraic numbers is which is, as a countable union of countable sets is itself countable. $\square$

1.10.79 Problem. (Principle of Mathematical Induction) If S is a subset of the set $\mathbb{N}$ of natural numbers such that $0 \in S$ and either

$$\begin{aligned} &(1)\ \ n \in S \Rightarrow n + 1 \in S\ \forall\, n \in \mathbb{N}; \\ or\quad &(2)\ \ m \in S\ \ \forall\, 0 \leq m < n \Rightarrow n \in S\ \forall\, n \in \mathbb{N}; \end{aligned}$$

then $S = \mathbb{N}$.

1.10.79.1 Solution. If $N \setminus S \neq \emptyset$, let $n \neq 0$ be its least element. Then for every $m < n$, we must have $m \notin \mathbb{N} \setminus S$ and hence $m \notin \mathbb{N} \setminus S$. Consequently either (i) or (ii) implies $n \in S$, which is a contradiction. Therefore $\mathbb{N} \setminus S = \emptyset$ and $\mathbb{N} = S$. $\square$

1.10.80 Problem. Let A, B, C be sets such that $C \subseteq A, A \cap B = \emptyset$ and B, C are denumerable. Prove that $A \cup B \simeq A$.

1.10.80.1 Solution. Let $B = \{b_1, b_2, ..., b_n..\}$ and $C = \{c_1, c_2, ..c_n, ..\}$. Define a mapping $f : A \cup B \to A$ by

$$f(x) = \begin{cases} c_i, & x = b_i \\ x, & \text{otherwise.} \end{cases} \quad \square$$

1.10.81 Problem. Let U be a partially ordered set. Show that we can write $U = S \cup T$, with $S \cap T = \emptyset$, such that S is well ordered (with respect to the ordering in U) and T has no least element.

1.10.81.1 Solution. Hint: Look at the union of all subsets of U that have no least element. $\square$

1.10.82 Problem. Show that $(0, 1]$ is equivalent to the unit square $(0, 1] \times (0, 1]$.

1.10.82.1 Solution. Let $P(x, y)$ be any point in this square. The coordinates x, y are real numbers and can be represented in a number system having base 2. We may then write

$$x = 0.x_1x_2..x_k...;\ y = 0.y_1y_2..y_k...$$

where x_k, y_k are either 0 or 1. In order to avoid ambiguity, we shall assume that the digits defining x, y contain an infinite number of times. With this understanding, the coordinates of $P(x, y)$ can be expressed in one and only one way as a dyadic fraction. Divide the digits composing these two decimal fractions into groups so that each group ends with the digit 1.Thus, if x and y have the particular values

$$x = 0.10010110001...$$
$$y = 0.01011001001...$$

The arrangements by groups is

$$x = 0.(1)(001)(01)(1)(0001)...$$
$$y = 0.(01)(01)(1)(001)(001)...$$

From this group arrangement, we form a new decimal number t by taking a group of digits first from one of the numbers x, y and then from the other. Thus, in the particular case just given we have

$$t = 0.(1)(01)(001)(01)(01)(1)(1)(001)(0001)(001)...$$
$$= 0.1010010101110010001001...$$

The number t thus defined is represented by a definite point of the segment (0,1), and thus to any point P of the square there corresponds one and only one point t of the required segment. Conversely, given any point t of the segment (0,1) represents uniquely a dyadic number, with the foregoing restriction as to the digit 1 contained in it. We can show by reversing the foregoing process that there corresponds one and only one point P of the given unit square, Since the required correspondence exists, the result follows. $\square$

1.10.82.2 Solution. To get the another solution we use **König binary Representation**: Let $n \in \mathbb{N}$, the **König-binary symbol** $(n]$ stands for $n-1$ zeros followed by a 1. Thus

$$\begin{aligned}
&(2] \text{ means } 01, (4] \text{ means } 0001, \text{and } (1] \text{ means } 1.\\
\text{e.g. } \frac{37}{128} &= \frac{1}{2^2} + \frac{1}{2^{2+3}} + \frac{1}{2^{2+3+2}}\\
&= .0100101 = 0.(2](3](2]\\
&= .010010011111.. = 0.(2](3](3](1](1](1]..
\end{aligned}$$

Thus for any $x \in (0,1], x$ has a unique binary form using infinitely many 1's. Then there is a unique infinite sequence $n_1, n_1, n_1, ...$ of positive integers such that

$$\begin{aligned}
x &= \frac{1}{2^{n_1}} + \frac{1}{2^{n_1+n_2}} + \frac{1}{2^{n_1+n_2+n_3}} + ...\\
&= 0.(n_1](n_2](n_3]...
\end{aligned}$$

This is called the **König-binary Representation** of x. Hence any $x \in (0,1]$ has one and only one König-binary Representation. For example $\frac{37}{128} = 0.(2](3](2]$ but the König-binary Representation is $0.(2](3](3](1](1](1]...$

We first prove that $(0,1] \simeq (0,1] \times (0,1]$. Given any $x \in (0,1]$, we have the unique König-binary form $0.(n_1](n_2](n_3]...$ Define

$$\begin{aligned}
f(x) &= 0.(n_1](n_3]...(n_{2k-1}]...\\
\text{and } g(x) &= 0.(n_2](n_4].....(n_{2k}]...
\end{aligned}$$

Let $F(x) = (f(x), g(x))$, then $F((0,1]) \subseteq (0,1] \times (0,1]$ and if $(x,y) \in (0,1] \times (0,1]$, so by the uniqueness of the König-binary form, we can write

$$\begin{aligned}
x &= 0.(m_1](m_2](m_3]...\\
y &= 0.(n_1](n_2](n_3]...
\end{aligned}$$

and the number t defined by $t = 0.(m_1](n_1](m_2](n_2](m_3](n_3]... \in (0,1]$ which implies F is onto. And F is 1-1 by the uniqueness of the König-binary form, hence

$$F : (0,1] \to (0,1] \times (0,1]$$

is bijective. Finally the function G defined for $0 \leq t \leq 1$ by

$$G(t) = \begin{cases} (4t, 0) & \text{if } 0 \leq t \leq \frac{1}{4} \\ (0, 4t-1) & \text{if } \frac{1}{4} < t \leq \frac{1}{2} \\ F(2t-1) & \text{if } \frac{1}{2} < t \leq 1. \end{cases}$$

is on $[0,1] \to [0,1] \times [0,1]$ and is one-one. $\square$

1.10.83 Problem. Show that the cardinality of the set P of all irrational numbers is $\mathfrak{c}$.

1.10.83.1 Solution. We propose two different approaches:

1. The set $\mathbb{N}^{\mathbb{N}}$ (which is of cardinality $\mathfrak{c}$, can be mapped one-to-one and onto the set of all irrational numbers between 0 and 1. This can be proved by using the concept of continued fractions; the required map ϕ is given by

$$\phi\left(n_1, n_2, ...\right) = \frac{1}{n_1 + \frac{1}{n_2 + \frac{1}{n_3} + ...}}, \ (n_1, n_2, ...) \in \mathbb{N}^{\mathbb{N}}.$$

2. Let ϕ be a one-to-one map from $\mathbb{R}$ onto $\mathbb{R}^2$, and call A the image of the set $\mathbb{Q}$. The set A is countable. Let ℓ be a vertical line in $\mathbb{R}^2$, that has the x-coordinate different from the x-coordinates of all the points in A. Then ℓ lies outside of A, and the cardinality of ℓ is $\mathfrak{c}$. Then the cardinality of $\phi^{-1}(\ell)$ is also $\mathfrak{c}$. The set $\phi^{-1}(\ell)$ lies outside of $\mathbb{Q}$. This shows that P contains a subset whose cardinality is $\mathfrak{c}$, and it is contained in a set whose cardinality is $\mathfrak{c}$. The Cantor–Bernstein–Schröder theorem shows that the cardinality of P is $\mathfrak{c}$. $\square$

1.10.84 Problem.

1. What is the cardinality of the family of all countable subsets of (0,1)?

2. What is the cardinality of the set of all continuous functions on [0,1]?

1.10.84.1 Solution.

1. Observe that there are as many countable subsets of (0,1) as mappings from $\mathbb{N}$ into (0,1). Accordingly, the cardinal of the family of all countable subsets of (0,1) is $card(0,1)^{\mathbb{N}} = \mathfrak{c}^{\aleph_0}$. Now compute $\mathfrak{c}^{\aleph_0} = \left(2^{\aleph_0}\right)^{\aleph_0} = 2^{\aleph_0 \cdot \aleph_0} = 2^{\aleph_0} = \mathfrak{c}$.

2. Use the fact that each continuous function is determined by its values at $\mathbb{Q} \cap [0,1]$. Then use (1). $\square$

1.10.85 Problem. A real number α is said to be algebraic if for some finite set of integers $a_0, ..., a_n$, not all 0, such that $a_0 + a_1\alpha + ... + a_n\alpha^n = 0$. Prove that the set of algebraic real numbers is countable.

1.10.85.1 Solution. Clearly the set of integers is countable. Since each polynomial corresponds to a finite selection of integers, the set of polynomials of degree n with integer coefficients is countable for each n. The set $\mathbb{Z}[x]$ of all polynomials with integer coefficients is the union over all n of sets of polynomials of degree n. As a countable union of countable sets, $\mathbb{Z}[x]$ is thus countable. Note that the set A of algebraic real numbers is

$$A = \bigcup_{p \in \mathbb{Z}[x]} \{x \in \mathbb{R}; p(x) = 0\}$$

Now each $p \in \mathbb{Z}[x]$ has some finite degree n, and then can only have at most n real roots. Thus the set $\{x \in \mathbb{R}; p(x) = 0\}$ is finite for each $p \in \mathbb{Z}[x]$. Hence as a countable union of finite sets, A is countable, as desired. $\square$

1.10.86 Problem. Let $\mathcal{A} = \{A_\alpha;\ \alpha \in \Lambda\}$ be non-empty family of pairwise disjoint non-empty subsets. Prove that there is an injection $\mathcal{A} \to \bigcup_{\alpha \in \Lambda} A_\alpha$.

1.10.86.1 Solution. Hint: Apply the axiom of choice to the family $\mathcal{A}$. $\square$

1.10.87 Problem. Let A and B be sets. Suppose that ρ is a relation from A to B. Prove that there is a mapping $\mu : A \to B$ that $\mu \subseteq \rho$.

1.10.87.1 Solution. Hint: Let $(a) = \{x; (x,a) \in \rho\}$. Apply the axiom of choice to the family $\mathcal{A} = \{(a); a \in A\}$. $\square$

1.10.88 Problem. Let ρ be a relation on a set X. Prove that

1. $\rho \cup \rho^{-1}$ is the smallest symmetric relation including ρ and
2. $\rho \cap \rho^{-1}$ is the largest symmetric relation included in ρ.

1.10.88.1 Solution. Hint:

1. Show that $\rho \cup \rho^{-1}$ is the symmetric relation including ρ, and every symmetric relation including ρ, includes $\rho \cup \rho^{-1}$.
2. Show rhat $\rho \cap \rho^{-1}$ is the symmetric relation included in ρ and every symmetric relation included ρ, is included in $\rho \cap \rho^{-1}$. $\square$

1.10.89 Problem. Let ρ be a reflexive and transitive relation on a set X. Prove that $\rho \circ \rho = \rho$.

1.10.89.1 Solution. Hint: The transitivity of ρ ensures that $\rho \circ \rho \subseteq \rho$ and use the reflexive property to prove the reverse inclusion. $\square$

1.10.90 Problem. Let ρ be a relation on a set X. Prove that ρ is an equivalence relation on X iff $\Delta_X \subseteq \rho$ and $\rho \circ \rho^{-1} \circ \rho = \rho$.

1.10.90.1 Solution. Hint: If ρ is an equivalence relation on X, so by definition $\Delta_X \subseteq \rho$ and we prove that $\rho \circ \rho^{-1} \circ \rho \subseteq \rho$ using symmetry and transitivity of ρ and the reverse inclusion using reflexive and transitive properties.
For the converse, reflexive is immediate. To prove the symmetricity and transitivity, choose the appropriate elements of ρ, ρ^{-1} and ρ and the fact $\rho \circ \rho^{-1} \circ \rho = \rho$. $\square$

1.10.91 Problem. Let X be a set with a binary operation "$*$" on X. We write $x * y = xy$ and $*$ satisfies

a. $x(yz) = (xy)z$,

b. $xy = yx$,

c. $xx = x$

for all $x, y, z \in X$. Define $\leq$ on X by $x \leq y$ iff $xy = y$. Prove that

1. X is a partially ordered set.
2. Each pair of elements of X has a least upper bound, that is if $x, y \in X$, then $\exists$ a $z \in X$ such that $x \leq z, y \leq z$ and if $x \leq w, y \leq w$ then $z \leq w$.
   As for example, for any set S, let $X = \mathcal{P}(S), *$ stands for $\cup$ and $\leq$ stands for $\subseteq$.

1.10.91.1 Solution. Let $x \in X$ then $xx = x \Rightarrow x \leq x$ showing that $\leq$ is reflexive. Suppose that $x \leq y$ and $y \leq x$ then $xy = y$ and $yx = x$, so by (b), $x = yx = xy = y$ gives $\leq$ is antisymmetric. Again, let $x \leq y$ and $y \leq z$ then $xy = y$ and $yz = z$, which shows $(xy)z = yz = z \Rightarrow x(yz) = z \Rightarrow xz = z \Rightarrow x \leq z$. Thus $\leq$ is transitive. Hence $(X, \leq)$ is a partially ordered set.
Let $x, y \in X$. Then $xy = yx = z$ (say). Hence $xz = x(xy) = (xx)y = xy = z$ and $yz = y(yx) = (yy)x = yx = z$ implies $x \leq z$ and $y \leq z$.
Now, let $x \leq w$ and $y \leq w$, then $xw = w, yw = w$ implies $yxw = yw = w$ and $xyw = xw = w$ implies $zw = w$, so $z \leq w$. $\square$

1.10.92 Problem. Let ρ be an equivalence relation on a set X. Let $q : X \to X/\rho$ be the corresponding quotient map. Then the equation $\bar{f} \circ q = f$ gives a one-to-one correspondence between the maps $\bar{f} : X/\rho \to Y$ from the quotient space and the maps $f : X \to Y$ and $x, y \in X$ satisfying $x\rho y \Rightarrow f(x) = f(y)$.

1.10.92.1 Solution. If $f : X \to Y$ is given by $f = \bar{f} \circ q$, then $x\rho y \Leftrightarrow q(x) = q(y) \Rightarrow f(x) = \bar{f}(q(x)) = \bar{f}(q(y)) = f(y)$.
Conversely, suppose $f : X \to Y$ has the property in the question. Then we show that $\bar{f}(q(x)) = f(x)$ uniquely defines a map $f : X \to Y$. Specifically, the requirement $\bar{f}(q(x)) = f(x)$ means that for any $[x] \in X/\rho$, and write $[x] = q(x)$. Then $\bar{f}([x]) = f(x)$. It remains to show that $\bar{f}$ is well defined. Let $[x] = [y]$, then $x\rho y \Rightarrow f(x) = f(y) \Rightarrow (\bar{f} \circ q)(x)) = (\bar{f} \circ q)(y) \Rightarrow \bar{f}(q(x)) = \bar{f}(q(y)) \Rightarrow \bar{f}([x]) = \bar{f}([y])$. Hence the result. □

1.10.93 Problem. Let $\mathbb{R}$ be the set of real numbers. Define a relation ρ on $\mathbb{R}$ by $a\rho b$ iff $a - b$ is an integer. Show that the quotient set can be identified with the interval $[0, 1)$.

1.10.93.1 Solution. Hint: The corresponding equivalence classes are

$$[x] = \{..., x - 2, x - 1, x, x + 1, x + 2, ...\} \ x \in [0, 1)$$

the quotient map q is $q(x) = x - n$ where n is the biggest integer $\leq x$. □

1.10.94 Problem. Let $\mathbb{R}$ be the set of real numbers. Define a relation ρ on $\mathbb{R}$ by $a\rho b$ iff $a - b$ is an integer. Show that the quotient set can be identified with the unit circle S^1 on the plane, which is also all the complex numbers of norm 1.

1.10.94.1 Solution. Hint: The map $E : \mathbb{R} \to S^1$ defined by $E(x) = e^{2\pi i x}$ is onto and satisfies $E(x) = E(y) \Leftrightarrow 2\pi x - 2\pi y \in 2\pi\mathbb{Z} \Leftrightarrow x - y \in \mathbb{Z}$. □

1.10.95 Problem. Let ρ_1 and ρ_2 be equivalence relations on X, such that $x\rho_1 y \Rightarrow x\rho_2 y$. How are the quotient sets X/ρ_1 and X/ρ_2 related?

1.10.95.1 Solution. Left to the reader. □

1.10.96 Problem. Let ρ_X and ρ_Y be equivalence relations on X and Y. A map $f : X \to Y$ is said to preserve the relation if $x_1\rho_X x_2 \Rightarrow f(x_1)\rho_Y f(x_2)$. Prove that such a map induces a unique map $\bar{f} : X/\rho_X \to Y/\rho_Y$ such that $q_Y \circ f = \bar{f} \circ q_X$, where q_X, q_Y are the corresponding quotient maps.

1.10.96.1 Solution. Left to the reader. □

1.10.97 Problem. Define two nonzero vectors u and v in a real vector space V to be equivalent if $v = ru$ for some real number $r > 0$. Show that the unit sphere SV of the vector space can be naturally identified with the quotient set of nonzero vectors under the relation. Then use this to identify homogeneous functions (with fixed degree) on $V \setminus \{0\}$ with functions on the unit sphere.

1.10.97.1 Solution. Left to the reader. □

1.10.98 Problem. Let F be the collection of all finite sets. For $A, B \in F$, define $A \sim B$ if there is a one-to-one correspondence $f : A \to B$. Prove that this is an equivalence relation. Moreover, identify the quotient set as the set of non-negative integers and the quotient map as the number of elements in a set. The exercise leads to a general theory of counting.

1.10.98.1 Solution. Left to the reader. □

1.10.99 Problem. A partition of a set X is a decomposition into a disjoint union of nonempty subsets

$$X = \bigsqcup_{i \in I} X_i, \ X_i \neq \emptyset$$

1. Define $x \sim y$ if x and y are in the same subset X_i. Prove that this is an equivalence relation.

2. Prove that X_i's are exactly the equivalence classes for the equivalence relation in the first part.

3. Prove that equivalence relations on X are in one-to-one correspondence with partitions of X.

1.10.99.1 Solution. Left to the reader. □

1.10.100 Problem. Previous problem shows that the concept of equivalence relations on X and the concept of partitions of X are equivalent concepts. Explain how the concept of quotient maps from X and the concept of partitions of X are equivalent.

1.10.100.1 Solution. Left to the reader. □

1.10.101 Problem. Suppose a relation '$\sim$' on X is reflexive and transitive. Prove that if we force the symmetry by adding $x \sim y$ (new relation) whenever $y \sim x$ (existing relation), then we get an equivalence relation.

1.10.101.1 Solution. Left to the reader. □

1.10.102 Problem. Find the quotient maps and the equivalence relations corresponding to the partitions.

1. $\mathbb{Z} = \{3n; n \in \mathbb{Z}\} \sqcup \{3n+1; n \in \mathbb{Z}\} \sqcup \{3n+2; n \in \mathbb{Z}\}$

2. $\mathbb{R} = (-\infty, 0) \cup \{0\} \cup (0, \infty)$

3. $X \times Y = \sqcup_{x \in X} \{x\} \times Y$.

1.10.102.1 Solution. Left to the reader. □

1.10.103 Problem. For the equivalence relation on X induced by a map $f : X \to Y$ defined by a relation $x_1 \sim x_2$ on X when $f(x_1) = f(x_2)$, show that the equivalence classes are $[x] = f^{-1}(f(x))$, and the corresponding partition is $X = \sqcup_{y \in f(X)} f^{-1}(y)$.

1.10.103.1 Solution. Left to the reader. □

1.10.104 Problem. Let $f : X \to Y$ and $g : Y \to X$ be any two maps. Show that X and Y can each be expressed as disjoint unions: $X = X_1 \cup X_2, Y = Y_1 \cup Y_2$, such that $f(X_1) = Y_1$ and $g(Y_2) = X_2$.

1.10.104.1 Solution. Hint: For each $E \subseteq X$, let $Q(E) = X \setminus g(Y \setminus f(E))$ and take $X_1 = \bigcap\{Q(E); Q(E) \subseteq E\}$. □

1.11 Additional Exercises on Chapter 1.

1.11.1 Exercise. Let $A = \{\emptyset\}$ and $B = \mathcal{P}(\mathcal{P}(A))$.

1. Is $\emptyset \in B,\ \emptyset \subseteq B$?
2. Is $\{\emptyset\} \in B,\ \{\emptyset\} \subseteq B$?
3. Is $\{\{\emptyset\}\} \in B,\ \{\{\emptyset\}\} \subseteq B$?

1.11.2 Exercise. Identify the equal and equivalent sets from the following

$$\begin{array}{l} (1)\ \{x \in \mathbb{N}; x \neq 2n, x \neq 2n+1,\ n \in \mathbb{N}\}. \quad (2)\ \{\emptyset\}. \\ (3)\ \emptyset. \quad (4)\ \{\{\emptyset\}\}. \quad (5)\ \{\{\emptyset\}, \{\emptyset\}\}. \quad (6)\ \{2,4,6,8,2,4\}. \\ (7)\ \{2,4,6,8\}. \quad (8)\ \{\{2,4,6,8\}\}. \\ (9)\ \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}\}. \quad (10)\ \{\{x; x \in \mathbb{N}\}\} \end{array}$$

1.11.3 Exercise. If $S = \{a, b, c\}$, A and B are subsets of S such that $A \cup B = S,\ A \cap B = \emptyset$. What is the number of possible solutions for A, B?

1.11.4 Exercise. What can be said about the sets A and B if $A \Delta B = A$?

1.11.5 Exercise. Let X be a set containing n elements and $\mathcal{C} \subseteq \mathcal{P}(X)$ such that for any two sets $A, B \in \mathcal{C}$ either $A \subseteq B$ or $B \subseteq A$. Show that the maximum cardinality of $\mathcal{C}$ is $n + 1$.

1.11.6 Exercise. Let X be a set containing n elements. Show that the number of pair of sets (A, B) that satisfy the condition $A \subseteq B \subseteq X$ is 3^n.

1.11.7 Exercise. Let A and B be finite sets such that $A \subseteq B$. Then the value of the expression

$$\sum_{C \setminus A \subseteq C \subseteq B} (-1)^{|C \setminus A|}$$

is always 1.

1.11.8 Exercise. Give an example of sets A, B and C such that $A \in B, B \in C$ and $A \notin C$.

1.11.9 Exercise. Give an example of sets A, B and C such that $A \in B, B \in C$ and $A \in C$.

1.11.10 Exercise. Give an example of sets A and B such that $A \in B$ and $A \subseteq B$.

1.11.11 Exercise. Determine whether each of following statements is true for arbitrary sets A, B, C.

1. If $A \in B$ and $B \subseteq C$, then $A \in C$.
2. If $A \in B$ and $B \subseteq C$, then $A \subseteq C$.
3. If $A \subseteq B$ and $B \in C$, then $A \in C$.
4. If $A \subseteq B$ and $B \in C$, then $A \subseteq C$.

1.11.12 Exercise. Show that any subset of 6 elements from the set $\{1, 2, 3, ..., 9\}$ must contain two elements whose sum is 10.
Hint: Consider the pigeon-holes $\{1 \textit{ or } 9\}\{2 \textit{ or } 8\}\{3 \textit{ or } 7\}\{4 \textit{ or } 6\}\{5\}$.

1.11.13 Note. Pigeonhole Principle: It may be interpreted as saying that if m pigeons are put into n pigeonholes and if $m > n$, then at least two pigeons must share one of the pigeonholes. This is a frequently-used result in combinatorial analysis.

1.11.14 Note. If m pigeon occupy n pigeon holes, then at least $\left\lfloor \frac{m-1}{n} \right\rfloor + 1 = \left\lceil \frac{m}{n} \right\rceil$ pigeons share the same pigeon hole. (where $\lceil x \rceil$ and $\lfloor x \rfloor$ are defined by $\lceil x \rceil$ least integer greater than or equal to x and $\lfloor x \rfloor$ the greatest integer less than or equal to x.

1.11.15 Exercise. Show that

1. for any subsets A, B and C of a set X, $A\Delta B \subseteq (A\Delta C) \cup (B\Delta C)$.
2. and for any subsets A_1, A_2, B_1, B_2 and C_1, C_2 of a set X,

$$(A_1\Delta B_1) \cup (A_2\Delta B_2) \supseteq \begin{cases} (A_1 \cup A_2)\,\Delta\,(B_1 \cup B_2) \\ (A_1 \cap A_2)\,\Delta\,(B_1 \cap B_2) \\ (A_1 \setminus A_2)\,\Delta\,(B_1 \setminus B_2). \end{cases}$$

1.11.16 Exercise. Prove that, for any subsets A, B, C and D of a set X,

1. $(A \cap B \cap C \cap D) \cup \left(A^C \cap B \cap C\right) \cup \left(B^C \cap C\right) \cup \left(C \cap D^C\right) = C.$
2. $\left\{(A \cap B \cap C) \cup \left[(A\Delta B) \cap C^C\right]\right\}^C \cap \left(\left(A^C \cap C\right) \cup C^C\right) \cap (B \cup (A \cap C))$
   $= B \cap (A\Delta C)$.

1.11.17 Exercise. Define a function $f : [0, 1] \to [0, 1]$ by

$$f(x) = \begin{cases} 0, & \text{if } x = 0, 1 \\ 1, & \text{if } x = \frac{1}{2} \\ \frac{(2[i/2]+1)}{2^{n-1}}, & \text{if } x = \frac{2i+1}{2^n} \\ x, & \text{otherwise.} \end{cases}$$

(Here $[x]$ is defined to be the largest integer smaller than x). Show that f is onto but not 1-1 on any interval.

1.11.18 Exercise. The basic laws of set equality or of subsets can be proved to be theorems of set theory. For all subsets X, Y and Z of any universal set U and A^C is the complement of $A \subseteq U$.

1. $X = X$, (reflexive property of equality)
2. $X \subseteq X$. (reflexive property of subset relation)
3. If $X = Y$, then $Y = X$. (symmetric property of equality)
4. $X = Y$ iff $X \subseteq Y$ and $Y \subseteq X$. (antisymmetric property of subsets)
5. If $X = Y$ and $Y = Z$ then $X = Z$. (transitive property of equality)
6. If $X \subseteq Y$ and $Y \subseteq Z$ then $X \subseteq Z$. (transitive property of of subset relation)
7. $\emptyset \subseteq X, \forall\ X \subseteq U$.

8. $X \cup X = X$. (idempotent law for union)
9. $X \cap X = X$. (idempotent law for intersection)
10. $X \cup \emptyset = X$. (identity for union)
11. $X \cap \emptyset = \emptyset$.
12. $X \cup U = U$.
13. $X \cap U = X$.
14. $X \cap Y = Y \cap X$.(commutative law for intersection)
15. $X \cup Y = Y \cup X$. (commutative law for union)
16. $X \cup (Y \cup Z) = (X \cup Y) \cup Z$. (associative law for union)
17. $X \cap (Y \cap Z) = (X \cap Y) \cap Z$. (associative law for intersection)
18. $X \subseteq X \cup Y$, and $Y \subseteq X \cup Y$.
19. $X \cap Y \subseteq X$, and $X \cap Y \subseteq Y$.
20. Properties of complementation:
    (i) $(X^C)^C = X$.
    (ii) $X \cup X^C = U$.
    (iii) $X \cap X^C = \emptyset$.
    (iv) $U^C = \emptyset$.
    (v) $\emptyset^C = U$.
21. Distributive laws
    (i) $X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$.
    (ii) $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$.
22. Set differences
    (i) $X \setminus Y = X \cap Y^C$.
    (ii) $X \setminus \emptyset = X$.
    (iii) $\emptyset \setminus X = \emptyset$.
    (iv) $X^C \setminus Y^C = Y \setminus X$.
    (v) $(X \setminus Y) \setminus Z = (X \setminus Z) \setminus (Y \setminus Z)$.
    (vi) De Morgan's laws $(X \cap Y)^C = X^C \cup Y^C$, $(X \cup Y)^C = X^C \cap Y^C$.
    (vii) $X \setminus (Y \cup Z) = (X \setminus Y) \cap (X \setminus Z)$.
    (viii) $X \setminus (Y \cap Z) = (X \setminus Y) \cup (X \setminus Z)$.
    (ix) $X = (X \cup Y) \cap (X \cup Y^C)$.
    (x) $X = (X \cap Y) \cup (X \cap Y^C)$.

(xi) $(X \cap Y) \cup (X^C \cap Y) \cup (X \cap Y^C) \cup (X^C \cap Y^C) = U$.

(xii) $X \cup (Y \setminus X) = X \cup Y$.

(xiii) $(X \setminus Y)^C = X^C \cup Y$.

23. Symmetric differences

    (i) $X \Delta Y = Y \Delta X$.

    (ii) $X \Delta (Y \Delta Z) = (X \Delta Y) \Delta Z$.

    (iii) $X \Delta X = \emptyset$.

    (iv) $X \Delta U = X^C$.

    (v) $X \Delta \emptyset = X$.

    (vi) $X \Delta Y = (X \cup Y) \setminus (X \cap Y)$.

    (vii) $X \cap (Y \Delta Z) = (X \cap Y) \Delta (X \cap Z)$.

    (viii) If $X \cap Y = \emptyset$ then $X \Delta Y = X \cup Y$.

24. (i) $X \subseteq Y$ iff $X \cap Y^C = \emptyset$.

    (ii) $X \subseteq Y$ iff $X^C \cup Y = U$.

    (iii) $X \subseteq Y$ and $X \subseteq Z$ implies $X \subseteq Y \cap Z$.

    (iv) $X \subseteq Z$ and $Y \subseteq Z$ implies $X \cup Y \subseteq Z$.

    (v) $X \subseteq Y$ implies $Y = X \cup (Y \setminus X)$.

    (vi) $X \subseteq Z$ implies $X \cup (Y \cap Z) = (X \cup Y) \cap Z$.

    (vii) If $X \cap Y = X \cap Z$ and $X \cup Y = X \cup Z$ then $Y = Z$.

    (ii) If $X \cap Y = X \cap Z$ and $X^C \cap Y = X^C \cap Z$ then $Y = Z$

    (viii) If $X \cup Y = X \cup Z$ and $X^C \cup Y = X^C \cup Z$ then $Y = Z$.

    (ix) $X \subseteq Y$ iff $Y^C \subseteq X^C$.

    (x) $X \subseteq Y$ iff $X \cup Y = Y$.

    (xi) $X \subseteq Y$ iff $X \cap Y = X$.

    (xii) $X \subseteq Y$ iff $X \setminus Y = \emptyset$.

1.11.19 Exercise.

1. $Y \times \emptyset = \emptyset \times Z = \emptyset$.
2. $(X \cup Y) \times Z = (X \times Z) \cup (Y \times Z)$.
3. $(X \cap Y) \times Z = (X \times Z) \cap (Y \times Z)$.
4. $(X \setminus Y) \times Z = (X \times Z) \setminus (Y \times Z)$.
5. $X \times Y = X \times Z$ and $X \neq \emptyset$ then $Y = Z$.
6. $X \times Y = Y \times X$, $X \neq \emptyset$ and $Y \neq \emptyset$ then $X = Y$.
7. If $Y \times Z = \emptyset$ then $Y = \emptyset$ or $Z = \emptyset$.

1.11.20 Exercise. Verify that

$$A_1 \cup ... \cup A_n = (A_1 \setminus A_2) \cup .... \cup (A_{n-1} \setminus A_n) \cup (A_n \setminus A_1) \cup (\cap_{i=1}^n A_i) .$$

1.11.21 Exercise. Prove that the system of equations

$$A \cup X = A \cup B, \ A \cap X = \emptyset$$

has at most one solution for X.

1.11.22 Exercise. If A, B are non-empty sets and $(A \times B) \cup (B \times A) = C \times C$, then prove that $A = B = C$.

1.11.23 Exercise. Let $A, B \subseteq X$ and $C, D \subseteq Y$. Prove that

1. $(A \times C) \cap (B \times D) = (A \cap B) \times (C \cap D)$.
2. $(A \times C) \cup (B \times D) \subseteq (A \cup B) \times (C \cup D)$; show that, in general, equality does not hold, by verifying

$$(A \cup B) \times (C \cup D) = (A \times C) \cup (B \times D) \cup (A \times D) \cup (B \times C).$$

3. $(X \times Y) \setminus (B \times D) = ((X \setminus B) \times Y) \cup (X \times (Y \setminus D))$.

1.11.24 Exercise. Let $\{A_n; n \in \mathbb{N}\}$ be a family of sets and $S_k = \bigcup_{i=1}^k A_i, \ k = 1, 2, ...$ Show that

$$\bigcup_{n=1}^{\infty} A_n = A_1 \cup (A_2 \setminus S_1) \cup ..... \cup (A_n \setminus S_{n-1}) \cup .....$$

and this is a pairwise disjoint union.

1.11.25 Exercise. Let $A_q = \{n \in \mathbb{N};\ n \text{ is divisible by } q\}$. What is $A_q \cup A_r, \ A_q \cap A_r$?

1.11.26 Exercise. Let $A \subset X$ and $f : X \to Y$. Let $i : A \to X$ be the map $i(a) = a$. Show that

1. $f_{|A} = f \circ i$.
2. If $g = f_{|A}$, Then $g^{-1}(B) = A \cap f^{-1}(B)$.

1.11.27 Exercise. Let $f : X \to Y$. Show that

1. f is injective $\Leftrightarrow \ \forall\, y \in Y; f^{-1}(y) = \emptyset$ or a singleton set $\Leftrightarrow \ \forall\, A; \ f(A^C) \subseteq (f(A))^C$.
2. f is surjective $\Leftrightarrow \ \forall\, y \in Y; f^{-1}(y) \neq \emptyset \Leftrightarrow \ \forall\, A; \ f(A^C) \supseteq (f(A))^C$.

1.11.28 Exercise.

1. If f is injective on A and B, then f is injective on $A \cup B$.
2. If f is injective on A and B, then f is injective on $A \cap B$.

1.11.29 Exercise. Show by an example that we can have maps f and g such that $f \circ g$ is one-one (onto) whereas f or g is not one-one (onto).

1.11.30 Exercise. If A be a finite set then show that any one-one map $f : A \to A$ is also onto.

1.11.31 Exercise. Let $f : A \to B$ be a map between two finite sets show that

1. If f is injective then $o(A) \leq o(B)$.
2. If f is surjective then $o(B) \leq o(A)$.
3. If $o(A) = o(B)$ then f is 1-1 and onto iff it is either 1-1 or onto

where $o(A)$ denotes the number of elements in the set A.

1.11.32 Exercise. Let $f : X \to Y$ be a function. Define $F : \mathcal{P}(X) \to \mathcal{P}(Y)$ by $F(S) = f(S)\ \forall\ S \in \mathcal{P}(X)$. Define $G : \mathcal{P}(Y) \to \mathcal{P}(X)$ by $G(S) = f^{-1}(S)\ \forall\ S \in \mathcal{P}(Y)$.

1. Show that the following statements are equivalent.
   (a) f is one-to-one.
   (b) F is one-to-one.
   (c) G is onto.
   (d) $f(M \cap N) = f(M) \cap f(N)\ \forall\ M, N \in \mathcal{P}(X)$.
   (e) $f(X \setminus M) \subseteq Y \setminus f(M)\ \forall\ M \in \mathcal{P}(X)$.
2. Show that the following statements are equivalent.
   (a) f is onto.
   (b) F is onto.
   (c) G is one-to-one.
   (d) $f(X \setminus M) \supseteq Y \setminus f(M)\ \forall\ M \in \mathcal{P}(X)$.
3. Show that the following statements are equivalent.
   (a) f is bijective.
   (b) F is bijective.
   (c) G is bijective.
   (d) $f(X \setminus M) = Y \setminus f(M)\ \forall\ M \in \mathcal{P}(X)$.

1.11.33 Exercise. Let X and Y be sets, $\mathcal{P}(X)$ and $\mathcal{P}(Y)$ their power sets. A function $f : X \to Y$ defines functions $F : \mathcal{P}(X) \to \mathcal{P}(Y)$ and $G : \mathcal{P}(Y) \to \mathcal{P}(X)$, by the formulas

$$F(A) = f(A) = \{f(x) : x \in A\}$$
$$G(B) = f^{-1}(B) = \{x \in X; f(x) \in B\}.$$

Discuss the composite functions $F \circ G$ and $G \circ F$.

1.11.34 Exercise. Let $\leq$ denote the product order on $\mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$, defined by for $(x_1, x_2) \leq (y_1, y_2)$ iff $x_1 \leq x_2$ and $y_1 \leq y_2$ for $(x_1, x_2), (y_1, y_2) \in \mathbb{N}^2$. Show that every subset $X \subseteq \mathbb{N}^2$ has only finitely many minimal elements.

1.11.35 Exercise. For any real $x, [x]$ denotes the greatest integer less than equal to x. Let ρ be a relation on $\mathbb{R}$, defined by $(x, y) \in \rho$ iff $[x] = [y]$, then show that ρ is an equivalence relation on $\mathbb{R}$ and for any positive integer $n, \langle n \rangle = [n, n+1)$.

1.11.36 Exercise. Let ρ, μ, τ be relations on X, then

1. $(\rho \cup \mu)^{-1} = \rho^{-1} \cup \mu^{-1}$
2. $(\rho \cap \mu)^{-1} = \rho^{-1} \cap \mu^{-1}$
3. $(\rho \circ \mu)^{-1} = \mu^{-1} \circ \rho^{-1}$
4. $(\rho \circ \mu) \circ \tau = \rho \circ (\mu \circ \tau)$.

1.11.37 Exercise. Let ρ be a relation on X, then $A \subseteq dom(\rho) \Leftrightarrow A \subseteq \rho^{-1}(\rho(A))$.

1.11.38 Exercise. Let ρ, μ be relations on X, then $dom(\mu) \subseteq dom(\rho) \Leftrightarrow \mu \subseteq \mu \circ \rho^{-1} \circ \rho$.

1.11.39 Exercise. Let ρ be a relation from X to Y, then ρ is a function iff $\rho\left(\rho^{-1}(A)\right) \subseteq A, \ \forall A \subseteq Y$.

1.11.40 Exercise. Let ρ be a relation from X to Y, then the following are equivalent.

1. ρ is function on X.
2. $\forall A, B \subseteq Y$ we have $\rho^{-1}(A \cap B) = \rho^{-1}(A) \cap \rho^{-1}(B)$.
3. $\forall A, B \subseteq Y$ with for $A \cap B = \emptyset$, we have $\rho^{-1}(A) \cap \rho^{-1}(B) = \emptyset$.

1.11.41 Exercise. Suppose ρ is an arbitrary relation on a set A, such that ρ is reflexive and transitive. Prove that the relation ϕ defined by $x\phi y$ iff $x\rho y$ and $y\rho x$, is an equivalence relation on A.

1.11.42 Exercise. How many distinct equivalence classes there are for each of the following equivalence relations?

1. Two people are equivalent if they are born in the same week.
2. Two people are equivalent if they are born in the same year.
3. Two people are equivalent if they are of the same sex.

1.11.43 Exercise. Let $f : X \to Y$ be a function. Show that if f is injective then there exists a set Z and functions $g, h : Z \to X$ with $f(g(z)) = f(h(z)) \ \forall z \in Z$ but $g(z) \neq h(z)$ for some $z \in Z$.

1.11.44 Exercise. Let A, B, C be non-empty sets, let $f : A \to B; g : B \to C$; let $h = g \circ f$. Prove that

1. if f and g are injective, then h is injective;
2. if f and g are surjective, then h is surjective;
3. if h is injective, then f is injective;
4. if h is surjective, then g is surjective;
5. if h is injective and f is surjective; then g is injective;

6. if h is surjective and g is injective; then f is surjective;

1.11.45 Remark. If $h = g \circ f$ is bijective, neither f nor g may be bijective.

1.11.46 Example. Let $X = \{1, 2, 3\}, Y = \{3, 4, 5, 6\}$. Define $f : X \to Y$ and $g : Y \to X$ by

$$f(1) = 3, f(2) = 4, f(3) = 5;$$
$$g(3) = 1, g(4) = 2, g(5) = 3, g(6) = 3.$$

Then $h = g \circ f = I_X$. As I_X is bijective, $h = g \circ f = I_X$ is bijective but neither f nor g is bijective.

1.11.47 Exercise. For every mapping f with domain A we have $f \circ \iota_A = f$ and for every mapping g with codomain A we have $\iota_A \circ g = g$.

1.11.48 Exercise. Let A, B, C be non-empty sets, let $f : A \to B$; $g : B \to C$; $h : C \to A$. If $h \circ g \circ f$ and $g \circ f \circ h$ are injections, and $f \circ h \circ g$ is a surjection, prove that f, g and h are all bijections.

1.11.49 Exercise. Let A, B, C be three non-empty sets, $f : A \to C$, and $g : B \to C$ be two mappings. If g is injective, prove that $\exists$ a mapping $h : A \to B$ such that $f = g \circ h$ if and only if $f(A) \subseteq g(B)$. Show that if this condition is satisfied then the mapping h is unique.

1.11.50 Exercise. Let $\{A_\alpha; \alpha \in \Lambda\}$ be a family in 2^X and $A \in 2^X$. Show that $\bigcup_{\alpha\in\Lambda} A_\alpha \setminus A = \bigcup_{\alpha\in\Lambda}(A_\alpha \setminus A)$ and $A \setminus \bigcup_{\alpha\in\Lambda} A_\alpha = \bigcap_{\alpha\in\Lambda}(A \setminus A_\alpha)$.

1.11.51 Exercise. Let $\{A_\alpha; \alpha \in \Lambda\}$ be a family of subsets of X. If each A_α has exactly one element, then $\prod_\alpha A_\alpha$ consists of a single element. If $\Lambda = \emptyset$ then $\prod_\alpha A_\alpha$ has exactly one element, the null set. If $\Lambda \neq \emptyset$ and some $A_\alpha = \emptyset$ then $\prod_\alpha A_\alpha = \emptyset$.

1.11.52 Exercise. Prove the following:

1. If S is any set, then $S \times S$ is an equivalence relation on S.
2. If a relation ρ on a set S is transitive, then ρ^{-1} is also transitive on S.
3. ρ is reflexive and transitive relation on $S \Rightarrow \rho \cap \rho^{-1}$ is an equivalence relation on S.
4. Show that, ρ is a symmetric and transitive relation on $S \Rightarrow \rho$ is an equivalence relation on S, is not true.
5. Prove that if the relation ρ on S is antisymmetric, so is the relation ρ^{-1}.
6. Prove that if the relation ρ on S is symmetric, then $\rho \cap \rho^{-1} = \rho$.
7. Prove that if the relations ρ and μ on S are antisymmetric, so is the relation $\rho \cap \mu$. What about $\rho \cup \mu$?
8. Can a relation ρ on a set S be both symmetric and antisymmetric?
9. Let $A = \{1, 2, 3\}$. Give an example of a relation ρ on A such that ρ is neither symmetric nor antisymmetric.
10. Prove that if the relation ρ on a set S is symmetric, then ρ^k is also symmetric for any $k > 0$, where ρ^k is the k-th power of the relation ρ^k. i.e. $\rho^k = \rho \circ .. \circ \rho(k \text{ times})$.

11. Prove that asymmetry implies antisymmetry.
12. Prove that it is false that antisymmetry implies asymmetry.
13. Prove that asymmetry and symmetry together imply transitivity.
14. Can a relation ρ on a set A be both asymmetric and antisymmetric? Why?
15. The intersection of two connected relations in S is connected in S.
16. The union of two connected relations in S is connected in S.

1.11.53 Exercise.

1. Show that $A \cup B$ is finite iff A and B are finite.
2. Let $\Lambda(\neq \emptyset)$ be finite and for each $\alpha \in \Lambda$, A_α be finite. Show that $\bigcup_{\alpha\in\Lambda} A_\alpha$ is finite. Is the converse of true?
3. Show that $A \times B$ is finite if A and B are finite. If $A \times B$ is finite, does it imply that A and B are both finite?

1.11.54 Exercise. Let ρ be an relation on a set X. Prove that ρ is a partial order relation on X iff $\rho \circ \rho = \rho$ and $\rho \cap \rho^{-1} = \Delta_X$.

1.11.55 Exercise.

1. Show that a set S is finite if and only if each nonempty subset of $\mathcal{P}(S)$ (partially ordered by inclusion) has a minimal element.
2. Show that a set S is infinite if and only if S is equivalent to some proper subset of itself. Hint: Recall that any infinite set contains a countably infinite subset.

1.11.56 Exercise.

1. Let $A, B \subseteq X$. Show that $A \times B$ is finite if A and B are finite. If $A \times B$ is finite, does it imply that both A and B are finite?
2. Let $\Lambda(\neq \emptyset)$ be finite and for each $\alpha \in \Lambda, A_\alpha$ be finite. Then show that $\prod_{\alpha\in\Lambda} A_\alpha$ is finite.
3. Is the converse of (2) true?

1.11.57 Exercise. Let $A \subseteq X$. Show that, if A is finite then the power set $\mathcal{P}(A)$ is also finite.

1.11.58 Exercise. Let $A, B \subseteq X$. Let $\mathcal{F}$ be the set of all functions $f : A \to B$. Show that if both A and B are finite then $\mathcal{F}$ is finite.

1.11.59 Exercise. Show that if B is not finite and $B \subseteq A$, then A is not finite.

1.11.60 Exercise. Let $\Lambda(\neq \emptyset)$ be countable and for each $\alpha \in \Lambda, A_\alpha$ be countable. Then show that $\bigcup_{\alpha\in\Lambda} A_\alpha$ is countable. Is the converse true?

1.11.61 Exercise. Verify whether the the following sets are countable or not?

1. The set of all two element subsets of $\mathbb{N}$.
2. The set of all finite subsets of $\mathbb{N}$.

1.11.62 Exercise. Show that, if A is an infinite set, then $|A \times \mathbb{N}| = |A|$.

1.11.63 Exercise. Let $F = \{A; A \in 2^{\mathbb{N}},\ \text{both } A \text{ and } A^C, \text{are infinite}\}$. Show that the set F is uncountable.

1.11.64 Exercise. Suppose that the sets X and Y are infinite and $f : X \to Y$ is an onto mapping such that, for each $y \in Y$, the set $f^{-1}(\{y\})$ is countable. Show that then $Card(X) = Card(Y)$.

1.11.65 Exercise. Let $F = \{A; A \in 2^{\mathbb{N}}\ A \neq \emptyset \text{ and finite}\}$. Let $\phi : F \to \mathbb{N}, \phi(A) = \sum_{n\in A} n$. Show that is ϕ onto and that, for each $n \in \mathbb{N}$, the set $\phi^{-1}(n)$ is finite. From this deduce another proof of the fact that F is countable.

1.11.66 Exercise. Let $(X, \leq)$ be an ordered set such that for any two elements $x, y \in X$, $\sup\{x, y\}$ and $\inf\{x, y\}$ exist. Let $f : X \to X$ be a mapping. Show that f is increasing iff $f(\inf\{x, y\}) \leq \inf f(\{x, y\})\ \forall\, x, y \in X$.

1.11.67 Exercise. Let $\mathcal{A} = \{A_\alpha; \alpha \in \Lambda\}$ be a collection of sets and $A = \bigcup\{A_\alpha; \alpha \in \Lambda\}$. Show that

1. A is finite if Λ is finite and each A_α is finite.
2. A is countable if Λ is countable and each A_α is countable.
3. $A \simeq \Lambda$ if Λ is infinite and each A_α is countable.
4. Suppose $\Lambda \simeq \mathbb{R}$ and $A_\alpha \simeq \mathbb{R}$ for each $\alpha \in \Lambda$. Then whether $A \simeq \mathbb{R}$ or not, justify.

1.11.68 Exercise. Let $A \neq \emptyset$. Then the following are equivalent:

1. $\exists$ a sujection $f : \mathbb{N} \to A$.
2. $\exists$ an injection $g : A \to \mathbb{N}$.
3. A is countable.

1.11.69 Exercise. Let $A \neq \emptyset$ and $n \in \mathbb{N}$, where $\mathbb{N}_n = \{1, 2, ..., n\}$. Then the following are equivalent:

1. $\exists$ a sujection $f : \mathbb{N}_n \to A$.
2. $\exists$ an injection $g : A \to \mathbb{N}_n$.
3. A is finite and at most n elements.

1.11.70 Exercise. Suppose A and B are two sets with $B \subseteq A$. Let $f : A \to B$ be injective. Then show that $\exists$ a bijection $h : A \to B$.

1.11.71 Exercise. Suppose A and B are two sets. Then $\exists$ a bijection $h : A \to B$ iff $\exists$ an injection $g : A \to B$ and a surjection $h : A \to B$.

1.11.72 Exercise. Let Y^X to denote the set of all functions $X \to Y$. Show that:

$$|\mathcal{P}(\mathbb{N})| = |\mathbb{N}^{\mathbb{N}}| = |(\mathbb{Q}^+)^{\mathbb{N}}| = |2^{\mathbb{N}}| = |\mathbb{R}|.$$

1.11.73 Exercise. Let $2^{\mathbb{N}}$ denote the set of all functions from $\mathbb{N}$ to $\{0, 1\}$. Let $\mathcal{B}$ denote the set of all countable subsets of $2^{\mathbb{N}}$. Then show that $|2^{\mathbb{N}}| = |\mathcal{B}|$.

1.11.74 Exercise. Let $A \subseteq \mathbb{R}$ be such that for every $x \in A$, $\exists\, \epsilon > 0$ with $(x, x+\epsilon) \cap A = \emptyset$, then prove that A is countable.

1.11.75 Exercise. Let $\mathcal{C} = 2^{\mathbb{N}}$.

1. Let $f : \mathcal{C} \to [0, 2]$ be the function defined by
$$f(x) = \sum_{n=0}^{\infty} \frac{x_n}{2^n}.$$
Prove that f is surjective and that every element of [0,2] has at most two preimages under f.

2. Find the set D of elements of [0,2] that have two preimages under f. Prove that D and $f^{-1}(D)$ are countably infinite.

3. Construct a bijection from $\mathcal{C} \to [0, 2]$, and a bijection from $\mathcal{C} \to \mathbb{R}$.

1.11.76 Exercise. For any function f there exists a function i which is one-to-one, a function j which is onto, and a function g which is one-to-one and onto, such that $f = i \circ g \circ j$.

1.11.77 Exercise. For the sets X, Y define $Y^X = \{f; f : X \to Y\}$ i.e. the set of all functions from X to Y. Let X be a set and $T = \{0, 1\}$.

1. If $X = \{a, b, c, d\}$, list all the elements of T^X.

2. Assume $X \neq \emptyset$. Is $|X| \leq |Y^X|, \forall\, X, Y$? Find a condition on Y such that $|X| \leq |Y^X|$. Prove that $|X| \leq |Y^X|$ by finding an injection $\phi : X \to Y^X$.

3. Assume that $A \cap B = \emptyset$. Prove that there exists a bijection
$$\phi : X^{A \cup B} \to X^A \times X^B.$$

4. Prove that there exists a bijection $\phi : (X^Y)^Z \to X^{Y \times Z}$.
Note: One can think of $\mathbb{R}^2$ as $\mathbb{R}^{\{1,2\}}$ and $\mathbb{R}^3$ as $\mathbb{R}^{\{1,2,3\}}$ and so on.

5. Assume $Y \neq \emptyset$. Prove that X^Y is uncountable iff X is uncountable or X has at least two elements and Y is infinite.

6. Let $X = \{x_1, x_2, ..., x_k\}$ be a finite set containing k elements, and $f : X \to X$. Let $x_1 \in X$ be fixed and $x_{n+1} = f(x_n)$. Suppose $f(x_{k+1}) = x_1$. Show that $x_2 = x_{k+2}$ and $x_3 = x_{k+3}$. Show that $x_{n+k} = x_n$.

1.11.78 Exercise. (Calkin and Wilf [16]). An explicit enumeration of the positive rational numbers can be produced by concatenating blocks of 2^n numbers ($n \in \mathbb{N}$). Start with $B_0 = \frac{1}{1}$ and continue with the blocks $B_1, B_2, B_3, ...$ following the rule
$$B_n = \left\{ \frac{p}{p+q}, \frac{p+q}{q}; \frac{p}{q} \in B_{n-1} \right\} \text{ for } n \geq 1:$$
$$\frac{1}{1}; \underbrace{\frac{1}{2}\ \frac{2}{1}}; \underbrace{\frac{1}{3}\ \frac{3}{2}\ \frac{2}{3}\ \frac{3}{1}}; \underbrace{\frac{1}{4}\ \frac{4}{3}\ \frac{3}{5}\ \frac{5}{2}\ \frac{2}{5}\ \frac{5}{3}\ \frac{3}{4}\ \frac{4}{1}};$$

Prove that the numerator and denominator of each ratio $p/q \in B_n$ are relatively prime, every reduced positive rational number occurs in some block, and no reduced positive rational number occurs at more than one block.

1.11.79 Exercise. Let $\mathcal{A} = \{A_n; n \in \mathbb{N}, |A_n| \geq 2 \ \forall\ n \in \mathbb{N}\}$. Consider the set

$$T = \left\{ f : \mathbb{N} \to \bigcup_{n=1}^{\infty} A_n; f(n) \in A_n \right\}.$$

Prove that T is uncountable.

1.11.80 Exercise. Show that $\left|\mathbb{R}^{\aleph_0}\right| = 2^{\aleph_0}$.

1.11.81 Exercise. Let A be an infinite set and B be a countable set. Then $|A| = |A \cup B|$ where $|A| = \alpha$ i.e.,$\alpha + \aleph_0 = \alpha,\ \forall\ |\alpha| \geq \aleph_0$.

1.11.82 Exercise. Show that the set of irrational numbers has the cardinality $\mathfrak{c}$ i.e. $|\mathbb{R} \setminus \mathbb{Q}| = |\mathbb{R}|$.

1.11.83 Exercise. Let $A \subseteq \mathbb{R}$ such that A contains an interval. Then prove that $|A| = |\mathbb{R}|$.

1.11.84 Exercise. Show that $|\mathbb{R}| = |\mathbb{R}^+|$.

1.11.85 Exercise. Is it true that $\aleph_0^{\aleph_0} = \aleph_0$?

1.11.86 Exercise. Let $\mathcal{C} = \{A_i; i \in \mathbb{N}\}$, is $|\prod_{i=1}^{\infty} A_i| = |\mathbb{N}|$?

1.11.87 Exercise. Show that $|\mathbb{N} \times \mathbb{R}| = |\mathbb{R}|$.

1.11.88 Exercise. Show that $|\mathbb{R}^{\mathbb{R}}| = |2^{\mathbb{R}}|$ i.e. $\mathfrak{c}^{\mathfrak{c}} = 2^{\mathfrak{c}}$.

1.11.89 Exercise. Show that $\mathfrak{c}^{\aleph_0} = \mathfrak{c}$.

1.11.90 Exercise. Suppose $f : \mathbb{N} \to \mathbb{N}$ is onto, $g : \mathbb{N} \to \mathbb{N}$ is one-to-one and $f(n) \geq g(n)$, for all $n \in \mathbb{N}$. Prove that $f(n) = g(n), n \in \mathbb{N}$.

1.11.91 Exercise. Consider an arbitrary function $f : X \to Y$ and denote the image $f(X)$ by B. Let $\alpha : B \to Y$ denote the inclusion function and define a function $\beta : X \to B$ by $\beta(x) = f(x) \in B$ for every $x \in X$. Show that α is injective and β is surjective, and that $\alpha \circ \beta = f$. Hence every function can be decomposed into the composition of a surjective function and an injective function.

1.11.92 Exercise. By the method of mathematical induction, prove that the subset $F_n = \{1, 2, ..., n\}$ of $\mathbb{N}$ is finite for every $n \in \mathbb{N}$. Establish the following statements:

1. The union $X \cup Y$ of two finite sets X and Y is finite.

2. If X is an infinite set, then there exists an injective function $f : \mathbb{N} \to X$.

1.11.93 Exercise. In $\mathbb{Z}^+$, define $m \prec n$ if n divides m. Show that this is a partial ordering, that every chain has an upper bound, and determine the set of maximal elements.

1.11.94 Exercise. Let F be the set of all real-valued functions of a real variable. Show that by defining $f \prec g$ to mean "$\forall\ x : f(x) \leq g(x)$", $(F, \prec)$ is a partially ordered set. If $f \prec' g$ denotes

$$f = g \text{ or } \lim_{x \to \infty} \frac{f(x)}{g(x)} = 0.$$

is $(F, \prec')$ partially ordered?

Chapter 2

Introduction to the Set of Real Numbers

No one shall expel us out of the paradise which Cantor has created for us.
—David Hilbert

2.1 Introduction

In the 19th century, mathematicians were more concerned with the construction of real numbers from the rationals than finding a system of axioms that would capture the essence of the set $\mathbb{R}$. In the year 1900, a German mathematician David Hilbert (1862–1943) gave a list of axioms that characterize the real numbers. What were Hilbert's axioms for $\mathbb{R}$? The first group of axioms took care of the usual operations on real numbers and their properties. These algebraic properties mean that the set of real numbers $\mathbb{R}$, together with the operations of addition and multiplication, is a field.
The field axioms and order axioms make the set $\mathbb{R}$ an ordered field. However, they are not sufficient to describe $\mathbb{R}$, and Hilbert was forced to include a third group. The problem is that the set $\mathbb{Q}$ (rational numbers) satisfies the same axioms, so it is also an ordered field. Yet, the Monotone Convergence Theorem is not true in $\mathbb{Q}$. What we mean is that if we believed (like the ancient Greek mathematicians did) that rational numbers are the only acceptable numbers, then the Monotone Convergence Theorem would fail. In order to show that the statement is not always true, as we shall see.

2.2 Real numbers: Axiomatic developement

There are several ways to introduce the real number system. We selected one of them assuming that it is easiest and shortest. The real number system is a set $\mathbb{R}$ together with two algebraic operations, denoted by $(+)$ (called sum or addition) and $(\cdot)$ (called product or multiplication), i.e. two mappings $+$ and $\cdot$ from $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$, and an order relation on $\mathbb{R}$) that satisfy the following axioms:

1. **Addition Axioms:**

- **A**(i) $p + q = q + p$ for all $p, q \in \mathbb{R}$. (**Commutative property**)
- **A**(ii) $p + (q + r) = (p + q) + r$ for all $p, q, r \in \mathbb{R}$. (**Associative property**)
- **A**(iii) There exists an element $0 \in \mathbb{R}$ such that $0 + p = p + 0 = p$ for every $p \in \mathbb{R}$. (**Existence of additive identity**)
- **A**(iv) For every $p \in \mathbb{R}$ there exists an element $-p \in \mathbb{R}$ such that $p + (-p) = (-p) + p = 0$. (**Existence of additive inverse**)

2. **Multiplication Axioms:**

- **M**(i) $p \cdot q = q \cdot p$ for all $p, q \in \mathbb{R}$.(**Commutative property**)
- **M**(ii) $p \cdot (q \cdot r) = (p \cdot q) \cdot r$ for all $p, q, r \in \mathbb{R}$.(**Associative property**)
- **M**(iii) There exists an element $1 \in \mathbb{R}$ such that $1 \cdot p = p \cdot 1 = p$ for every $p \in \mathbb{R}$. (**Existence of multiplicative identity**)
- **M**(iv) For every $p \in \mathbb{R} \setminus \{0\}$, there exists an element $p^{-1} \in \mathbb{R}$ such that $p \cdot (p^{-1}) = (p^{-1}) \cdot p = 1$. (**Existence of multiplicative inverse**)

3. **Distributive property of multiplication over addition:**

- $p \cdot (q + r) = (p \cdot q) + (p \cdot r)$ and $(p + q) \cdot r = (p \cdot r) + (q \cdot r)\ \forall\, p, q, r \in \mathbb{R}$.

4. **Order Axioms:** The subset $\mathbb{P} \subset \mathbb{R}$ satisfies the following:

- O(i) $0 \notin \mathbb{P}$,
- O(ii) If $p, q \in \mathbb{P}$, then $p + q$ and $pq \in \mathbb{P}$.
- O(iii) If $p \in \mathbb{R}$, then one and only one of the following statements hold: $p \in \mathbb{P},\ p = 0$ or $-\,p \in \mathbb{P}$. (**Law of trichotomy**)

2.2.1 Remark. The elements of $\mathbb{P}$ are are called *positive* elements and the elements of $-\mathbb{P} = \{-p; p \in \mathbb{P}\}$ are are called *negative* elements. Since $0 = -0$, so 0 cannot be an element of $\mathbb{P}$. There exist several systems satisfying above axioms. But all are algebraically and order isomorphic. We choose one of them and call it the set of real numbers, and denoted it by $\mathbb{R} = (\mathbb{R},\ +,\ \cdot\ ,\ \leq)$. An element in $\mathbb{R}$ is said to be a **real number**.

2.2.2 Note. We write $a > 0$ if $a \in \mathbb{P}$ and $a > b$ if $a - b \in \mathbb{P}$ or $a < b$ if $b - a \in \mathbb{P}$.

2.3 Special subsets of Real Numbers:

The real number system $\mathbb{R}$ contains certain special subsets obtained by beginning with $1 \in \mathbb{R}$ and successively adding 1's to form 2=1+1, 3=2+1, etc. the set of **natural numbers**

$$\mathbb{N} = \{1, 2, ..., n, ...\}.$$

Since $\mathbb{R}$ is the additive group, so considering the additive inverses of the members of $\mathbb{N}$, together with 0, the additive subgroup as the set of **integers**

$$\mathbb{Z} = \{.., -2, -1, 0, 1, 2, ..., n, ...\},$$

(*Zahlen* is German for number); and **non-negative integers**

$$\mathbb{N}_0 = \{0, 1, 2, ..., n, ...\},$$

the set of **rational numbers** (or fractions or quotients)

$$\mathbb{Q} = \{m/n; m, n \in \mathbb{Z}, n \neq 0\},$$

and the set of **irrational numbers**

$$\mathbb{Q}^C = \mathbb{R} \setminus \mathbb{Q}.$$

Equality in $\mathbb{Q}$ is defined by $m/n = p/q$ if and only if $mq = np$. Recall that each of the sets $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$, and $\mathbb{R}$ is a proper subset of the next; i,e.,

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}.$$

2.3.1 Definition. A nonempty subset $X \subseteq \mathbb{R}$ is **inductive** if for every $x \in X$ it holds that the real number $x + 1$ is also in X.

2.3.2 Note. The set of natural numbers, denoted by $\mathbb{N}$, is the smallest inductive set containing the real number 1.

2.4 Bounds of a set

2.4.1 Definition. Let $A \subseteq \mathbb{R}$. We say that A is **bounded above** if there exists $u \in \mathbb{R}$ so that $a \leq u$ for all $a \in A$. The real number u is called an **upper bound** for the set A. If there exists no $u \in \mathbb{R}$ so that $a \leq u$ for all $a \in A$, we say that A is **unbounded above**. We say that A is **bounded below** if there exists $l \in \mathbb{R}$ so that $l \leq a$ for all $a \in A$. The real number l is called a **lower bound** for the set A. If there exists no $l \in \mathbb{R}$ so that $l \leq a$ for all $a \in A$, we say that A is **unbounded below**. If A is simultaneously bounded above and bounded below we say that A is **bounded**.

2.4.2 Definition. Let $A \subseteq \mathbb{R}$. We say that $U \in \mathbb{R}$ is the **least upper bound,** also called the **supremum** for the set A, if the two following conditions hold true:

(a) The number U is an upper bound for the set A.

(b) If u is another upper bound for the set A, then $U \leq u$.

If a supremum U of A exists, we write $U = \sup A$. Analogously, we say that $L \in \mathbb{R}$ is the **greatest lower bound**, also called the **infimum** for the set A, if the two following conditions hold true:

(a) The number L is a lower bound for the set A.

(b) If l is another lower bound for the set A, then $L \geq l$. If an infimum L of A exists, we write $L = \inf A$.

5. **LUB axiom or Completeness axiom:**

- Every nonempty subset which is bounded above, has a supremum, Equivalently,
- For every ordered pair (A, B) of nonempty subsets of $\mathbb{R}$ having the property that $a \leq b$ for every $a \in A$ and $b \in B$ there exists an element $\xi \in \mathbb{R}$ such that $a \leq \xi \leq b$, for every $a \in A$ and $b \in B$.

This is also referred to as the completeness axiom. A set S endowed with two algebraic operations $+$ and $\cdot$ and an order relation $\leq$, satisfying the addition axioms, the product axioms, the distributive axiom, and the order axioms above is called an **ordered field**.

2.5 Extended Real numbers

2.5.1 Definition. We choose two new objects, which we name $+\infty$ (∞) and $-\infty$, and consider the set $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$. The elements of $\overline{\mathbb{R}}$ are called **extended real numbers**.

2.5.2 Remark. [1] These definitions come with a warning: $\pm\infty$ are not real numbers! They are just symbols suggestively standing for a certain type of behavior of a subset of $\mathbb{R}$. This extension is primarily order-theoretic: that is, we may extend the $\leq$ relation to the extended real numbers in the obvious way:

$$\forall\, x \in \mathbb{R}, -\infty < x < +\infty.$$

Conversely much of the point of the extended real numbers is to give the real numbers, as an ordered set, the pleasant properties of a closed, bounded interval $[a, b]$: namely we have a largest and smallest element. The extended real numbers $[-\infty, +\infty]$ is not a field. In fact, we cannot even define the operations of $+$ and $\cdot$ unrestrictedly on them. However, it is useful to define some of these operations:

$$\begin{aligned}
&x + \infty = \infty + x = \infty \text{ if } -\infty < x \leq +\infty\\
&x - \infty = -\infty + x = -\infty \text{ if } -\infty \leq x < +\infty\\
&x.\infty = \infty.x = \infty, \;\; x.(-\infty) = (-\infty).x = -\infty \text{ if } 0 < x \leq +\infty\\
&x.\infty = \infty.x = -\infty, \;\; x.(-\infty) = (-\infty).x = +\infty \text{ if } -\infty \leq x < 0.
\end{aligned}$$

Only $-\infty + \infty, +\infty - \infty, 0.\infty, \infty/0, 0/\infty, \pm\infty/\pm\infty$ cannot be defined.
None of these definitions are really surprising. If we think about it, they correspond to facts we have learned about manipulating infinite limits, e.g. if $\lim_{x\to c} f(x) = \infty$ and $\lim_{x\to c} g(x) = 10$, then $\lim_{x\to c}(f(x) + g(x)) = \infty$. However, certain other operations with the extended real numbers are not defined, for similar reasons. In particular we do not define

$$\infty - \infty,\ 0.\infty,\ \frac{\pm\infty}{\pm\infty}.$$

Why not? Again we might think in terms of associated limits. The above are indeterminate forms: if you that $\lim_{x\to c} f(x) = \infty$ and $\lim_{x\to c} g(x) = -\infty$, then what about $\lim_{x\to c}(f(x) + g(x))$? Answer: nothing, unless you know what specific functions f and g are. As a simple example, suppose

$$f(x) = \frac{1}{x^2} + 1 \text{ and } g(x) = \frac{-1}{x^2},$$

[1] It is advised to omit for the first reading.

then $\lim_{x\to 0}\left(\frac{1}{x^2}+1\right)=\infty$ and $\lim_{x\to 0}\frac{-1}{x^2}=-\infty$, but $\lim_{x\to 0}(f(x)+g(x))=1$,
So $\infty-\infty$ cannot have a universal definition independent of the chosen functions. In a similar way, when evaluating limits $0.\infty$ is an indeterminate form: if $\lim_{x\to a} f(x)=0$ and $\lim_{x\to a} g(x)=\infty$, then $\lim f(x)g(x)$ depends on how fast f approaches zero compared to how fast g approaches infinity. Again, consider something like $f(x)=(x-c)^2, g(x)=2/(x-c)^2$. And similarly for ∞/∞.
However, there are also more purely algebraic reasons: there is no way to define the above expressions in such a way to make the field axioms work out. For instance, let $a\in\mathbb{R}$. Then $a+\infty=\infty$. If therefore we were allowed to substract ∞ from ∞ we would deduce $a=\infty-\infty$, and thus could be any real number: that's not a well-defined operation.
The extended real number system provides a certain case of expressing our ideas in a different fashion. For instance, we can now define the supremum or infimum of any subset of real numbers not necessarily bounded. Thus, if A is a set of real numbers which is not bounded above then $\sup A$ is defined to be equal to ∞ whereas if it is bounded above then its supremum is as defined earlier. Similarly, a set which is not bounded below has its infimum equal to $-\infty$. Another advantage of having extended real number system is that even an empty set of real number has a lub now, viz., $-\infty$. For every real number is an upper bound for elements of $\emptyset$. Therefore the set of upperbounds is unbounded below and hence the smallest one is $-\infty$. Likewise, every real number is a lower bound for $\emptyset$ and hence the largest one is ∞. We can therefore say that every subset of $\mathbb{R}$ has a lub and a glb in $[-\infty,\infty]$.

2.6 Archimedean Principle

2.6.1 Theorem. (**The Archimedean Property**) Let $u>0$. Then, for every $x\in\mathbb{R}$, there is a unique integer n such that

$$nu\le x<(n+1)u.$$

Equivalent formulations are given in the problems.

In particular, when $u=1$, we obtain that for every $x\in\mathbb{R}$, there is a unique integersuch that $n\le x<n+1$. This number is called the integral part of x and is denoted by $[x]$. Then

$$[x]\le x<[x]+1$$

and thus

$$[x]=\max\{n\in\mathbb{Z};n\le x\}\ \forall\, x\in\mathbb{R}.$$

The difference $x-[x]$, called the fractional part of x, is denoted by $\{x\}$.

2.6.2 Lemma. For every real number x and every integer n,

$$[x+n]=[x]+n.$$

2.6.3 Theorem. (Gelfond-Schneider, 1934) If $\alpha\neq 0,1$ is algebraic number and $\beta\in\mathbb{Q}^C$ then α^β is transcendental.

2.7 Arithmetic Modulo One

Let $\mathbb{N}_0 = \{0, 1, 2, ...\}$. Denote the fractional part of a real number a by $\{a\}$. If $a \geq 0$ then $\{a\} = a - [a]$ where $[a] = \max\{n \in \mathbb{N}_0; n \leq a\}$. Alternatively, $\{a\}$ is the unique real such that $0 \leq \{a\} < 1$, and $a = \{a\} + n$ for some integer n. It is immediate that

$$\{a+b\} = \begin{cases} \{a\} + \{b\}, & \text{if } \{a\} + \{b\} < 1 \\ \{a\} + \{b\} - 1, & \text{if } \{a\} + \{b\} \geq 1 \end{cases} \tag{2.1}$$

To see this write $a+b = \{a\}+\{b\}+[a]+[b]$ and use the alternative characterization of the fractional part. Alternatively (2.1) can be written as

$$\{a\} + \{b\} = \begin{cases} \{a+b\}, & \text{if } \{a\} + \{b\} < 1 \\ 1 + \{a+b\}, & \text{if } \{a\} + \{b\} \geq 1 \end{cases} \tag{2.2}$$

Also, if $\{a\} > \{b\}$, then

$$\{a-b\} = \begin{cases} \{a-b\}, & \text{if } a \geq b \\ 1 - \{b-a\}, & \text{if } a < b \end{cases} \tag{2.3}$$

The first follows from

$$a - b = \{a\} - \{b\} + ([a] - [b])$$

and the second from

$$b - a = 1 + \{b\} - \{a\} + ([b] - [a] - 1)$$

and the alternative characterization of the fractional part.

2.8 Problems and Solutions on Chapter 2

Warning! *Never Never Ever read this solution unless you tried problems quite long time and gave up. Doing so may impair your ability to think and solve problems.*

2.8.1 Problem. Prove that $\sqrt{2}$ is irrational.

2.8.1.1 Solution. Supose that $\sqrt{2}$ is rational, that is $\sqrt{2} = \frac{p}{q}$ for some $p, q \in \mathbb{N}$. So $q\sqrt{2} = p$, that is, it is possible to multiply $\sqrt{2}$ by a natural number so that the result is an integer. Let

$$m = \min\{q; q\sqrt{2} \text{ is an integer}, \}$$

then $m\sqrt{2}$ is an integer. Now

$$\begin{aligned} &1 < \sqrt{2} < 2 \\ \Rightarrow &0 < \sqrt{2} - 1 < 1 \\ \Rightarrow &0 < m(\sqrt{2} - 1) < m. \end{aligned}$$

Consider the number $k = m(\sqrt{2}-1) = m\sqrt{2} - m$ which an integer as $m\sqrt{2}, m$ are both integers. Again,

$$\begin{aligned} k\sqrt{2} &= (\sqrt{2}m - m)\sqrt{2} \\ &= 2m - m\sqrt{2}, \text{ and} \\ k - m &= m\sqrt{2} - 2m < 0 \\ &\Rightarrow 0 < k < m. \end{aligned}$$

We see that k is a natural number such that $k\sqrt{2}$ is an integer which is less than m contradicts the minimality of m. So $\sqrt{2}$ is a rational number. □

2.8.2 Problem. Prove that the decimal expansion of x will end in zeros (or in nines) if and only if, x is a rational number whose denominator is of the form 2^n5^m, where m and n are non-negative integers.

2.8.2.1 Solution. "⇐" Suppose that $x = \frac{k}{2^n5^m}$, if $n \geq m$, we have

$$\frac{k5^{n-m}}{2^n5^n} = \frac{k5^{n-m}}{10^n}.$$

So, the decimal expansion of x will end in zeros. Similarly for $m \geq n$.

"⇒" Suppose that the decimal expansion of x will end in zeros (or in nines). For case $x = a_0.a_1a_2......a_n$. Then

$$x = \frac{\sum_{k=0}^{n} 10^{n-k}a_k}{10^n} = \frac{\sum_{k=0}^{n} 10^{n-k}a_k}{2^n5^n}.$$

For case $x = a_0.a_1a_2......a_n99999.....$ Then

$$\begin{aligned} x &= \frac{\sum_{k=0}^{n} 10^{n-k}a_k}{2^n5^n} + \frac{9}{10^{n+1}} + ... + \frac{9}{10^{n+m}} + ... \\ &= \frac{\sum_{k=0}^{n} 10^{n-k}a_k}{2^n5^n} + \frac{9}{10^{n+1}}\sum_{i=0}^{\infty} 10^{-i} \\ &= \frac{\sum_{k=0}^{n} 10^{n-k}a_k}{2^n5^n} + \frac{1}{10^n} \\ &= \frac{1 + \sum_{k=0}^{n} 10^{n-k}a_k}{2^n5^n}. \end{aligned}$$

So, in both cases, we proved that x is a rational number whose denominator is of the form 2^n5^m, where m and n are non-negative integers. □

2.8.3 Problem. Prove that $p = \sqrt{n+1} + \sqrt{n-1}$ is irrational for every integer $n \geq 1$.

2.8.3.1 Solution. Suppose that $p = \sqrt{n+1} + \sqrt{n-1}$ is rational, and then

$$\left(\sqrt{n+1} + \sqrt{n-1}\right)\left(\sqrt{n+1} - \sqrt{n-1}\right) = 2$$

which implies that $\sqrt{n+1} + \sqrt{n-1}$ is rational. Hence, $\sqrt{n+1}$ and $\sqrt{n-1}$ are rational. So, $n-1 = k^2$ and $n+1 = h^2$, where k and h are positive integers. It implies that $h = 3/2$ and $k = 1/2$ which is absurd. So, $\sqrt{n+1} + \sqrt{n-1}$ is irrational for every integer $n \geq 1$. □

2.8.4 Problem. Let x be a positive rational number of the form

$$x = \sum_{k=1}^{n} \frac{a_k}{k!}$$

where each a_k is non-negative integer with $a_k \leq k-1$ for $k \geq 2$ and $a_n > 0$. Let $[x]$ denote the largest integer in x. Prove that $a_1 = [x]$, that $a_k = [k!x] - k[(k-1)!x]$ for $k = 2, ..., n$, and that n is the smallest integer such that $n!x$ is an integer. Conversely, show that every positive rational number x can be expressed in this form in one and only one way.

2.8.4.1 Solution. ($\Rightarrow$) First,

$$\begin{aligned}
[x] &= \left[a_1 + \sum_{k=2}^{n} \frac{a_k}{k!} \right] \\
&= a_1 + \left[\sum_{k=2}^{n} \frac{a_k}{k!} \right], \text{ since } a_1 \in \mathbb{N} \\
&= a_1, \text{ since } \sum_{k=2}^{n} \frac{a_k}{k!} \leq \sum_{k=2}^{n} \frac{k-1}{k!} = \sum_{k=2}^{n} \left(\frac{1}{(k-1)!} - \frac{1}{k!} \right) = 1 - \frac{1}{n!} < 1.
\end{aligned}$$

Second, fixed k and consider

$$\begin{aligned}
k!x &= k! \sum_{i=1}^{n} \frac{a_i}{i!} \\
&= k! \sum_{i=1}^{k-1} \frac{a_i}{i!} + a_k + k! \sum_{i=k+1}^{n} \frac{a_i}{i!} \\
\text{and } \ (k-1)!x &= (k-1)! \sum_{i=1}^{n} \frac{a_i}{i!} \\
&= (k-1)! \sum_{i=1}^{k-1} \frac{a_i}{i!} + (k-1) \sum_{i=k}^{n} \frac{a_i}{i!}. \\
\text{So } \ [k!x] &= \left[k! \sum_{i=1}^{n} \frac{a_i}{i!} \right] \\
&= \left[k! \sum_{i=1}^{k-1} \frac{a_i}{i!} + a_k + k! \sum_{i=k+1}^{n} \frac{a_i}{i!} \right] \\
&= k! \sum_{i=1}^{k-1} \frac{a_i}{i!} + a_k, \text{ since } k! \sum_{i=k+1}^{n} \frac{a_i}{i!} < 1
\end{aligned}$$

and

$$\begin{aligned} k[(k-1)!x] &= k\left[(k-1)!\sum_{i=1}^{k-1}\frac{a_i}{i!} + (k-1)!\sum_{i=k}^{n}\frac{a_i}{i!}\right] \\ &= k(k-1)!\sum_{i=1}^{k-1}\frac{a_i}{i!}, \text{ since } k(k-1)!\sum_{i=k}^{n}\frac{a_i}{i!} < 1 \\ &= k!\sum_{i=1}^{k-1}\frac{a_i}{i!} \end{aligned}$$

which implies that $a_k = [k!x] - k[(k-1)!x]$ for $k = 2, ..., n$.
Last, in order to show that n is the smallest integer such that $n!x$ is an integer. It is clear that

$$n!x = n!\sum_{k=1}^{n}\frac{a_k}{k!} \in \mathbb{Z}.$$

In addition,

$$\begin{aligned} (n-1)!x &= (n-1)!\sum_{k=1}^{n}\frac{a_k}{k!} \\ &= (n-1)!\sum_{k=1}^{n-1}\frac{a_k}{k!} + \frac{a_n}{n} \notin \mathbb{Z}, \text{ since } \frac{a_n}{n} \notin \mathbb{Z}. \end{aligned}$$

($\Leftarrow$) It is clear that, every a_n is uniquely determined. Hence the result follows. $\square$

2.8.5 Problem. Show that, if $0 < |b| < |a|$ then, for all $n \geq 1$,

$$n\left|\frac{b}{a}\right|^{n-1} < \frac{|a|}{|a| - |b|}.$$

2.8.5.1 Solution. Hint: Put $\left|\frac{b}{a}\right| = c$. Then

$$1 + c + c^2 + ... + c^{n-1} = \frac{1 - c^n}{1 - c} < \frac{1}{1 - c}.$$

The left hand side is not greater than n times its smallest term, so not greater than nc^{n-1}. Hence $nc^{n-1} < \frac{1}{1-c}$ implies

$$n\left|\frac{b}{a}\right|^{n-1} < \frac{|a|}{|a| - |b|}. \quad \square$$

2.8.6 Problem. Let a, b be real numbers such that $|a|, |b| > k > 0$. Show that

$$\left|\frac{1}{a} - \frac{1}{b}\right| \leq \frac{|a - b|}{k^2}.$$

2.8.6.1 Solution. Since $|ab| > k^2$, $\left|\frac{1}{a} - \frac{1}{b}\right| < \frac{|a-b|}{k^2}$. $\square$

2.8.7 Problem. For every real number x and every integer n,

$$[x+n] = [x] + n.$$

2.8.7.1 Solution. The formula $[x] \le x < [x] + 1$ implies $[x] + n \le x + n < [x] + n + 1$, from which we conclude that $[x+n] = [x] + n$ since $[x] + n$ and $[x] + n + 1$. are consecutive integers. $\square$

2.8.8 Problem. For $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$, we have,

1. $[x+y] \ge [x] + [y]$.
2. $\left[\frac{[x]}{n}\right] = \left[\frac{x}{n}\right]$.
3. $\sum_{k=0}^{n} \left[x + \frac{k}{n}\right] = [nx]$.

2.8.8.1 Solution.

1. Since $[x] + [y]$ is an integer and satisfies $[x] + [y] \le x + y$, therefore $[x] + [y] \le [x+y]$.
2. We claim that $[x/n] \le [x]/n$. Indeed, if this were not the case we would have $[x/n] < [x]/n \le ([x] + \epsilon)/n$, for some $0 < \epsilon < 1$. Therefore $[x] \le n[x/n] \le [x] + \epsilon$, a contradiction since $n[x/n]$ is an integer. It follows that $[x/n] \le [[x]/n]$. The converse inequality is obvious.
3. Let

$$f(x) = \sum_{k=0}^{n-1} \left[x + \frac{k}{n}\right] - [nx].$$

Then f is periodic with period $1/n$ and vanishes on the interval $[0, 1/n]$. So, $f = 0$ identically. $\square$

2.8.9 Problem. The equality $\{x+n\} = \{x\}$ holds for every real number x and for every integer n.

2.8.9.1 Solution. We infer that

$$\begin{aligned}\{x+n\} &= x + n - [x+n] \\ &= x + n - ([x] + n) = \{x\}. \quad \square\end{aligned}$$

2.8.10 Problem. Consider a sequence (a_n) of positive numbers such that $(a_1 + a_2 + .. + a_n)^2 = a_1^3 + a_2^3 + ... + a_n^3,\ n \ge 1$. Prove that $a_n = n\ \forall\, n \ge 1$.

2.8.10.1 Solution. Use induction. $\square$

2.8.11 Problem.

1. For any two real numbers a, b, prove that

$$\max\{a,b\} = \sup\{a,b\} = \frac{a+b+|a-b|}{2} \text{ and}$$
$$\min\{a,b\} = \inf\{a,b\} = \frac{a+b-|a-b|}{2}.$$

2. If $a, b, c, d \in \mathbb{R}$, then show that

$$\sup\{a+c, b+d\} \leq \sup\{a, b\} + \sup\{c, d\}.$$

2.8.11.1 Solution.

1. For any two points $a, b \in \mathbb{R}$, $\frac{a+b}{2}$ is the midpoint of a and b. So, adding half of their distance $\frac{|a-b|}{2}$ to $\frac{a+b}{2}$, we get

$$\sup\{a, b\} = \frac{a+b+|a-b|}{2}$$

and subtracting $\frac{|a-b|}{2}$ from $\frac{a+b}{2}$, we get

$$\inf\{a, b\} = \frac{a+b-|a-b|}{2}.$$

2.

$$\begin{aligned}
\sup\{a+c, b+d\} &= \frac{a+c+b+d+|a+c-b-d|}{2} \\
&= \frac{a+b+c+d+|a-b+c-d|}{2} \\
&\leq \frac{a+b+c+d+|a-b|+|c-d|}{2} \\
&\leq \frac{a+b+|a-b|}{2} + \frac{c+d+|c-d|}{2} \\
&\leq \sup\{a, b\} + \sup\{c, d\}. \quad \square
\end{aligned}$$

2.8.12 Problem.

1. $|ab| = |a||b|$ and $ab \leq |ab|$ for all $a, b \in \mathbb{R}$.

2. $|a-b| = |b-a|$ for all $a, b \in \mathbb{R}$.

3. $|a|^2 = a^2$ for all $a \in \mathbb{R}$,

4. If $b > 0$, then $|a| \leq b$ iff $-b \leq a \leq b$.

5. $-|a| \leq a \leq |a|$ for all $a \in \mathbb{R}$.

6. $|a+b| \leq |a| + |b|$ for all $a, b \in \mathbb{R}$.

7. If both a and b are non-negative, then $|a-b| < \max\{a, b\} < a+b = |a| + |b|$.

8. If both a and b are non-positive, then $|a-b| \leq \max\{|a|, |b|\} \leq |a| + |b|$.

9. If one of a or b is positive and the other negative, then $|a-b| = |a| + |b|$.

10. $||a| - |b|| \leq |a-b|$ for all $a, b \in \mathbb{R}$.

2.8.12.1 Solution. We prove only (5) and (9). By (2) and (4), we have

$$\begin{aligned} |a+b|^2 &= (a+b)^2 = a^2 + 2ab + b^2 \\ &\le |a|^2 + |a||b| + |b|^2 \\ &\le (|a|+|b|)^2 \\ \Rightarrow |a+b| &\le |a| + |b|. \end{aligned}$$

Again, by (5)

$$\begin{aligned} &|a| = |a-b+b| \le |a-b| + |b|, \text{ and} \\ &|b| = |b-a+a| \le |b-a| + |a| \\ \Rightarrow &|a| - |b| \le |a-b|, \text{ and} \\ &|b| - |a| \le |b-a| = |a-b| \\ &-(|a|-|b|) \le |b-a| = |a-b| \\ \Rightarrow &||a| - |b|| \le |a-b|, \end{aligned}$$

proves (5) and (9). □

2.8.13 Problem. Show that the triangle inequality $|x+y| \le |x| + |y|$ can be stated more sharply for real numbers as

$$\begin{aligned} |x+y| &= |x| + |y|, \qquad \text{if } xy \ge 0 \\ &< \max\{|x|, |y|\}, \text{ if } xy < 0. \end{aligned}$$

2.8.13.1 Solution. Left to the reader. □

2.8.14 Problem. The positive part of an $x \in \mathbb{R}$ is defined by

$$x^+ = \frac{|x|+x}{2} \text{ and the negative part by } x^- = \frac{|x|-x}{2}$$

Prove that

1. $x^+ = \max\{x, 0\}$ and $x^- = \min\{-x, 0\}$.
2. $x = x^+ - x^-$ and $|a| = a^+ + a^-$.
3. $x^+ = \begin{cases} x, & \text{if } x \ge 0 \\ 0, & \text{if } x \le 0 \end{cases}$ and $x^- = \begin{cases} 0, & \text{if } x \ge 0 \\ -x, & \text{if } x \le 0. \end{cases}$
4. $x + y = \max\{x, y\} + \min\{x, y\}$.
5. $\max\{x, y\} + z = \max\{x+z, y+z\}$.
6. $\min\{x, y\} + z = \min\{x+z, y+z\}$.
7. $x \le y$ iff $x^+ \le y^+$ and $x^- \le y^-$.
8. $\max\{x, y\} = -\min\{-x, -y\}$.

9. $\max\{x, y\} = \max\{x - y, 0\} + y = (x - y)^{+} + y = \dfrac{(x + y) + |x - y|}{2}$.

10. $\min\{x, y\} = \min\{x - y, 0\} + y = -(x - y)^{-} + y = \dfrac{(x + y) - |x - y|}{2}$.

2.8.14.1 Solution. Left to the reader. □

2.8.15 Problem. Let $a, b, c \in \mathbb{R}$. Prove that:

1. $|\min\{a, b\} - \min\{a, c\}| \leq |b - c|$;

2. $|\max\{a, b\} - \max\{a, c\}| \leq |b - c|$;

2.8.15.1 Solution.

1. We have, by definition

$$\begin{aligned}
&|\min\{a, b\} - \min\{a, c\}| \\
=&\left|\frac{a+b}{2} - \frac{|a-b|}{2} - \frac{a+c}{2} + \frac{|a-c|}{2}\right| \\
=&\frac{1}{2}\,|b - c - |a - b| + |a - c|| \\
\leq&\frac{1}{2}|b - c| + ||a - c| - |a - b|| \\
\leq&\frac{1}{2}\left(|b - c| + |b - c|\right) = |b - c|.
\end{aligned}$$

2. Left to the reader. □

2.8.16 Problem. Show that $|a + b| = |a| + |b|$ if and only if $ab \geq 0$.

2.8.16.1 Solution. Hint: After squaring, the equation is equivalent to $ab = |ab|$. Note that if a and b are nonzero, then $ab > 0$ means that a and b have the same sign. □

2.8.17 Problem. Prove that $|x| + |y| + |z| \leq |x + y - z| + |y + z - x| + |z + x - y|$ for all $x, y, z \in \mathbb{R}$.

2.8.17.1 Solution. We write

$$\begin{aligned}
2x &= (x - y + z) + (x + y - z) \text{ to get} \\
|2x| &= |(x - y + z) + (x + y - z)| \\
&\leq |x - y + z| + |x + y - z|
\end{aligned}$$

Similarly for $2y$ and $2z$, and then adding the desired result follows. □

2.8.18 Problem. Show that $1/3$ is not a dyadic number. (A rational number is **dyadic** if it is of the form $m/2^n$; where $m \in \mathbb{Z}$ and $n \in \mathbb{N}$.)

2.8.18.1 Solution. The base-2 expansion of $1/3$ is 0.010101....; prove then that a number is dyadic, if and only if, it has a terminating base-2 expansion. □

2.8.18.2 Solution. Assume that $1/3 = k/2^n$ for some $k \in \mathbb{Z}$ and $n \in \mathbb{N}$. This implies $3k = 2^n$, and this is impossible due to the uniqueness of the prime number factorization. □

2.8.19 Problem. Give an example of a subset M of $\mathbb{R}$ such that $M + M$ is not a subset of $2M$.

2.8.19.1 Solution. $M = \{0, 1/2\}, M = \mathbb{Z}$. $\square$

2.8.20 Problem. Show that for a convex subset M of $\mathbb{R}$ we have $M + M \subseteq 2M$. For which subsets M of $\mathbb{R}$ we have $2M \subseteq M+M$? A subset S of $\mathbb{R}$ is **convex** if $\mu x+(1-\mu)y \in S \; \forall \; x, y \in S, 0 \leq \mu \leq 1$.

2.8.20.1 Solution. Let $x, y \in M$, then for $\mu = 1/2$, $\frac{x+y}{2} \in M$ implies $x + y = 2m \in 2M$ for some $m \in M$, which implies that $M+M \subseteq 2M$, and all non-empty subsets of $\mathbb{R}$ satisfy $2M \subseteq M+M$. $\square$

2.8.21 Problem. Let $x, y, a \in \mathbb{R}$.

1. $x < y + \epsilon \; \forall \; \epsilon > 0$ if and only if $x \leq y$.
2. $x > y + \epsilon \; \forall \; \epsilon > 0$ if and only if $x \geq y$.
3. $|a| < \epsilon \; \forall \; \epsilon > 0$ if and only if $a = 0$.

2.8.21.1 Solution. We prove only (1). If possible, let $x < y + \epsilon \; \forall \; \epsilon > 0$ but $x > y$. Now, choose $\epsilon = (x-y)/2$ and observe that $y+\epsilon = (x+y)/2 < x$ (as $y < x$), a contradiction. Thus $x \leq y$. Other parts are similar. $\square$

2.8.22 Problem. A real number U is the **supremum** of a set $A \subseteq \mathbb{R}$ if and only if

1. for every $x \in A, x \leq U$.
2. for every $\epsilon > 0$ there is an element $y \in A$ such that $y > U - \epsilon$.

2.8.22.1 Solution. (1) shows that U is an upper bound of A. Let $\epsilon > 0$, so $U - \epsilon < U$ and $U - \epsilon$ cannot be an upper bound of A. This means $\exists \; y \in A$ such that

$$U - \epsilon < y \leq U.$$

Conversely, let u be any upper bound of A, but $u < U$. Choose $\epsilon < (U-u)/2$, then by the condition

$$\exists \; y > U - \epsilon > U - (U-u)/2 = (U+u)/2 > u \Rightarrow y > u$$

showing that u is not an upper bound, a contradiction. Hence $U \leq u$.

2.8.23 Problem. A real number L is the **infimum** of a set $A \subseteq \mathbb{R}$ if and only if

1. for every $x \in A, L \leq x$.
2. or every $\epsilon > 0$ there is an element $y \in A$ such that $y < U + \epsilon$.

2.8.23.1 Solution. Similar to above.

2.8.24 Problem. The supremum (infimum) of a nonempty bounded set is unique.

2.8.24.1 Solution. Let $A \subseteq \mathbb{R}$ and $\sup A = U_1$ and $\sup A = U_2$. Then either $U_1 > U_2$ or $U_2 > U_1$. Suppose that $U_1 > U_2$, then taking $\epsilon < (U_1 - U_2)/2$ we get an $y \in A$ such that $y > U_1 - \epsilon = U_1 - (U_1 - U_2)/2 = (U_1 + U_2)/2 > U_2$ implies U_2 is not an upper bound of A, a contradiction. Similarly, if $U_2 > U_1$ then U_1 is not an upper bound of A, again a contradiction.Thus $U_1 = U_2$.

2.8.25 Problem. The following are equivalent:

1. Every nonempty subset $A \subseteq \mathbb{R}$ that is bounded above has a supremum.
2. For every ordered pair (A, B) of nonempty subsets of $\mathbb{R}$ having the property that $a \leq b$ for every $a \in A$ and $b \in B$ there exists an element $\xi \in \mathbb{R}$ such that $a \leq \xi \leq b$, for every $a \in A$ and $b \in B$.

2.8.25.1 Solution. (1) $\Rightarrow$ (2) Let $A, B \subseteq \mathbb{R}$ with $A \neq \emptyset, B \neq \emptyset$ and $a \leq b \; \forall \; a \in A, b \in B$. Now, we observe that A is bounded above by every element of B, so by (1) $\sup A$ exists and let $\sup A = \xi$, then $a \leq \xi \; \forall \; a \in A$. We claim $\xi \leq b \; \forall \; b \in B$. Suppose $\exists \; b_0 \in B$ such that $b_0 < \xi$ then $a \leq b_0 < \xi \; \forall \; a \in A$ contradicts that ξ is the least of all such upper bounds. Thus $a \leq \xi \leq b \; \forall \; a \in A, b \in B$.
(2) $\Rightarrow$ (1) Suppose that a nonempty subset $A \subseteq \mathbb{R}$ is bounded above. Let

$$B = \{b \in \mathbb{R}; b \text{ is an upper bound of } A\}.$$

Now, the ordered pair (A, B) of nonempty subsets of $\mathbb{R}$ have the property that $a \leq b \; \forall \; a \in A, b \in B$. Then by (2) there exists $\xi \in \mathbb{R}$ such that $a \leq \xi \leq b$ for every $a \in A$ and $b \in B$. We have to show that $\xi = \sup A$. For, suppose there exists $b_0 \in B$ such that $b_0 < \xi$. Then b_0 is an upper bound of A strictly less then ξ, contradicting assumption $\xi = \sup A$. $\square$

2.8.26 Problem. Let $(A_\lambda)_{\lambda \in M}$ be a non-empty family of non-empty sets of real numbers such that the set $A = \bigcup_\lambda A_\lambda$ is bounded above. Prove that $\sup A = \sup_{\lambda \in M}\{\sup A_\lambda\}$, $\inf A = \inf_{\lambda \in M}\{\inf A_\lambda\}$.

2.8.26.1 Solution. Since A is bounded, so $A_\lambda \subseteq A$ is bounded $\forall \; \lambda \in M$. Let $\sup A = \alpha$ and $\sup A_\lambda = \alpha_\lambda \; \forall \; \lambda \in M$. We show that $\alpha = \sup_{\lambda \in M} \alpha_\lambda$. Since $A_\lambda \subset A \; \forall \; \lambda \in M$, then $\sup A_\lambda \leq \sup A$ i.e. $\alpha_\lambda \leq \alpha \; \forall \; \lambda \in M$. i.e. α is an upper bound of $\{\alpha_\lambda\}_{\lambda \in M}$. Suppose $\epsilon > 0$, then $\exists \; a \in A$ such that $a > \alpha - \epsilon$ and $a \in A_\mu$ for some $\mu \in M \Rightarrow a \leq \sup A_\mu = \alpha_\mu$, hence $\alpha_\mu \geq a > \alpha - \epsilon$. Thus $\alpha = \sup_{\lambda \in M} \alpha_\lambda$. Hence the result. Other part is similar. $\square$

2.8.27 Problem. $1 \in \mathbb{P}$, i.e. $1 > 0$.

2.8.27.1 Solution. Suppose that $1 \notin \mathbb{P}$. Then either $1 = 0$ or $-1 \in \mathbb{P}$. If $1 = 0$, then for all $p \in \mathbb{R}$, $p = p \cdot 1 = p \cdot 0 = 0 \Rightarrow \mathbb{R} = \{0\}$, which is impossible. Again, $-1 \in \mathbb{P}$ and using O(ii), we get for $p \in \mathbb{P}$ implies $(-1).p \in \mathbb{P} \Rightarrow -p \in \mathbb{P}$. Thus $p \in \mathbb{P}$ and $-p \in \mathbb{P} \Rightarrow p + (-p) \in \mathbb{P} \Rightarrow 0 \in \mathbb{P}$. But this contradicts O(i). Hence $1 \in \mathbb{P}$, i.e. $1 > 0$. $\square$

2.8.28 Problem. (P.G.L. Dirichlet) Suppose that α is irrational and $N \in \mathbb{N}$. Then there exist $m, n \in \mathbb{Z}$, such that $1 \leq n \leq N$, and

$$\left|\alpha - \frac{m}{n}\right| \leq \frac{1}{nN}.$$

2.8.28.1 Solution. For a number $\alpha \in \mathbb{R}$, denote $n\alpha - [n\alpha]$ by $(n\alpha)$ the fractional part of $n\alpha$, then the numbers $(\alpha), (2\alpha), .., (N\alpha)$ are irrational, distinct, and all belong to (0,1). Now dividing the interval (0,1) into N equal parts, we get that the numbers belong to the following disjoint intervals

$$\left(0, \frac{1}{N}\right) \bigcup .. \bigcup \left(\frac{N-1}{N}, 1\right).$$

If the interval $(0, 1/N)$ contains the number $(n\alpha)$, then $0 < n\alpha - [n\alpha] < 1/N$, i.e. $0 < \alpha - [n\alpha]/n < 1/(nN)$. If none of the numbers $(\alpha), (2\alpha), ..., (N\alpha)$ is in the interval $(0, 1/N)$, then two of them (say $(h\alpha)$ and $(k\alpha)$, with $h > k$) will belong to the same interval (Pigeonhole Principle). This gives us $|(h\alpha) - (k\alpha)| < 1/N$, that is,

$$|(h-k)\alpha - ([h\alpha] - [k\alpha])| < \frac{1}{N}.$$

Let $n = h - k$ and $m = [h\alpha] - [k\alpha]$. Then $n, m \in \mathbb{Z}, 1 \leq n \leq N$ and

$$|\alpha - m/n| \leq 1/(nN). \quad \square$$

2.8.29 Problem. Suppose that α is irrational, then there are infinitely many rational numbers p/q with $q > 0$ and such that $|\alpha - p/q| < 1/q^2$.

2.8.29.1 Solution. Assume there are finitely many, say, $p_1/q_1, .., p_n/q_n$. Then, by the preceding exercise, there exists p/q such that $|\alpha - p/q| \leq 1/(qN)$ with $q \leq N$ and $1/N < \min\{|\alpha - p_i/q_i|; 1 \leq i \leq n\}$. (The minimum is positive because α is irrational.) $\square$

2.8.30 Problem. Let $S \subseteq \mathbb{R}$ be nonempty. Show that $u \in \mathbb{R}$ is an upper bound of S if and only if the conditions $t \in \mathbb{R}$ and $t > u$ imply that $t \notin S$.

2.8.30.1 Solution. Suppose that $u \in \mathbb{R}$ is an upper bound of S, and the conditions hold and $t \in S$. By definition, $t \leq u$ contradicts $t > u$. Thus $t \notin S$. Conversely, suppose that u is not an upper bound then $\exists\, s \in S$ such that $u < s$ implies $s \notin S$, again a contradiction. $\square$

2.8.31 Problem. (**The Archimedean Property**) Let $u > 0$. Then, for every $x \in \mathbb{R}$, there is a unique integer n such that

$$nu \leq x < (n+1)u.$$

2.8.31.1 Solution. First, we show that there is $q \in \mathbb{Z}$ such that $x < qu$. Assuming the contrary, it follows that the set $A = \{nu; n \in \mathbb{Z}\}$ is bounded above and the Completeness Axiom assures the existence of $z = \sup A$. As z is the least upper bound, there must be an integer number m such that $mu > z - u$, whichmeans $(m+1)u > z$, contradicting the choice of z. By applying the above remark to $-x$, we get the existence of an integer number p such that $pu < x$. Consequently, there exist two integers p and q such that $pu \leq x < qu$. On the other hand, $[pu, qu)$ can be represented as a finite union of pairwise disjoint intervals of length u,

$$[pu, (p+1)u) \cup [(q-1)u, qu),$$

which ensures that x belongs to one (and only one) of the component intervals. $\square$

2.8.32 Problem. Show that in $\mathbb{R}$, $\sup \emptyset = -\infty$ and $\inf \emptyset = \infty$.

2.8.32.1 Solution. A possible argument can be given to justify the statement: The sup and inf are defined for non-empty bounded subsets of $\mathbb{R}$. Since $\emptyset \subseteq S\ \forall\, S \subseteq \mathbb{R}$, thus

$$\begin{aligned} &\emptyset \subseteq \{x\}\ \forall\, x \in \mathbb{R} \\ \Rightarrow &\inf \emptyset \geq \inf\{x\} = x\ \forall\, x \in \mathbb{R} \\ \Rightarrow &\inf \emptyset = \infty \end{aligned}$$

Similarly,

$$\begin{aligned} &\emptyset \subseteq \{x\}\ \forall\, x \in \mathbb{R} \\ \Rightarrow &\sup \emptyset \leq \sup\{x\} = x\ \forall\, x \in \mathbb{R} \\ \Rightarrow &\sup \emptyset = -\infty. \quad \square \end{aligned}$$

2.8.33 Problem.

1. Let A and B be two subsets of $\mathbb{R}$. Prove that $\sup A < \inf B \Rightarrow A \cap B = \emptyset$.

2. Give an example that the converse is not true.

2.8.33.1 Solution.

1. Suppose that $x \in A \cap B$. Then $x \in A \Rightarrow x \leq \sup A$, and $x \in B \Rightarrow x \geq \inf B$. Hence $x \leq \sup A < \inf B \leq x \Rightarrow x < x$, a contradiction.

2. Consider the sets $A = (1, 2), B = (2, 3)$. □

2.8.34 Problem. Let A be a non-empty set of positive real numbers, which is bounded above, and $B = \{x \in \mathbb{R}; 1/x \in A\}$. Prove that B is bounded below $\inf B = 1/\sup A$.

2.8.34.1 Solution. Let $\sup A = \alpha$, then $\alpha \geq a\; \forall\, a \in A$ implies $\frac{1}{a} \geq \frac{1}{\alpha}\; \forall\, a \in A$ which shows that $\frac{1}{\alpha}$ is a lower bound of B.
Now, choose $\epsilon > 0$ then $\exists\, a' \in A$ such that

$$\begin{aligned} a' &> \alpha - \frac{\epsilon\alpha^2}{1+\epsilon\alpha} = \frac{\alpha}{1+\epsilon\alpha} \\ \Rightarrow \frac{1}{a'} &< \frac{1+\epsilon\alpha}{\alpha} = \frac{1}{\alpha} + \epsilon \\ \Rightarrow \inf B &= \frac{1}{\alpha} = \frac{1}{\sup A}. \quad \square \end{aligned}$$

2.8.35 Problem (Archimedean Principle)**.** For every two real numbers x and y such that $x > 0$, there exists a natural number n such that $nx > y$.

2.8.35.1 Solution. Under the above-mentioned assumptions define

$$A = \{u \in \mathbb{R}; \exists\, n \in \mathbb{N}, nx > u\}$$

and note that $A \neq \emptyset$ (because at least $x \in A$). We show that $A = \mathbb{R}$. Suppose that $A \neq \mathbb{R}$ and denote $B = \mathbb{R} \setminus A$. Obviously, $B \neq \emptyset$. Note that for every $u \in A$ and $v \in B, u < v$. Indeed, for every $u \in A$ there exists a natural n such that $nx > u$. Because $v \in B$ and the real number set is a totally ordered set, it follows that $nx \leq v$. Then $u < nx \leq v \Rightarrow u < v$. Part (2) implies that for the ordered pair (A, B) there exists a real number z such that

$$u \leq z \leq v\; \forall\, u \in A, v \in B. \tag{2.4}$$

The real number $z - x \in A$, because otherwise $z - x \in B$, and then by (2.1)

$$z \leq z - x \Rightarrow x \leq 0,$$

contradicting the hypothesis. Therefore $z - x \in A$. Then we can find a natural number n such that $z - x < nx$. We also have

$$z + x = (z - x) + 2x < (n + 2)x,$$

and it follows that $z + x \in A$. Then $z + x \leq z$, thus $x \leq 0$. The contradiction shows that $A = \mathbb{R}$ and the theorem is proved.

2.8.35.2 Solution. Suppose $nx \le y \; \forall \; n \in \mathbb{N}$, i.e. y is an upper bound for $A = \{nx; n \in \mathbb{N}\}$. Then A is bounded above, so by the Completeness Axiom A has a supremum. Let $s = \sup(A)$. Then $s - x$ is not an upper bound for A since $x > 0$, so there is an $n \in \mathbb{N}$ with $nx > s - x$. But then $(n+1)x \in A$ and $(n+1)x = nx + x > (s - x) + x = s$, contradicting the fact that s is an upper bound for A. Thus the assumption that $nx \le y$ for all $n \in \mathbb{N}$ is false.

We now prove some equivalent conditions on $\mathbb{R}$.

2.8.36 Problem. Let $\mathbb{R}$ be the set of real numbers. Then the following conditions are equivalent:

1. For every two real numbers x and y such that $x > 0$ there exists a natural number n such that $nx > y$.
2. $\mathbb{N}$ is unbounded above in $\mathbb{R}$.
3. If $x \in \mathbb{R}$, then there exists an $n_x \in \mathbb{N}$ such that $x \le n_x$.
4. If $x \in \mathbb{R}$, then there exist $m_x, n_x \in \mathbb{Z}$ such that
$$m_x \le x \le n_x.$$
5. $\inf\{1/n; n \in \mathbb{N}\} = 0$.
6. $\mathbb{Z}$ is neither bounded above nor bounded below.
7. If $x \in \mathbb{R}$, there exists $n_x \in \mathbb{N}$ such that $0 < 1/n_x < |x|$.
8. If $x \in \mathbb{R}$, there exists $n_x \in \mathbb{N}$ such that $n_x \le x < n_x + 1$. The integer n_x is called the **integral part** of x, denoted by $[x]$.
9. For every $x, y \in \mathbb{R}, x < y$, there exists a rational number r, such that
$$x < r < y.$$

2.8.36.1 Solution. We prove some of the implications, other implications are left to the reader to complete the cycle.

1. $(1) \Rightarrow (2)$: Suppose that $\mathbb{N}$ is bounded above, so $\exists \; B \in \mathbb{R}$ such that
$$n \le B \; \forall \; n \in \mathbb{N}.$$
Now by (1), $\exists \; m \in \mathbb{N}$ such that $m.1 > B \Rightarrow m > B$, which shows that B cannot be an upper bound of $\mathbb{N}$. Thus $\mathbb{N}$ is unbounded above in $\mathbb{R}$.
2. $(1) \Rightarrow (3)$: If possible, let no such n_x exist. So, $n \le x \; \forall \; n \in \mathbb{N}$, i.e. $\mathbb{N}$ is bounded above by x, so $\mathbb{N}$ has a sup. in $\mathbb{R}$. Let $\sup \mathbb{N} = G$, then $G - 1$ is not an upper bound of $\mathbb{N}$ hence $\exists \; K \in \mathbb{N}$ such that $K > G - 1 \Rightarrow G < K + 1$, which is a contradiction that G is an upper bound of $\mathbb{N}$. Thus $\exists \; n_x \in \mathbb{N}$ such that $x \le n_x$.
3. $(3) \Rightarrow (4)$: Now $x \in \mathbb{R} \Rightarrow -x \in \mathbb{R}$, so by (3) $\exists \; n_x$ and $n_{-x} \in \mathbb{N}$ such that
$$x \le n_x \text{ and } -x \le n_{-x}.$$
Then $m_x = -n_{-x} \in \mathbb{Z}$ and $x \ge m_x$. Thus for $x \in \mathbb{R}$ there exist $m_x, n_x \in \mathbb{Z}$ such that
$$m_x \le x \le n_x.$$

4. (4) $\Rightarrow$ (5) : Let $A = \{1/n; n \in \mathbb{N}\}$. Since $1/n > 0 \ \forall \ n \in \mathbb{N}$, so $\inf A \geq 0$. If possible, let $\inf A = a > 0$. Then by (4) $\exists \ n_1 \in \mathbb{N}$ such that $1/a < n_1$, then $a > 1/n_1 > 1/(1+n_1)$, as $1/(1+n_1) \in A$ and $1/(1+n_1) < a$ so, a cannot be infimum, a contradiction. Thus $\inf A = 0$.

5. (5) $\Rightarrow$ (2) : Suppose that $\inf\{1/n; n \in \mathbb{N}\} = 0$ and $\mathbb{N}$ is bounded above by $M \in \mathbb{R}$. Let $\epsilon = 1/M$ then $\exists \ N \in \mathbb{N}$ such that $1/N < \epsilon = 1/M$ implies $N > M$ shows that M is not an upper bound of $\mathbb{N}$. Alternatively, let $\sup \mathbb{N} = A$, then $\inf\{\frac{1}{n}; n \in \mathbb{N}\} = \frac{1}{\sup \mathbb{N}} = \frac{1}{A} \neq 0$.

6. (1) $\Rightarrow$ (7) : Suppose that $x \neq 0$ then $|x| > 0$, now, for the reals 1 and $|x|$, by (1) $\exists \ n_x \in \mathbb{N}$ such that $n_x.|x| > 1 \Rightarrow |x| > 1/n_x$.

7. (1) $\Rightarrow$ (8) : By (1) there exists an $m \in \mathbb{N}$ such that $m.1 > x \Rightarrow m > x$. Suppose first that $x \geq 0$. Then the set $\{n \in \mathbb{N}; n > x\}$ is nonempty; let m_x be the smallest element of this set, and let $n_x = m_x - 1$. Then $n_x \leq x < n_x + 1$. If $x < 0$, there is an $r_x \in \mathbb{N}$ such that $r_x \leq -x < r_x + 1$, so $-(r_x+1) < x \leq -r_x$. If $x = -r_x$, then let $n_x = -r_x$; otherwise let $n_x = -(r_x + 1)$.
The n_x is unique since, if $n_x \leq x < n_x+1$ and $p_x \leq x < p_x+1$, we have $-(p_x+1) < -x \leq -p_x$, so
$$\begin{aligned} & n_x - (p_x+1) < 0 < (n_x+1) - p_x \\ \Rightarrow & -1 < p_x - n_x < 1, \\ \Rightarrow & \ p_x - n_x = 0 \\ \Rightarrow & \ p_x = n_x. \end{aligned}$$

8. (1) $\Rightarrow$ (9) : Since $y - x > 0$, so by (1) $\exists \ n \in \mathbb{N}$ such that $n(y-x) > 1$ i.e. $ny - nx > 1$ which implies that $\exists \ m \in \mathbb{N}$ such that
$$\begin{aligned} & m \leq nx < m+1 \\ \Rightarrow & \frac{m}{n} \leq x < \frac{m+1}{n} \\ \Rightarrow & \ x < \frac{m}{n} + \frac{1}{n} < x + (y-x) = y. \end{aligned}$$
Let $r = \frac{m+1}{n}$. Thus $x < r < y$. $\square$

2.8.37 Problem. [2] Let $\mathbb{R}$ be the set of real numbers, then prove that the following are equivalent:

1. **Least Upper Bound Axiom**: Each non-empty subset of $\mathbb{R}$ that is bounded above has a supremum.

2. **Nested intervals Theorem (Cantor)**: If $([a_n, b_n])$ is a nested sequence of closed and bounded intervals, such that $\inf_n(b_n - a_n) = 0$. then there exists a unique point ξ such that $\xi \in \bigcap_{n=1}^{\infty}[a_n, b_n]$, i.e. $\bigcap_{n=1}^{\infty}[a_n, b_n] = \{\xi\}$.

3. **Dedekind's Theorem**: Let $A, B \subseteq \mathbb{R}$ with $A \cup B = \mathbb{R}, A \cap B \neq \emptyset, A \neq \emptyset, B \neq \emptyset$ and $a < b \ \forall \ a \in A, b \in B$. Then $\exists! \ \xi \in \mathbb{R}$ such that $a \leq \xi \leq b \ \forall \ a \in A, b \in B$.

2.8.37.1 Solution.

[2] It is advised to omit for the first reading.

1. $(1) \Rightarrow (2)$:
Let $S = \{a_n; n \in \mathbb{N}\}$. Then $S \neq \emptyset$. So by LUB axiom, let $\xi = \sup S$. Then $a_n \leq \xi \ \forall\ n \in \mathbb{N}$. Since S bounded above by $b_n \ \forall\ n \in \mathbb{N}$ hence $\sup S \leq b_n \ \forall\ n \in \mathbb{N}$, thus $a_n \leq \xi \leq b_n \ \forall\ n \in \mathbb{N}$. We show that ξ is unique. Suppose $\eta (\neq \xi) \in F$ satisfying $a_n \leq \eta \leq b_n \ \forall\ n \in \mathbb{N}$ and $\xi > \eta$. Then
$$\begin{aligned} & a_n \leq \eta < \xi \leq b_n \ \forall\ n \in \mathbb{N} \\ \Rightarrow\ & \xi - \eta \leq b_n - a_n \ \forall\ n \in \mathbb{N} \\ \Rightarrow\ & \xi - \eta \leq \inf_n (b_n - a_n) = 0 \\ \Rightarrow\ & \xi = \eta. \end{aligned}$$
Thus $\xi \in \mathbb{R}$ is unique.

2. $(2) \Rightarrow (1)$:
Let $S (\neq \emptyset) \subseteq \mathbb{R}$ be bouded above. We show that $\sup S$ exists in $\mathbb{R}$. Let $b_0 \in \mathbb{R}$ is an upper bound of S. Then $a_0 \leq b_0$ for some $a_0 \in S$. then either $\frac{a_0+b_0}{2}$ is an upper bound of S or $\exists\ a' \in S$ such that $a' > \frac{a_0+b_0}{2}$. If $\frac{a_0+b_0}{2}$ is an upper bound of S then we write $a_1 = a_0$ and $b_1 = \frac{a_0+b_0}{2}$. If $\frac{a_0+b_0}{2}$ is not an upper bound of S then we write $a_1 = a'$ and $b_1 = b_0$. In both cases, $[a_1, b_1] \subseteq [a_0, b_0]$, so $b_1 - a_1 \leq \frac{b_0 - a_0}{2}$. Continuing in the same way, we obtain a nested sequence of intervals $([a_n, b_n])$ where $a_n \in S$ and b_n is an upper bound of S. Also, $b_n - a_n \leq \frac{b_0 - a_0}{2^n}$, and $\inf_n \{b_n - a_n\} = 0$. So, by (2) there a unique point $\xi \in \mathbb{R}$ such that $\xi \in \bigcap_{n=1}^{\infty} [a_n, b_n]$, i.e. $a_n \leq \xi \leq b_n \ \forall\ n \in \mathbb{N}$. Now we claim ξ is the lub of S. Let $x \in S$, then either $x < a_{n_0}$ for some $n_0 \in \mathbb{N}$ or $x \geq a_n \ \forall\ n \in \mathbb{N}$, hence $x < \xi$ or $x = \xi$. Thus $x \leq \xi \ \forall\ x \in S$ which shows that ξ is an upper bound of S. Let $\epsilon > 0$. Then $\exists\ k \in \mathbb{N}$ such that $b_{n_k} - a_{n_k} < \epsilon$, and since $\xi \leq b_n \ \forall\ n \in \mathbb{N}$, so
$$a_{n_k} > b_{n_k} - \epsilon \geq \xi - \epsilon,$$
shows that ξ is the lub of S.

3. $(2) \Rightarrow (3)$:
Let $A, B \subseteq \mathbb{R}$ with $A \cup B = \mathbb{R}, A \cap B \neq \emptyset, A \neq \emptyset, B \neq \emptyset$ and $a < b \ \forall\ a \in A, b \in B$. Then, we show that $\exists!\ \xi \in \mathbb{R}$ such that $a \leq \xi \leq b \ \forall\ a \in A, b \in B$. Let $a_0 \in A$ and $b_0 \in B$ then $\frac{a_0+b_0}{2} \in \mathbb{R} = A \cup B$. If $\frac{a_0+b_0}{2} \in A$, we write $\frac{a_0+b_0}{2} = a_1$ and $b_0 = b_1$. If $\frac{a_0+b_0}{2} \in B$, we write $a_0 = a_1$, $\frac{a_0+b_0}{2} = b_1$. Then
$$a_0 \leq a_1 \leq b_1 \leq b_0.$$
Considering $a_1 \in A$ and $b_1 \in B$ to obtain as before $a_2 \in A$ and $b_2 \in B$ such that
$$a_1 \leq a_2 \leq b_2 \leq b_1.$$
So, continuing the same proceedure, we obtain a nested sequence of intervals $([a_n, b_n])$ where $a_n \in A$ and $b_n \in B \ \forall\ n \in \mathbb{N}$, and $\inf_n \{b_n - a_n\} = 0$. Then by (2), $\exists$ a unique point $\xi \in F$ such that $a_n \leq \xi \leq b_n \ \forall\ n \in \mathbb{N}$. We show that $a \leq \xi \leq b \ \forall\ a \in A$ and $b \in B$. It is true that $a_0 \leq \xi \leq b_0$. Let $a \in A$, then either $a \leq a_0$ or $a > a_0$. If $a \leq a_0$ then $a \leq a_0 \leq \xi$. If $a > a_0$ then either $a < a_n$ for some $n \in \mathbb{N}$ or $a \geq a_n \ \forall\ n \in \mathbb{N} \Rightarrow a \in [a_n, b_n] \ \forall\ n \in \mathbb{N}$. Since $\bigcap_{n=1}^{\infty} [a_n, b_n] = \{\xi\}$, so $a = \xi$, i.e. $a \leq \xi$. Similarly, we can show that $\xi \leq b$. Hence $\exists!\ \xi \in \mathbb{R}$ such that $a \leq \xi \leq b \ \forall\ a \in A, b \in B$.

4. (3) $\Rightarrow$ (2) :
Suppose that, $([a_n, b_n])$ is a nested sequence of closed and bounded intervals in $\mathbb{R}$. Let $A = \{x \in \mathbb{R}; x < a_n \text{ for some } n \in \mathbb{N}\}$ and $b \in \mathbb{R} \setminus A$. Thus, we get
(i) $A \cup B = \mathbb{R}, A \cap B = \emptyset$.
(ii) $A \neq \emptyset$ and $B \neq \emptyset$ for $b_n \in B \; \forall \; n \in \mathbb{N}$.
(iii) Let $a \in A$ and $b \in B$. Then $a < a_k$ for some $k \in \mathbb{N}$ and by definition of $B, a_n \leq b \; \forall \; n \in \mathbb{N}$ so $a < b$. Thus by Dedekind's Theorem $\exists! \; \xi \in F$ such that $a \leq \xi \leq b \; \forall \; a \in A, b \in B$. As $a_n \in A$ and $b_n \in B \; \forall \; n \in \mathbb{N}$, so, we get $a_n \leq \xi \leq b_n \; \forall \; n \in \mathbb{N} \Rightarrow \xi \in \bigcap_{n=1}^{\infty}[a_n, b_n]$.

5. (3) $\Rightarrow$ (1) :
Let $S \subseteq \mathbb{R}$ and $S \neq \emptyset$ be bounded above. Let $A = \{x \in \mathbb{R}; x < s \text{ for some } s \in S\}$ and $B = \mathbb{R} \setminus A$. Note that,
(i) $A, B \subseteq \mathbb{R}$ with $A \cup B = \mathbb{R}, A \cap B = \emptyset$.
(ii) as $S \neq \emptyset$ so $\exists \; s \in S$ and so all elements $x \in A$ for which $x < s$, thus $A \neq \emptyset$, and for S is bounded above so $\exists \; M \in \mathbb{R}$ such that $s \leq M \; \forall \; s \in S$ implies $M \in B \Rightarrow B \neq \emptyset$.
(iii) Let $a \in A$ and $b \in B$. Then, by definition of $A, a < s$ for some $s \in S$ and by definition of $B, s \leq b \; \forall \; s \in S$ implies $a < b$, hence by Dedekind's Theorem $\exists \; \xi \in \mathbb{R}$ such that $a \leq \xi \leq b \; \forall \; a \in A, b \in B$. Now $\xi \in A \cup B \Rightarrow \xi \in A$ or $\xi \in B$. If $\xi \in A$ then there exists some $s \in S$ such that $\xi < s$. We select an element $x \in \mathbb{R}$ such that $\xi < x < s$. Then $x \in A$ and it contradicts the fact that $a \leq \xi \; \forall \; a \in A$. Thus $\xi \in B$.
We claim $\xi = \sup S$. Since $a \leq \xi \; \forall \; a \in A$, so ξ is an upper bound of S, if $y \in \mathbb{R}$ is any upper bound of S, then $s \leq y \; \forall \; s \in S \Rightarrow y \in B$ and so $\xi \leq y$, thus ξ is the least of such upper bounds. Thus each non-empty subset of F that is bounded above has a supremum.

6. (1) $\Rightarrow$ (3) :
Let $A, B \subseteq \mathbb{R}$ with $A \cup B = \mathbb{R}, A \cap B \neq \emptyset, A \neq \emptyset, B \neq \emptyset$ and $a < b \; \forall \; a \in A, b \in B$. Now, we observe that A is bounded above by every element of B, so by (1) $\sup A$ exists and let $\sup A = \xi$, then $a \leq \xi \; \forall \; a \in A$. We claim $\xi \leq b \; \forall \; b \in B$. Suppose $\exists \; b_0 \in B$ such that $b_0 < \xi$ then $a \leq b_0 \; \forall \; a \in A$ contradicts that ξ is the least of all such upper bounds. Thus $a \leq \xi \leq b \; \forall \; a \in A, b \in B$. □

2.8.38 Problem. [3] Show that the Least Upper Bound Property (every set bounded above has a least upper bound) implies the Cauchy Completeness Property (every Cauchy sequence has a limit) of the real numbers.

2.8.38.1 Solution. Let (x_n) be a Cauchy sequence of real numbers. We first show that (x_n) is bounded. Let $\epsilon > 0$ and let $N \in \mathbb{N}$ be such that $|x_n - x_m| < \epsilon$ for $n, m \geq N$. Let $R = \max\{|x_N - x_1|, ..., |x_{N-1} - x_N|\}$. Then the entire sequence (x_n) is contained in $B(x_n; 2R)$. Thus (x_n) is bounded.
By the least upper bound property, we can define

$$z_n = \sup\{x_k; k \geq n\}$$

for each $n \geq 1$. Clearly (z_n) is decreasing and bounded since (x_n) is bounded. The least upper bound property implies the greatest lower bound property, thus we can define $x = \inf\{z_n; n \geq 1\} = \lim_{n \to \infty} z_n$. We show that $x_n \to x$. First, we exhibit a subsequence of (x_n) which converges to x.

[3] It is advised to omit for the first reading.

Let k be a positive integer. Since x is the limit of the z_n, there exists N_k such that for all $n \geq N_k$,

$$|z_n - x| < \frac{1}{2k},$$

Since $z_{N_k} = \sup\{x_i; i > N_k\}$ so $\exists\, n_k > N_k$ such that

$$|z_{N_k} - x_{n_k}| < \frac{1}{2k}$$

Thus we obtain infinitely many distinct n_k ; re-index the distinct n_k to obtain the subsequence (x_{n_j}). By construction,

$$|x_{n_j} - x| \leq |x_{n_j} - z_{N_j}| + |z_{N_j} - x| < \frac{1}{2j} + \frac{1}{2j} = \frac{1}{j}.$$

Thus (x_{n_j}) converges to x.
Now since (x_n) is a Cauchy sequence, it follows that the whole sequence converges to x. Let $\epsilon > 0$. There exists $N \in \mathbb{N}$ such that for all $k > N$, $|x_{n_k} - x| < \epsilon/2$. There also exists N' such that if $p, q > N'$ then $|x_p - x_q| < \epsilon/2$. Hence for all $n \geq \max\{n_N, N'\}$

$$|x_n - x| \leq |x_n - x_{n_N}| + |x_{n_N} - x| < \epsilon/2 + \epsilon/2 = \epsilon$$

Thus (x_n) converges to x. $\square$

2.8.39 Problem. A, B be non-empty subsets of $\mathbb{R}$ with the property that $x \leq y$ whenever $x \in A$ and $y \in B$. Prove that $\exists\, c \in \mathbb{R}$ (not necessarily unique) such that $x \leq c \leq y$ whenever $x \in A$ and $y \in B$.

2.8.39.1 Solution. Use Dedekind's theorem above. $\square$

2.8.40 Problem. Let F be an ordered field. Greatest Lower Bound (GLB) Axiom: Let S be a nonempty subset of F which is bounded below. Then S admits a greatest lower bound, or infimum.

1. Then F satisfies (LUB) iff it satisfies (GLB).
2. In particular $\mathbb{R}$ satisfies both (LUB) and (GLB) and is (up to isomorphism) the only ordered field with this property.

2.8.40.1 Solution.

1. (LUB)$\Rightarrow$(GLB): Let $S \subseteq F$ be nonempty and bounded below by m. Consider

$$-S = \{-x; x \in S\}.$$

Then S is nonempty and bounded above by $-m$. By (LUB), it has a least upper bound $\sup(S)$. We claim that in fact $-\sup(-S)$ is a greatest lower bound for S, or more symbolically: $\inf S = -\sup(-S)$.
(GLB)$\Rightarrow$(LUB): Let S be nonempty and bounded above by M. Consider

$$U(S) = \{x \in F; x \text{ is an upper bound for } S\}.$$

Then $U(S)$ is nonempty: indeed $M \in U(S)$. Also $U(S)$ is bounded below: indeed any $s \in S$ (there is at least one such s, since $S \neq \emptyset$) is a lower bound for $U(S)$. By (GLB) axiom $U(S)$ has a greatest lower bound $\inf U(S)$. We claim that in fact $\inf U(S)$ is a least upper bound for S, i.e. $\sup S = \inf U(S)$.

2. Left to the reader.

2.8.40.2 Solution. (LUB)$\Rightarrow$(GLB): Let an ordered field satisfy the least upper bound property. Let A be a non-empty set and bounded below. Let B be the set of lower bounds of A. Then B is non-empty. Any member of A is an upper bound of B. Thus B is bounded above. So by the least upper bound property, B has lub$B = x_0$. We claim that $x_0 \leq a \;\forall\; a \in A$. If possible, let $x_0 > a$ for some $a \in A$, then lub$B = x_0 > a$ an upper bound of B, which is a contradiction. Thus x_0 is a lower bound of A. Next claim is that x_0 is the glb of A. Let $a \in A$ and $a > x_0$, if $\exists$ no element $a' \in A$ such that $a' < a$, then a is a lower bound of A and $a \in B$ implies $a < x_0$, which is a contradiction. Hence x_0 is the GLB of A. Similarly, (GLB)$\Rightarrow$(LUB). $\square$

2.8.41 Problem. Every nonempty subset $A \subseteq \mathbb{R}$ that is bounded below has infimum.

2.8.41.1 Solution. Denote by A_0 the set of lower bounds of A. Because A is bounded below, $A_0 \neq \emptyset$. Notice that the ordered system (A_0, A) has the property that for every $a \in A_0$ and $b \in A$ implies $a \leq b$. From Dedekind's Theorem (2.8.37) it follows there exists a real number ξ such that $a \leq \xi \leq b$, for every $a \in A_0$ and $b \in A$. It results that number ξ is the greatest element in A_0, that is, an infimum of A.

2.8.42 Problem. Let A and B be non-empty bounded set of positive real numbers, and define C by $C = \{ab; a \in A, b \in B\}$. Prove that C is bounded and that $\sup C = \sup A. \sup B$ and $\inf C = \inf A. \inf B$. Prove also that if AB is the set of real numbers of the form ab where $a \in A, b \in B$, then $\sup AB$ need not exist, and if it does exist it is not necessarily equal to $\sup A. \sup B$.

2.8.42.1 Solution. Let $\sup A = \alpha, \sup B = \beta$, then $\alpha \geq a \;\forall\; a \in A$ and $\beta \geq b \;\forall\; b \in B$.
So $\alpha\beta \geq ab \;\forall\; a \in A$ and $\forall\; b \in B$. Thus $\alpha\beta$ is an upper bound of C. Choose $\epsilon > 0$ then $\exists\; a' \in A$ such that $a' \geq \alpha - \frac{\epsilon}{2\beta}$ and $\exists\; b' \in B$ such that $b' \geq \beta - \frac{\epsilon}{2a'}$. Hence $a'\beta \geq \alpha\beta - \frac{\epsilon}{2}$ and $a'b' \geq a'\beta - \frac{\epsilon}{2}$. Thus $a'b' \geq \alpha\beta - \epsilon$. Which shows that $\alpha\beta = \sup C$.
Let $A = (-\infty, -3)$ and $B = (-\infty, 5)$ then $\sup A = -3$ and $\sup B = 5$, but $AB = (-15, \infty)$, so $\sup AB$ does not exist, and if $A = (-3, -5), B = (-4, -2)$ then $\sup A = -5$ and $\sup B = -2$ but $AB = (10, 12)$ so $\sup AB \neq \sup A. \sup B$. $\square$

2.8.43 Problem. Let $a, b > 0$, and $a^2 = nb^2, n > 1$ then

$$0 < a - b < nb \text{ and } \frac{a}{b} = \frac{nb - a}{a - b}.$$

2.8.43.1 Solution. From the given condition, we have

$$a^2 = nb^2 > b^2 \Rightarrow a > b.$$
$$\text{Now, } a^2 = nb^2 < (nb)^2 \Rightarrow a < nb$$
$$\text{and } a < nb < (n+1)b \Rightarrow a - b < nb.$$

Therefore, $0 < a - b < nb$. Again,

$$\begin{aligned} &a^2 = nb^2 \\ \Rightarrow &a^2 - ab = nb^2 - ab \\ \Rightarrow &a(a - b) = b(nb - a) \\ \Rightarrow &\frac{a}{b} = \frac{nb - a}{a - b}. \quad \square \end{aligned}$$

2.8.44 Problem. Let $a, b > 0$, and $a^2 = 2b^2$, then

$$0 < a - b < b \text{ and } \frac{a}{b} = \frac{2b - a}{a - b}.$$

2.8.44.1 Solution. Hint: Same as above.

2.8.45 Problem. Prove that if A is the set of rational numbers x such that $x \leq c$, where c is any fixed real number, then $\sup A = c$. Prove the same result for the set of irrational numbers x such that $x \leq c$.

2.8.45.1 Solution. By definition, A is bounded above by c, to prove that c is $\sup A$, let $\epsilon > 0$ then $\exists\, r \in \mathbb{Q}$ such that $c - \epsilon < r < c$ by definition of A, $r \in A$. Hence $\sup A = c$. □

2.8.46 Problem. Let $\emptyset \neq H \subseteq \mathbb{R}$, and $p \in \mathbb{R}$. Define $H_1 = (-\infty, p) \cap H$ and $H_2 = (p, \infty) \cap H$. And for each $n \in \mathbb{N}$, define

$$A_n = H \cap (n, n+1]; B_n = H \cap [-(n+1), -n)$$

and

$$C_n = H \cap \left[p - \frac{1}{n}, p - \frac{1}{n+1}\right); D_n = H \cap \left(p + \frac{1}{n+1}, p + \frac{1}{n}\right]$$

1. Prove that H has no upper(lower) bound iff given any $N \in \mathbb{N}, \exists\, n_0 > N + 1$ such that $A_{n_0} \neq \emptyset$ ($B_{n_0} \neq \emptyset$).
2. Suppose that H has no upper bound. Let i_1 be the smallest positive integer such that $A_{i_1} \neq \emptyset$. Now define a subsequence (A_{i_n}) of (A_n) by Induction. If (A_{i_k}) has been defined, and let i_{k+1} is the smallest integer greater than $1 + i_k$ such that $A_{i_{k+1}} \neq \emptyset$. Finally $\forall\, n \in \mathbb{N}$, define $K_n = A_{i_n}$. Prove that the sequence (K_n) has the following property
   (a) K_n is a non-empty subset of H $\forall\, n \in \mathbb{N}$.
   (b) If $x \in K_n$ and $y \in K_{n+1}$, then $x + 1 < y$.
   (c) Suppose that H has no lower bound. Prove that there is a sequence K_n of non-empty subsets of H such that $x \in K_n$ and $y \in K_{n+1}$, then $x - 1 > y$.
   (d) Prove that p is a cluster point of $H_1[H_2]$ iff there exists a subsequence (C_{i_n}) of $(C_n)[(D_{i_n})$ of $(D_n)]$ such that $(C_{i_n}) \neq \emptyset$ $[D_{i_n} \neq \emptyset]$ for all $n \in \mathbb{N}$.

2.8.46.1 Solution.

1. Suppose H has no upper bound, so for any integer $N, \exists\, h \in H$ such that $h > N + 2$ *i.e.* $H \cap (N + 2, \infty) \neq \emptyset$ and

$$\begin{aligned}
&H \cap (N+2, \infty) = H \cap [(N+2, N+3] \cup (N+3, N+4] \cup ...] \neq \emptyset \\
\Rightarrow &\bigcup_{p=2}^{\infty} H \cap (N+p, N+p+1] \neq \emptyset \\
\Rightarrow &H \cap (N+p, N+p+1] \neq \emptyset \text{ for some } p \in \mathbb{N} \\
\Rightarrow &A_{N+p} \neq \emptyset \text{ for } p \in \mathbb{N} \\
\Rightarrow &A_{n_0} \neq \emptyset \text{ for } n_0 = N + p > N + 1.
\end{aligned}$$

Other part is similar.

2. By the above, $A_{i_n} \neq \emptyset \ \forall \ n \in \mathbb{N}$
and $i_1 < 1 + i_1 < i_2 < 1 + i_2 < i_3 .... < i_n < 1 + i_n < i_{n+1} < ....$ Since $K_n = A_{i_n} \neq \emptyset \ \forall \ n \in \mathbb{N}$, suppose $x \in K_n = A_{i_n}$ and $y \in K_{n+1} = A_{i_{n+1}}$, then $A_{i_n} \cap A_{i_{n+1}} = \emptyset$ and $1 + i_n < i_{n+1} \Rightarrow \sup A_{i_{n+1}} - \sup A_{i_n} > 1 \Rightarrow x < \sup A_{i_n}, y > \inf A_{i_{n+1}} \Rightarrow y - x \geq \inf A_{i_{n+1}} - \sup A_{i_n} > 1 \Rightarrow y > x + 1$. Other part is left to the reader. □

2.8.47 Problem. Find $\bigcup_{n=1}^{\infty} I_n$ and $\bigcap_{n=1}^{\infty} I_n$, where

1. $I_n = \left\{x \in \mathbb{R}; -\frac{1}{n} < x < \frac{1}{n}\right\}$
2. $I_n = \left\{x \in \mathbb{R}; -1 + \frac{1}{n} \leq x \leq 2 - \frac{1}{n}\right\}$

2.8.47.1 Solution.

1. We observe that $I_{n+1} \subseteq I_n, \forall \ n \in \mathbb{N}$.
For

$$\begin{aligned} & x \in I_{n+1} \\ \Rightarrow & -\frac{1}{n+1} < x < \frac{1}{n+1} \\ \Rightarrow & -\frac{1}{n} < -\frac{1}{n+1} < x < \frac{1}{n+1} < \frac{1}{n} \\ \Rightarrow & \ x \in I_n. \end{aligned}$$

Hence

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq .... \supseteq I_n \supseteq I_{n+1} \supseteq ...,$$

and cosequently $\bigcup_{n=1}^{\infty} I_n \subseteq I_1 = (-1, 1)$.
On the other hand, let $x \in (-1, 1)$, since $x + 1 > 0$ and $1 - x > 0$, by Archimedean Principle $\exists \ p, q \in \mathbb{N}$ such that $p(x + 1) > 1$ and $q(1 - x) > 1$

$$\begin{aligned} & \Rightarrow \ -1 + \frac{1}{p} < x < 1 - \frac{1}{q} \\ & \Rightarrow \ -1 + \frac{1}{m} \leq -1 + \frac{1}{p} < x < 1 - \frac{1}{q} \leq 1 - \frac{1}{m} \text{ where } m = \max\{p, q\} \\ & \Rightarrow x \in I_m \subseteq \bigcup_{n=1}^{\infty} I_n \Rightarrow (-1, 1) \subseteq \bigcup_{n=1}^{\infty} I_n. \end{aligned}$$

Hence $\bigcup_{n=1}^{\infty} I_n = (-1, 1)$.
Again, let $x \in \bigcap_{n=1}^{\infty} I_n$. If $x > 0$, then by Archimedian principle, $\exists \ m \in \mathbb{N}$ such that $\frac{1}{m} < x \Rightarrow x \notin \left(-\frac{1}{m}, \frac{1}{m}\right) = I_m \Rightarrow x \notin \bigcap_{n=1}^{\infty} I_n$. Similarly, if $x < 0$ then also $x \notin \bigcap_{n=1}^{\infty} I_n$. Hence $x = 0$. Thus $\bigcap_{n=1}^{\infty} I_n = \{0\}$.

2. Here we see that $I_n \subseteq I_{n+1} \ \forall \ n \in \mathbb{N}$. For

$$\begin{aligned} & x \in I_n \\ \Rightarrow & -1 + \frac{1}{n} \leq x \leq 2 - \frac{1}{n} \\ \Rightarrow & -1 + \frac{1}{n+1} \leq -1 + \frac{1}{n} \leq x < 2 - \frac{1}{n} \leq 2 - \frac{1}{n+1} \\ \Rightarrow & \ x \in I_{n+1}. \end{aligned}$$

Hence

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq .... \subseteq I_n \subseteq I_{n+1} \subseteq ...,$$

and $\bigcap_{n=1}^{\infty} I_n = [0,1]$. We show that $(-1,2) = \bigcup_{n=1}^{\infty} I_n$. Let

$$\begin{aligned} & x \in \bigcup_{n=1}^{\infty} I_n \\ & \Rightarrow x \in I_m \text{ for some } m \in \mathbb{N} \\ & \Rightarrow -1 + \frac{1}{m} \leq x \leq 2 - \frac{1}{m} \\ & \Rightarrow -1 < -1 + \frac{1}{m} \leq x \leq 2 - \frac{1}{m} < 2 \\ & \Rightarrow x \in (-1,2). \end{aligned}$$

On the other hand, let $x \in (-1,2)$, since $x+1>0$ and $2-x>0$, by Archimedean Principle $\exists\, p,q \in \mathbb{N}$ such that $p(x+1) > 1$ and $q(2-x) > 1$

$$\begin{aligned} & \Rightarrow\ -1 + \frac{1}{p} < x < 2 - \frac{1}{q} \\ & \Rightarrow\ -1 + \frac{1}{m} \leq -1 + \frac{1}{p} < x < 2 - \frac{1}{q} \leq 2 - \frac{1}{m} \text{ where } m = \max\{p,q\} \\ & \Rightarrow x \in I_m \subseteq \bigcup_{n=1}^{\infty} I_n. \end{aligned}$$

Thus $\bigcup_{n=1}^{\infty} I_n = (-1,2)$. □

2.8.48 Problem. Show that, for $n \neq 10^k, k \in \mathbb{N}$, $\log_{10} n$ is not a rational number.

2.8.48.1 Solution. Let $\log_{10} n = \frac{p}{q}$ where $(p,q)=1$. Then

$$\begin{aligned} & \log_{10} n = \frac{p}{q} \\ & \Rightarrow 10^{\frac{p}{q}} = n = 2^r 5^s (2k+1) \text{ for some } r,s \in \mathbb{N} \\ & \Rightarrow 10^p = [2^r 5^s (2k+1)]^q, \text{ where } 2k+1 \text{ is not a multiple of 5} \\ & \Rightarrow 2^{p-rq} 5^{p-sq} = (2k+1)^q \\ & \Rightarrow \text{an even integer} = \text{an odd integer not multiple of 5 (a contradiction).} \end{aligned}$$

2.8.49 Problem. Let S be a bounded subset of $\mathbb{R}$ and $\sup S = b, \inf S = a$ with $a \neq b$. Show that $[a,b]$ is the smallest closed interval containing the set S.

2.8.49.1 Solution.

Case 1. $b = \sup S \in S$ and $a = \inf S \in S$
Case 2. $b = \sup S \in S$ and $a = \inf S \notin S$
Case 3. $b = \sup S \notin S$ and $a = \inf S \in S$
Case 4. $b = \sup S \notin S$ and $a = \inf S \notin S$
We shall do only the case 2. (other cases can be done in a similar way.) i.e. $b = \sup S \in S$ and

$a = \inf S \notin S$. We show that $a \notin S \Rightarrow$ a is a limit point of S. Since a is $\inf S$, so $\forall\, \epsilon > 0\ \exists\, c \in S$ such that $c < a + \epsilon$. Thus $S \cap (a - \epsilon, a + \epsilon) \neq \emptyset \Rightarrow$ a is a limit point of S. Again, b may or may not be a limit point of S. Consequently, S is a subset of the closed interval $[a, b]$. Next suppose that T be a closed set containing S. We claim $S \subseteq [a, b] \subseteq T$. Now $b \in S \subseteq T \Rightarrow b \in T$. But $S \subseteq T \Rightarrow S' \subseteq T' \subseteq T \Rightarrow a \in S' \subseteq T \Rightarrow a \in T$. Hence $[a, b] \subseteq T$. $\square$

2.8.50 Problem. If $a, b \in \mathbb{R}$ and $0 \leq a - b < \epsilon$, $\forall\, \epsilon > 0$, then prove that $a = b$.

2.8.50.1 Solution. We always have $a \geq b$. If possible, let $a > b$, and choose $\epsilon = \frac{1}{2}(a - b)$, then $a - b < \frac{1}{2}(a - b) \Rightarrow a < b$, a contradiction. Thus $a = b$. $\square$

2.8.51 Problem. If $a \in \mathbb{R}$ and $0 \leq a < \frac{1}{n}\ \forall\, n \in \mathbb{N}$, then prove that $a = 0$.

2.8.51.1 Solution. If $a > 0$ then by Archimedean Principle, $\exists\, m \in \mathbb{N}$ such that $ma > 1 \Rightarrow a > \frac{1}{m} \Rightarrow a$ is not less than $\frac{1}{m}$, a contradiction. Hence $a = 0$. $\square$

2.8.52 Problem. Let S be a bounded subset of $\mathbb{R}$ and $\emptyset \neq T \subseteq S$. Prove that

$$\inf S \leq \inf T \leq \sup T \leq \sup S.$$

2.8.52.1 Solution. Since $\inf S$ is a lower bound of S, so $\inf S \leq s\ \forall\, s \in S \Rightarrow \inf S \leq t\ \forall\, t \in T \subseteq S \Rightarrow \inf S$ is a lower bound of $T \Rightarrow \inf S \leq \inf T$. Again $\inf T \leq t \leq \sup T\ \forall\, t \in T \Rightarrow \inf T \leq \sup T$. Furthermore $\sup S \geq s\ \forall\, s \in S \Rightarrow \sup S \geq t\ \forall\, t \in T \subset S$. Hence $\sup S$ is an upper bound for $T \Rightarrow \sup S \geq \sup T$. $\square$

2.8.53 Problem. Let S and T be bounded subsets of $\mathbb{R}$ and $S + T = \{x + y; x \in S, y \in T\}$. Prove that $\sup(S + T) = \sup S + \sup T$ and $\inf(S + T) = \inf S + \inf T$.

2.8.53.1 Solution. Let $\sup S = \alpha$ and $\sup T = \beta$. Then $\alpha + \beta \geq x + y\ \forall\, x \in S, \forall\, y \in T$. Hence $\alpha + \beta$ is an upper bound for $S + T$. Again, let $\epsilon > 0$, then $\exists\, x_1 \in S$ and $y_1 \in T$ such that $x_1 > \alpha - \frac{1}{2}\epsilon$ and $y_1 > \beta - \frac{1}{2}\epsilon$, so that $x_1 + y_1 > \alpha + \beta - \epsilon$. Hence $\sup(S + T) = \alpha + \beta = \sup S + \sup T$.
Let $\inf S = \gamma$ and $\inf T = \delta$. Then $\gamma + \delta \leq x + y\ \forall\, x \in S, \forall\, y \in T$. Hence $\alpha + \beta$ is a lower bound for $S + T$. Again, let $\epsilon > 0$, then $\exists\, x_2 \in S$ and $y_2 \in T$ such that $x_2 < \gamma + \frac{1}{2}\epsilon$ and $y_2 < \delta + \frac{1}{2}\epsilon$, so that $x_2 + y_2 < \gamma + \delta + \epsilon$. Hence $\inf(S + T) = \gamma + \delta = \inf S + \inf T$. $\square$

2.8.54 Problem. Let S be a non-empty subset of $\mathbb{R}$, bounded below and $T = \{-x; x \in S\}$. Prove that T is bounded above and $\sup T = -\inf S$.

2.8.54.1 Solution. Let $\inf S = \gamma$, then $\gamma \leq x\ \forall\, x \in S \Rightarrow -\gamma \geq -x\ \forall\, x \in S$ implies $-\gamma$ is an upper bound for T and for $\epsilon > 0, \exists\, x_2 \in S$ such that $x_2 < \gamma + \epsilon \Rightarrow -x_2 > -\gamma - \epsilon$. Hence $-\gamma$ is the supremum of T i.e, $\sup T = -\inf S$. $\square$

2.8.55 Problem. Let S be a bounded subset of $\mathbb{R}$ and with $\sup S = M$ and $\inf S = m$. Prove that the set $T = \{x - y; x, y \in S\}$ is a bounded set and $\sup T = M - m, \inf T = m - M$.

2.8.55.1 Solution. We observe that $T = S + (-S)$, hence by the previous problems $\sup T = \sup S + \sup(-S) = \sup S - \inf S = M - m$ and $\inf T = \inf S + \inf(-S) = \inf S - \sup S = m - M$. $\square$

2.8.55.2 Solution. We get by the problem, $M \geq x\ \forall\, x \in S$ and $-m \geq -y\ \forall\ \in S \Rightarrow M - m \geq x - y\ \ \forall x - y \in T \Rightarrow M - m$ is an upper bound of T. Let $\epsilon > 0$, then $\exists\, s, t \in S$ such that $s > M - \frac{1}{2}\epsilon$ and $t < m + \frac{1}{2}\epsilon$. This implies $s - t > M - m - \epsilon$ which shows that $M - m$ is the supremum of T. The other part can be shown in the same way. $\square$

2.8.56 Problem. Let S be a non-empty subset of $\mathbb{R}$, bounded below. A lower bound l of S is such that for each $n \in \mathbb{N}\ \exists\ x_n \in S$ satisfying $x_n < l + \frac{1}{n}$. Prove that $l = \inf S$.

2.8.56.1 Solution. Suppose that l is a lower bound of S then $l \leq \inf S$. Now if $l < \inf S$, then by Archimedean Principle, $\exists\ m \in \mathbb{N}$ such that $m(\inf S - l) > 1 \Rightarrow \inf S > l + \frac{1}{m}$ and $\exists\ x_m$ such that $x_m < l + \frac{1}{m}$. Hence

$$\inf S > l + \frac{1}{m} > x_m \in S,$$

a contradiction that infimum cannot be greater than an element of a set. □

2.8.57 Problem. Let S and T be bounded subsets of $\mathbb{R}$. Prove that

$$\sup(S \cup T) = \max\{\sup S, \sup T\}, \inf(S \cup T) = \min\{\inf S,\ \inf T\}.$$

2.8.57.1 Solution. Since $S \subseteq S \cup T \Rightarrow \sup S \leq \sup(S \cup T)$ and $T \subseteq S \cup T \Rightarrow \sup T \leq \sup(S \cup T)$, hence $\max\{\sup S, \sup T\} \leq \sup(S \cup T)$.
Let $\max\{\sup S, \sup T\} = \sup S$. Then $\sup S \geq \sup T \Rightarrow \sup S \geq s\ \forall\ s \in S \cup T \Rightarrow \sup S$ is an upper bound of $S \cup T \Rightarrow \sup S \geq \sup(S \cup T) \Rightarrow \max\{\sup S, \sup T\} \geq \{\sup S \cup T\}$. Thus $\sup(S \cup T) = \max\{\sup S, \sup T\}$. The other part can be shown in the same way. □

2.8.58 Problem. Let $I_n = [a_n, b_n]$ and $I_1 \supseteq I_2 \supseteq I_3 \supseteq ...$, If

$$\alpha = \sup\{a_n; n \in \mathbb{N}\} \text{ and } \beta = \inf\{b_n; n \in \mathbb{N}\},$$

prove that

1. $[\alpha, \beta] = \bigcap_{n=1}^{\infty} I_n$ if $\alpha \neq \beta$
2. $\{\alpha\} = \bigcap_{n=1}^{\infty} I_n$ if $\alpha = \beta$.

2.8.58.1 Solution.

1. Let $x \in [\alpha, \beta]$, then $a_n < x < b_n\ \forall\ n \in \mathbb{N}$ implies $x \in \bigcap_{n=1}^{\infty} I_n$. Again, let $y \in \bigcap_{n=1}^{\infty} I_n$ and either $y < \alpha$ or $y > \beta$. If $y < \alpha$ and $\sup a_n = \alpha$ then $\exists\ m \in \mathbb{N}$ such that $a_m > y$ this implies $y \notin \bigcap_{n=1}^{\infty} I_n$, a contradiction. If $y > \beta$ and $\inf b_n = \beta$, then $\exists\ p \in \mathbb{N}$ such that $b_p < y$ then also $y \notin \bigcap_{n=1}^{\infty} I_n$, again a contradiction. Hence $y \in [\alpha, \beta]$.
2. Let $\alpha = \beta$. Suppose that $x \neq \alpha$ and $x \in \bigcap_{n=1}^{\infty} I_n$. Then, if $x > \alpha\ \exists\ k \in \mathbb{N}$ such that $b_k < x$ which shows that $x \notin [a_k, b_k]$ i.e. $x \notin \bigcap_{n=1}^{\infty} I_n$, a contradiction. Similarly, if $x < \alpha$, we also arrive at a contradiction. Thus $\{\alpha\} = \bigcap_{n=1}^{\infty} I_n$. □

2.8.59 Problem. Let a, b, c, d be rational numbers and x an irrational number such that $cx + d \neq 0$. Prove that $(ax + b)/(cx + d)$ is irrational if and only if $ad \neq bc$.

2.8.59.1 Solution. Suppose that $(ax+b)/(cx+d) = p/q$, where $p, q \in \mathbb{Z}$. Then $(aq-cp)x = dp-bq$, and so we must have $dp - bq = aq - cp = 0$, since x is irrational. It follows that $ad = bc$. Conversely, if $ad = bc$ then $(ax + b)/(cx + d) = b/d \in \mathbb{Q}$. □

2.8.60 Problem. Let $0 < x < 1$. Then x has a terminating decimal expansion if and only if there exist nonnegative integers m and n such that $2^m 5^n x$ is an integer.

2.8.60.1 Solution. If x has a terminating decimal expansion, then $x = p/10^k = p/(2^k 5^k)$. Conversely, if $2^m 5^n x = N \in \mathbb{N}$ for some $m \leq n$, then $x = 2^{n-m} N/10^n$.

2.8.61 Problem. (SHIGA MEDICAL UNIVERSITY) Given $a^2 \geq b$, where a, b are natural numbers, prove that the necessary and sufficient condition for $\sqrt{a+\sqrt{b}} + \sqrt{a-\sqrt{b}}$ to be a natural number is that there exists a natural number n such that

$$n^2 < a \leq 2n^2 \text{ and } b = 4n^2(a-n^2).$$

2.8.61.1 Solution. Suppose $m = \sqrt{a+\sqrt{b}} + \sqrt{a-\sqrt{b}}$ is a natural number where $a^2 \geq b$ and a, b are natural numbers. Squaring both sides, we have

$$\begin{aligned} m^2 &= 2a + 2\sqrt{a^2-b} \\ \Rightarrow m^2 - 2a &= 2\sqrt{a^2-b} \qquad (1) \\ \text{Squaring both sides, we have } (m^2-2a)^2 &= 4(a^2-b) \\ \Rightarrow m^4 &= 4(am^2-b). \qquad (2) \end{aligned}$$

Since the right side is an even number, $m = 2n$ for some natural number n. Then (2) becomes $4n^4 = 4an^2 - b$, hence

$$b = 4n^2(a-n^2). \qquad (3)$$

(3) and $b \geq 1$ imply $a - n^2 > 0$, therefore $n^2 < a$. From (1) $(2n)^2 - 2a \geq 0$, therefore $a \leq 2n^2$. Together we have

$$n^2 < a \leq 2n^2 \text{ and } b = 4n^2(a-n^2).$$

Converse part is left to the reader. □

2.8.62 Problem. Show that if $\cos a + \sin a$ is rational for some $a \in \mathbb{R}$, then for any $n \in \mathbb{N}$, $\cos^n a + \sin^n a$ is rational.

2.8.62.1 Solution. We observe that $\cos a + \sin a$ is rational implies $\cos a . \sin a$ is rational, for $\cos a . \sin a = \frac{1}{2}\left(1 - (\cos a + \sin a)^2\right)$ Taking $\cos a = p,\ \sin a = q$, we let $p^n + q^n = s_n$ and $p + q = h,\ pq = k$, so s_1, s_2 are rational, thus p, q are the roots of $x^2 - hx + k = 0$. Hence, we get

$$\begin{aligned} &x^3 - hx^2 + kx = 0 \Rightarrow s_3 - hs_2 + ks_1 = 0 \\ &x^4 - hx^3 + kx^2 = 0 \Rightarrow s_4 - hs_3 + ks_2 = 0 \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ &x^n - hx^{n-1} + kx^{n-2} = 0 \Rightarrow s_n - hs_{n-1} + ks_{n-2} = 0 \end{aligned}$$

Using the above equations, we can find s_n and s_n is rational. □

2.8.62.2 Solution. Since the number $\cos a + \sin a$ is rational, its square must be rational. Thus $1 + 2\cos a \sin a = (\cos a + \sin a)^2$ is rational. This shows that both the sum and the product of $\cos a$ and $\sin a$ are rational and the fact that $\cos^n a + \sin^n a$ is rational can be proved inductively using the formula

$$\begin{aligned} &\cos^{n+1} a + \sin^{n+1} a \\ &= (\cos a + \sin a)(\cos^n a + \sin^n a) - \cos a \sin a (cos^{n-1} a + \sin^{n-1} a). \quad \square \end{aligned}$$

2.8.63 Problem. Let $1 \leq \alpha < \beta$ be real numbers. Prove that $\exists\ m, n \in \mathbb{N}, m, n > 1$ such that $\alpha < \sqrt[n]{m} < \beta$.

2.8.63.1 Solution. Let $c = \beta - \alpha$. Then for $p \in \mathbb{N}$, we get

$$\beta^p - \alpha^p = (\alpha + c)^p - \alpha^p = \binom{p}{1}\alpha^{p-1}c + ... + c^p > p\alpha^{p-1}c > pc, \text{ for } \alpha > 1.$$

Now, take an integer $n > \frac{1}{c}$. Then $\beta^n - \alpha^n > 1$. Since the difference between two real numbers is greater than 1, so $\exists$ an integer $m > 1$ such that $\alpha^n < m < \beta^n$ i.e., $\alpha < \sqrt[n]{m} < \beta$ as desired. $\square$

2.8.64 Problem. Prove that $\cos 1^\circ$ is irrational.

2.8.64.1 Solution. Assume that $\cos 1^\circ$ is a rational number. Consider the complex number $z = \cos 1^\circ + i \sin 1^\circ$. Since $z + 1/z = 2\cos 1^\circ$ is rational, as in the introduction we conclude that $z^{45} + 1/z^{45}$ is rational. But $z^{45} + 1/z^{45} = 2\cos 45^\circ = \sqrt{2}$, a contradiction. This shows that $\cos 1^\circ$ is irrational. $\square$

2.8.65 Problem. Can a rational or an irrational number raised to a rational or an irrational power be rational or irrational? All the eight possibilities are as follows:

1. $(\text{irrational})^{\text{irrational}} = \text{rational}$
   Yes. Observe that,
   $$\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \left(\sqrt{2}\right)^{\sqrt{2}\sqrt{2}} = \left(\sqrt{2}\right)^2 = 2.$$
   We note that $\sqrt{2}^{\sqrt{2}}$ is irrational and it was actually proved to be transcendental by Kuzmin in 1930.
2. $(\text{irrational})^{\text{irrational}} = \text{irrational}$; Yes; $\sqrt{2}^{\sqrt{2}}$ is irrational, as above.
3. $(\text{irrational})^{\text{rational}} = \text{rational}$; Yes; $\left(\sqrt{2}\right)^2 = 2$.
4. $(\text{irrational})^{\text{rational}} = \text{irrational}$; Yes; $\left(\sqrt{2}\right)^1 = \sqrt{2}$.
5. $(\text{rational})^{\text{irrational}} = \text{irrational}$; Yes; $2^{\sqrt{2}}$ is irrational,
6. $(\text{rational})^{\text{irrational}} = \text{rational}$; Yes; $2^{\log_2 3} = 3$.
7. $(\text{rational})^{\text{rational}} = \text{rational}$; Yes; $2^3 = 8$.
8. $(\text{rational})^{\text{rational}} = \text{irrational}$; Yes; $2^{\frac{1}{2}} = \sqrt{2}$.

2.8.66 Problem. Let F be the family of all real-valued functions on the real line $\mathbb{R}$. Then the cardinality $|F|$ of the set F is larger than $\mathfrak{c}$.

2.8.66.1 Solution. The mapping $\phi : \mathbb{R} \to F$, where for $a \in \mathbb{R}$, $\phi(a)$ denotes the constant function a, is one-to-one, showing that the cardinality of F is larger than or equal to $\mathfrak{c}$. To show that the cardinality of F is indeed larger than $\mathfrak{c}$, assume that a mapping $a \mapsto f_a$ is one-to-one and maps $\mathbb{R}$ onto F. We use the Cantor diagonal procedure to derive a contradiction. Indeed, define the element $f \in F$ by $f(x) = f_x(x) + 1$, for $x \in \mathbb{R}$. We show that for no $a \in \mathbb{R}$, $f = f_a$. For it, assume that, on the contrary, $f = f_a$ for some $a \in \mathbb{R}$. Compare the values of f and f_a at the point a: we have $f(a) = f_a(a) + 1 = f(a) + 1$, a contradiction. $\square$

2.8.67 Problem. The supremum s of a set $A \subseteq (0,1)$ is less than 1, and A has the property that if $x, y \in A$ and $x < y$, then $\frac{x}{y} \in A$. Prove that $s \in A$, and s is an isolated point of A.

2.8.67.1 Solution. Suppose $s = \sup A \notin A$. So $s < 1 \Rightarrow s^2 < s$. Choose $\epsilon > 0$ such that $0 < \epsilon < s - s^2$, then $\exists\, x \in A$ such that $s^2 < s - \epsilon < x < s$ and $\exists\, y \in A$ such that $s^2 < x < y < s$. Thus, by the condition, $x/y \in A$, so $x/y < s \Rightarrow x < ys$ and $ys < s^2 \Rightarrow x < ys < s^2$, which contradicts $s^2 < x$. Hence $(s - \epsilon, s) \cap A = \emptyset$, which shows that s is not a limit point of A. Thus the only member of A which is greater than $s - \epsilon$ is the $\sup A = s$. $\square$

2.8.68 Problem. Solve the equation $\cos^n x - \sin^n x = 1$, where n is a given positive integer.

2.8.68.1 Solution. For $n \geq 2$ we have

$$\begin{aligned} 1 &= \cos^n x - \sin^n x \leq |\cos^n x - \sin^n x| \\ &\leq |\cos^n x| + |\sin^n x| \leq \cos^2 x + \sin^2 x = 1. \end{aligned}$$

Hence $\sin^2 x = |\sin^n x|$ and $\cos^2 x = |\cos^n x|$, from which it follows that $\sin x, \cos x \in \{1, 0, -1\}$ implies $x \in \pi\mathbb{Z}/2$. By inspection one obtains the set of solutions $\{m\pi; m \in \mathbb{Z}\}$ for even n and $\{2m\pi, 2m\pi - \pi/2; m \in \mathbb{Z}\}$ for odd n. For $n = 1$ we have $1 = \cos x - \sin x = -\sqrt{2}\sin(x - \pi/4)$, which yields the set of solutions $\{2m\pi, 2m\pi - \pi/2; m \in \mathbb{Z}\}$. $\square$

2.8.69 Problem. Let n and k be positive integers such that $1 \leq n, k \leq N + 1$. Show that

$$\min_{n \neq k} |\sin n - \sin k| < 2/N.$$

2.8.69.1 Solution. Since $\sin 1, \sin 2, ..., \sin(N + 1) \in (-1, 1)$, two of these $N + 1$ numbers have distance less than $2/N$. Therefore $|\sin n - \sin k| < 2/N$ for some integers $1 \leq k, n \leq N+1, n \neq k$. $\square$

2.8.70 Problem. If p and q are integers satisfying $p^3 + pq^2 + q^3 = 0$, then prove that p is even iff q is even. Deduce that both p and q are even. Hence show that if x is a real number for which $x^3 + x + 1 = 0$, then x is irrational.

2.8.70.1 Solution. Let p be even, i.e. $p = 2m$ (say), then $(2m)^3 + 2mq^2 + q^3 = 0 \Rightarrow q^3 = 2(-mq^2 - 4m^3)$= even integer implies q is even. Similarly q is even implies p is even. Consider the cases

1. p is odd, q is even; then in the expression $p^3 = -(p + q)q^2, p^3$ is even and $-(p + q)q^2$ odd;
2. p is odd, q is odd, then in the expression $p^3 = -(p + q)q^2, p^3$ is odd and $-(p + q)q^2$ is even;
3. p is even, q is odd, then in the expression $p^3 = -(p + q)q^2, p^3$, p^3 is even and $-(p + q)q^2$ odd;

Hence p and q are both even.
Again, let $x = p/q$, where p and q have no common factors, be a root of the equation $x^3 + x + 1 = 0$. Then $(p/q)^3 + (p/q) + 1 = 0 \Rightarrow p^3 + pq^2 + q^3 = 0$, so by the above, p and q are both even i.e. p and q have a common factor 2, which is a contradiction. Hence, if the equation $x^3 + x + 1 = 0$, has a real root, then it must be irrational. $\square$

2.8.71 Problem. Prove that for $0 < x < \pi/4, (\sin x)^{\sin x} < (\cos x)^{\cos x}$.

2.8.71.1 Solution. We use the relation

$$\log(\lambda a + (1-\lambda)b) > \lambda \log a + (1-\lambda)\log b.$$

Let $a = \sin x, b = \sin x + \cos x, \lambda = \tan x$, so that for $0 < x < \pi/4, \lambda \in (0,1)$. Then inequality becomes

$$\begin{aligned}&\log(\tan x.\sin x + (1-\tan x)(\sin x + \cos x))\\ &> \tan x.\log(\sin x) + (1-\tan x).\log(\sin x + \cos x)\end{aligned}$$

i.e., $\log(\cos x) > \tan x.\log(\sin x) + (1-\tan x).\log(\sin x + \cos x) > \tan x.\log(\sin x)$, (since in the second term, $1 - \tan x > 0$ and $\sin x + \cos x = \sqrt{2}.\cos(\pi/4 - x) > 1$), i.e. $\log(\cos x) > \log\left((\sin x)^{\tan x}\right)$, i.e., $\cos x > (\sin x)^{\sin x/\cos x}$, i.e., $(\cos x)^{\cos x} > (\sin x)^{\sin x}$. $\square$

2.8.72 Problem. Prove that, $\cos(\sin x) > \sin(\cos x), \forall\, x \in \mathbb{R}$.

2.8.72.1 Solution. Without loss of generality assume that $x \in (-\pi, \pi]$. We first show that $\cos(\sin x) \neq \sin(\cos x)$. For, otherwise,

$$\begin{aligned}\cos(\sin x) = \sin(\cos x) &= \cos(\pi/2 - \cos x)\\ \text{and so } \sin x &= \pm(\pi/2 - \cos x),\end{aligned}$$

since both $\sin x$ and $\pi/2 - \cos x \in (-\pi, \pi]$. Thus, either

$$\sin x + \cos x = \pi/2 \text{ or } \sin x - \cos x = -\pi/2.$$

But $\sin x \pm \cos x = \sqrt{2}.\sin(x \pm \pi/4)$ and we get that

$$-\sqrt{2} \leq \sin x \pm \cos x \leq \sqrt{2}$$

or, since $-\pi/2 < -\sqrt{2}$ and $\sqrt{2} < \pi/2$, we have,

$$-\pi/2 < -\sqrt{2} \leq \sin x \pm \cos x \leq \sqrt{2} < \pi/2$$

a contradiction. Now, by continuity and the fact that $\cos(\sin 0) > \sin(\cos 0)$, it follows that $\cos(\sin x) > \sin(\cos x)$. $\square$

2.8.73 Problem. Prove that, $\cos(\sin^{-1} x) < \sin^{-1}(\cos x), 0 \leq x \leq 1$.

2.8.73.1 Solution. Let $\sin a = \cos x$; then $a = \sin^{-1}(\cos x)$. But $\sin a < a$, i.e.,

$$\cos x < \sin^{-1}(\cos x). \tag{1}$$

Also $x > \sin x$ and so $\sin^{-1} x > \sin^{-1}(\sin x) = x$ and therefore

$$\cos(\sin^{-1} x) < \cos x. \tag{2}$$

Now (1) and (2) imply $\cos(\sin^{-1}(x)) < \sin^{-1}(\cos x)$. $\square$

2.8.74 Problem. Prove that, $|\sin rx| < r|\sin x|$, for integer $r \geq 2$, $\sin x \neq 0$.

2.8.74.1 Solution.

$$\begin{aligned}|\sin rx| &= |\sin((r-1)x + x)| \\ &= |\sin(r-1)x\cos x + \cos(r-1)x\sin x| \\ &\le |\sin(r-1)x| + |\sin x|.\end{aligned}$$

and the result follows by induction. □

2.8.75 Problem.

1. For all $n \in \mathbb{N}$ we have

$$\sqrt{n+1} - \sqrt{n} \le \frac{1}{\sqrt{n}} < \sqrt{n} - \sqrt{n-1}.$$

2. If $n \in \mathbb{N}$ and $n > 1$ then

$$2\sqrt{n+1} - 2 < \sum_{k=1}^{n} \frac{1}{\sqrt{k}} < 2\sqrt{n} - 1.$$

2.8.75.1 Solution.

1. We have

$$\begin{aligned}\sqrt{n+1} - \sqrt{n} &= \frac{1}{\sqrt{n+1}+\sqrt{n}} < \frac{1}{2\sqrt{n}}. \\ \sqrt{n} - \sqrt{n-1} &= \frac{1}{\sqrt{n}+\sqrt{n-1}} > \frac{1}{2\sqrt{n}}\end{aligned}$$

2. Sum the inequalities in (1) for $k = 2, 3, ..n$. □

2.8.76 Problem. Let $a, b, c \in \mathbb{R}$ with ac and $b \in \mathbb{Q}$, such that the equation $ax^2 + bx + c = 0$ has a solution $r \in \mathbb{Q}$. Prove that for any $n \in \mathbb{N}, \exists\, b_n \in \mathbb{Q}$ for which r^n is a solution of $a^n x^2 + b_n x + c^n = 0$.

2.8.76.1 Solution. The conclusion is trivial for $ac = 0$, so we may assume that $a \ne 0$ and $c \ne 0$. Then $ar^2 + br + c = 0$ implies that $r \ne 0$ and that $ar + c/r = -b$ is a rational number. Using the identity

$$a^{n+1}r^{n+1} + \frac{c^{n+1}}{r^{n+1}} = \left(ar + \frac{c}{r}\right)\left(a^n r^n + \frac{c^n}{r^n}\right) - ac\left(a^{n-1}r^{n-1} + \frac{c^{n-1}}{r^{n-1}}\right)$$

it follows by induction that for all positive integers $n, a^n r^n + \dfrac{c^n}{r^n}$ is a rational number, say b_n. Then $a^n(r^n)^2 + b_n r^n + c^n = 0$, and the problem is solved. □

2.9 Additional Exercises on Chapter 2.

2.9.1 Exercise. Define an equivalence relation in the set $\mathbb{R}^+ = (0, \infty)$ by taking $x \sim y$ if $x/y \in \mathbb{Q}$. Prove that the intersection of each equivalence class with any (nonempty) open interval contained in $\mathbb{R}^+$ is nonempty.

2.9.2 Exercise. If x and y are two real numbers, we define $\max\{x, y\}$ to be x if $x > y$ and y if $y > x$. We often denote $\max\{x, y\}$ by $x \vee y$. Similarly, we define $\min\{x, y\}$ to be the smaller of x and y, and denote it by $x \wedge y$. Show that

1. $(x \wedge y) \wedge z = x \wedge (y \wedge z)$.
2. $(x \wedge y) + (x \vee y) = x + y$.
3. $(-x) \wedge (-y) = -(x \vee y)$.
4. $(x \vee y) + z = (x + z) \wedge (y + z)$.
5. $z(x \vee y) = (zx \vee zy)$.

2.9.3 Exercise. Prove that $\tan 1^\circ$ is irrational.

2.9.4 Exercise. If $\{a\}$ denotes the fractional part of $a \in \mathbb{R}$, show that,

1. $\{a + \{na\}\} = \{(n + 1)a\}\ \forall\, n \in \mathbb{N}$.
2. $\{m\{na\}\} = \{mna\}\ \forall\, m, n \in \mathbb{N}$.

2.9.5 Exercise. Answer the following questions. Explain your answer.

1. Can the intersection of a sequence of nested intervals be empty?
2. Can the intersection of a sequence of nested closed intervals be empty?
3. Can the intersection of a sequence of nested closed intervals be a one point set?
4. Can the intersection of a sequence of nested open intervals be nonempty?
5. Can the intersection of a sequence of nested open intervals be a closed interval?

2.9.6 Exercise. Using the Cantor axiom give a direct proof of the fact that the subset of irrational numbers is dense in the real line: every open interval contains an irrational number.

2.9.7 Exercise. Which axioms of the reals are satisfied for the set of rational numbers (with the usual operations and ordering)? [Only the Cantor's axiom is not satisfied.]

2.9.8 Exercise. Prove that if an ordered field satisfies the completeness theorem, then the Archimedean axiom holds. Hint: What is the supremum of the set of positive integers?

2.9.9 Exercise. In this problem, prove the existence and uniqueness of the number α such that $\alpha^2 = a$ for $a > 0$. Define $\alpha = \sup\{r; 0 \le r,\ r^2 < a\}$.

1. Show that for any $0 < \epsilon < 1$ we have that $(1+\epsilon)^2 < 1+3\epsilon$. Similarly, show that $(1-\epsilon)^2 > 1-2\epsilon$. Use this to show that $(a(1 + \epsilon))^2 < a^2 + 3\epsilon a^2$ and $(a(1 - \epsilon))^2 > a^2 - 2\epsilon a^2$.
2. Show that if $\alpha^2 < a$ then there is some $\epsilon > 0$ with $(\alpha(1 + \epsilon))^2 < a$. Use this to prove that $\alpha^2 \ge a$.
3. Show that if $\alpha^2 > a$ then there is some $\epsilon > 0$ with $(\alpha(1 - \epsilon))^2 > a$. Use this to prove that $\alpha^2 \le a$.

4. Prove that there is at most one number $r \in \mathbb{R}$ with $r > 0$ and $r^2 = a$.

2.9.10 Exercise. Construct proofs for the following in the style demanded; in each case it is more the presentation and style of the proof. Correct mathematical ideas however are not discouraged.

1. (Direct proof) For all real numbers x and y, if $x < y$ then $x^3 < y^3$.
2. (Contrapositive) If x is irrational then $x + r$ is irrational for all rational numbers r.
3. (Indirect proof) The number $\sqrt[3]{2}$ is irrational.
4. (Counterexample) For any natural number n the equation $4x^2 + x - n = 0$ has no rational root.
5. (Induction) For every $n = 1, 2, 3, ..$ Prove that $2^n > n$.

2.9.11 Exercise. If a, b, c, d are rational and if x is irrational, prove that $(ax+b)/(cx+d)$ is usually irrational. When do exceptions occur?

2.9.12 Exercise. Given any real $x > 0$, prove that there is an irrational number between 0 and x.

2.9.13 Exercise. Determine for which values of the integer $n \geq 1$ the number $\sqrt{n+1} + \sqrt{n-1}$ is rational and for which values it is irrational.

2.9.14 Exercise. Let a and b be positive integers. Prove that $\sqrt{2}$ always lies between the two fractions a/b and $(a + 2b)/(a + b)$. Which fraction is closer to $\sqrt{2}$?

2.9.15 Exercise. If p, q, r, s are all rational numbers, and

$$(ps - qr)^2 + 4(p - r)(q - s) = 0,$$

prove that either (i) $p = r, q = s$ or else (ii) $1 - pq$ and $1 - rs$ are squares rational numbers or are zero.

2.9.16 Exercise. If all the solutions of the equations

$$ax^2 + 2bxy + cy^2 = 1, \; lx^2 + 2mxy + ny^2 = 1$$

are rational solutions, prove that both

$$\sqrt{(b-m)^2 - (a-l)(c-n)} \text{ and } \sqrt{(an-cl)^2 + 4(am-bl)(cm-bn)}$$

are rational, when a, b, c, l, m, n are rational numbers.

2.9.17 Exercise. Let S be a non-empty subset of $\mathbb{R}$ that is bounded above and let $\mu = \sup S$. Suppose that $\mu \notin S$. Prove that for each $\epsilon > 0$, the set $\{x \in S; x > \mu - \epsilon\}$ is infinite.

2.9.18 Exercise. For each rational number x, write $x = p/q$, where $p, q \in \mathbb{Z}, (p, q) = 1, q \geq 1$, let $d(x) = q$. Note that $d(n) = 1 \; \forall \; n \in \mathbb{N}$. Let $[a, b]$ be a closed interval and r be a positive real number. Prove that the set $\{x \in [a, b] \cap \mathbb{Q}; d(x) \leq r\}$ is a finite set.

2.9.19 Exercise. Prove that the union of two disjoint countably infinite sets is countably infinite by finding a one-to-one correspondence between the union of two sets and the set $\mathbb{Z}$.

2.9.20 Exercise. Let A and B be two sets.

1. Suppose that A and B are both countably infinite sets. Prove that there is a one-to-one correspondence between A and B.

2. Suppose that A is countably infinite and that there is a one-to-one correspondence between A and B. Prove that B is countably infinite.

2.9.21 Exercise. For a set $E \subseteq \mathbb{R}$,

1. E is denumerable $\iff$ card $E = \aleph_0$;

2. E is countable $\iff$ card $E \leq \aleph_0$;

3. E is finite $\iff$ card $E < \aleph_0$;

4. E is infinite $\iff$ card $E \geq \aleph_0$;

5. E is uncountable $\iff$ card $E > \aleph_0$.

2.9.22 Exercise. If $u > 0$ is any real number and $x < y$, show that there exists a rational number r such that $x < ru < y$. (Hence the set $\{ru; r \in \mathbb{Q}\}$ is dense in $\mathbb{R}$.)

2.9.23 Exercise. Give the binary representations of $\frac{3}{8}$, $\frac{13}{16}$ and $-\frac{5}{7}$.

2.9.24 Exercise. Give an example of a countable bounded subset A of $\mathbb{R}$ whose glb and lub are both in $\mathbb{R} \setminus A$.

2.9.25 Exercise. If A is a nonempty bounded subset of $\mathbb{R}$, and B is the set of all upper bounds for A, prove that $\inf B = \sup A$.

2.9.26 Exercise. If E is not empty, then $\inf E \leq \sup E$; there is strict inequality if E contains at least two points. If equality holds, what can you say about E?

2.9.27 Exercise. Explore the consequences of introducing numbers $+\infty$ and $-\infty$ such that $a/0 = +\infty$ if $a > 0$, and $a/0 = -\infty$ if $a < 0$. Can a reasonable meaning be given to $(+\infty) + (-\infty)$? $0.(+\infty)$?

2.9.28 Exercise. The set of all points inside a circle is called a disk. Let S be a set of nonoverlapping disks in the plane – that is, no disk in S has a nonempty intersection with any other disk in S. Show that S must be countable.

Chapter 3

Metric Structure on $\mathbb{R}$ and Point Set Topology

In Mathematics the art of proposing a question
must be held of higher value than solving it.
– A thesis defended in Cantor's doctoral examination.

3.0.1 Definition. A **metric** "d" on a set X is a real valued function

$$d : X \times X \to \mathbb{R}$$

which has the following properties:

1. $d(x, y) \geq 0 \; \forall \; x, y \in X;\; d(x, y) = 0$ iff $x = y$ (non-negativity),
2. $d(x, y) = d(y, x) \; \forall \; x, y \in X$ (symmetric property),
3. $d(x, y) \leq d(x, z) + d(z, y) \; \forall \; x, y$ and $z \in X$ (triangle inequality).

If d is a metric on X, the number $d(x, y)$ is called the distance between the elements x and y. In $\mathbb{R}$, we define

$$d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

by $d(x, y) = \sqrt{(x - y)^2}$, which is equal to $|x - y|$.
Now we define

$$d(0, x) = \sqrt{x^2} = |x| = \begin{cases} x, & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ -x, & \text{if } x < 0. \end{cases}$$

Some properties are described in the problems:

3.1 Open Set, Derived set, Closed set:

3.1.1 Definition. A **metric space** (X, d) is a non-empty set X and a metric d defined on X.

3.1.2 Example. $(\mathbb{R}, d)$ is a metric space with $d(x, y) = |x - y|$

3.1.3 Definition. In (X, d), we define an **open** r**-ball** with center a and radius r by

$$B(a; r) = \{x \in X; d(x, a) < r\}.$$

3.1.4 Note. In $\mathbb{R}$, for every $\epsilon > 0$, ϵ**-neighborhood or** ϵ**-nbhd.** of a point $a \in \mathbb{R}$ or ϵ**-ball** at a is the set

$$\begin{aligned} B(a; \epsilon) &= \{x \in \mathbb{R}; |x - a| < \epsilon\} \\ &= \{x \in \mathbb{R}; a - \epsilon < x < a + \epsilon\} \\ &= (a - \epsilon, a + \epsilon). \end{aligned}$$

3.1.5 Definition (Neighborhood of a point). Let $a \in D \subseteq \mathbb{R}$, then D is said to be **nbhd.** of a iff $\exists\ \epsilon > 0$ such that $B(a; \epsilon) \subseteq D$.

3.1.6 Definition (Interior point of a set). A point $a \in D$ is said to be an **interior point** of D iff D is a nbhd. of a.

3.1.7 Definition (Interior of a set). The set of all interior points of D is called the interior of D and is denoted by $int(D)$ or D°.

3.1.8 Definition (Open set). A set S is said to be an **open set** in $\mathbb{R}$ iff each point of S is an interior point of S.

3.1.9 Proposition.

1. $A^\circ \subseteq A\ \forall\ A \subseteq \mathbb{R}$.
2. If $A \subseteq B$ then $A^\circ \subseteq B^\circ$ for $A, B \subseteq \mathbb{R}$.
3. $A^\circ \cup B^\circ \subseteq (A \cup B)^\circ$ for $A, B \subseteq \mathbb{R}$.
4. $A^\circ \cap B^\circ = (A \cap B)^\circ$ for $A, B \subseteq \mathbb{R}$.
5. $(A^\circ)^\circ = A^\circ\ \forall\ A \subseteqq \mathbb{R}$.
6. A is open iff $A^\circ = A\ \forall\ A \subseteq \mathbb{R}$.
7. A° is the largest open set contained in $A\ \forall\ A \subseteq \mathbb{R}$.

3.1.10 Definition (Neighborhood of $\infty, -\infty$). Any set which contains a set of the form (a, ∞) is called a nbhd. of ∞ and which contains a set of the form $(-\infty, b)$ is called a nbhd. of $-\infty$ where $a, b \in \mathbb{R}$. Thus $[5, 6) \cup [7, \infty)$ or $(-\infty, 2) \cup [3, 4]$ are also the nbhds of ∞ and $-\infty$ respectively.

3.1.11 Note. It is important to notice that nbhds of $\infty, -\infty$. are not nbhds in the sense used in topology, since a nbhd. of a point p in a topological space necessarily has the point as an element. However, this deviation from the standard topological usage concerning nbhds, should cause no confusion, since there are no elements $\infty, -\infty$ in $\mathbb{R}$.

3.1.12 Definition. A point $a \in \mathbb{R}$ is said to be a **limit point** of D iff every nbhd. of a contains at least one point of D other than a.
In other words $\forall\ \epsilon > 0, \hat{B}(a; \epsilon) \cap D \neq \emptyset$. The set $\hat{B}(a; \epsilon) = B(a; \epsilon) \setminus \{a\}$ is called a **deleted nbhd.** of a. The set of all limit points of D is denoted by D' and is called the **derived set** of D.

3.1.13 Definition. A set $A \subseteq \mathbb{R}$ is said to be **closed** iff its complement is open.

3.1.14 Note. The closed sets ("abgeschlossen" in German) were introduced by Cantor in 1884 through his work on the Continuum Hypothesis. Namely, he proved that no closed set can have a cardinal number strictly between the cardinal numbers of $\mathbb{N}$ and $\mathbb{R}$.

3.1.15 Definition. The **closure** of a set A is denoted by $\overline{A}$ and is defined by $\overline{A} = A \cup A'$.

3.1.16 Definition. A point $a \in \mathbb{R}$ is said to be a **boundary point** of $S \subseteq \mathbb{R}$ iff every nbhd. U of a intersects both S and $\mathbb{R} \setminus S$. The set of all boundary points of S is denoted by ∂S.

3.1.17 Example. Boundary points of the set $S = \{1, 2, 3, 4, 5, \}$ is the set itself, i.e. $\partial S = S$. If $S = (a, b)$, then $\partial(a, b) = \{a, b\}$, here $\partial S \cap S = \emptyset$ and $\partial \mathbb{R} = \emptyset$.

3.2 Dense set, Perfect set

3.2.1 Definition.

1. Let $E \subset \mathbb{R}$. The set E is said to be **dense** at a point "a" if a is a limit point of E.
   Example: The set $A = \{1, 1/2, 1/3, ...\}$ is dense at 0.

2. A set E is said to be **dense** in a set F if E is dense at each point of F i.e. $F \subseteq E'$ (the derived set of E.) Thus E is **dense-in-itself** if $E \subseteq E'$.
   Example: $\mathbb{Q}$ is dense in $\mathbb{Q}$, $\mathbb{Q}^C$; $\mathbb{Q}^C$ is dense in $\mathbb{Q}$, $\mathbb{Q}^C$ and $\mathbb{R}$.

3. A set E is said to be a **perfect set** if $E' = E$, in other words if E is closed and dense-in-itself.
   Example: $[a, b]$

4. A set E is said to be **somewhere dense** if $(\overline{E})^\circ \neq \emptyset$. That is, a set E is said to be **nowhere dense** if $(\overline{E})^\circ = \emptyset$.
   Example: The sets $\mathbb{N}$, $\mathbb{Z}$ and $\left\{1, \frac{1}{2}, ..., \frac{1}{n}, ...\right\}$ are nowhere dense and the set $\mathbb{N} \cup (1, 2)$ is somewhere dense.

5. A point "a" is said to be a **condensation point** of E if for every open set U containing a, $E \cap U$ is uncounable.

6. A point $a \in D$ is said to be an **isolated point** of $D \subseteq \mathbb{R}$ iff $\exists$ a ϵ-nbhd. $B(a; \epsilon)$ of a such that $B(a; \epsilon) \cap D = \{a\}, i.e.\ a \notin D'$.

7. A point $a \in \mathbb{R}$ is said to be an **exterior point** of $S \subseteq \mathbb{R}$ iff a is an interior point of $\mathbb{R} \setminus S$. The set of all exterior points of S is denoted by $Ext(S)$.

3.3 Intervals

Let $a, b \in \mathbb{R}$ such that $a \leq b$. The set of real numbers $\{x \in \mathbb{R}; a < x < b\}$ and $\{x \in \mathbb{R}; a \leq x \leq b\}$ are called open and closed intervals in $\mathbb{R}$ with endpoints a, b and are denoted by (a, b) and $[a, b]$ respectively. Similarly the sets $\{x \in \mathbb{R}; a < x \leq b\}$ and $\{x \in \mathbb{R}; a \leq x < b\}$ are called left-half-open and right-half-open intervals in $\mathbb{R}$ with endpoints a, b and are denoted by $(a, b]$ and $[a, b)$ respectively. If $a = b$, the four sets are called degenerate; in this case $[a, b]$ is singleton and other three intervals

are empty. Now we assume that all intervals are non-empty. the four intervals defined above are bounded sets. We also require unbounded intervals with the end point a are

$$[a,\infty) = \{x \in \mathbb{R}; x \geq a\},\ (a,\infty) = \{x \in \mathbb{R}; x > a\},$$
$$(-\infty,a] = \{x \in \mathbb{R}; x \leq a\}, (-\infty,a) = \{x \in \mathbb{R}; x < a\},$$

We also define $(-\infty,\infty) = \mathbb{R}$. A simple consequence of the order completeness, we have to characterize an interval.

3.4 Characterization of an interval

3.4.1 Proposition. Let $a \in \mathbb{R}$ and $A \subseteq (a,\infty)$ with the property that if $y \in A$ and $a < x < y$ then $x \in A$. Then A is either (a,∞) or $(a,b]$ or (a,b) for some $b \in \mathbb{R}$ and $b > a$.

Proof. Case 1. Let A be unbounded above and $x \in (a,\infty)$, then $\exists\, y \in A$ such that $a < x < y$, hence $x \in A$. Thus $(a,\infty) \subseteq A$. Since $A \subseteq (a,\infty)$. we have $A = (a,\infty)$.
Case 2. Let A be bounded above, then A has a least upper bound b (say), and $b > a$. Let $x \in (a,b)$, since b is a least upper bound of $A, \exists\, y \in A$ such that $x < y \leq b$, then $x \in A$ and thus $(a,b) \subseteq A$. Further, if $x > b$, then $x \in A$ and by definition of lub, $A \subseteq (a,b]$. Hence $A = (a,b)$ or $(a,b]$ according as $b \notin A$ or $b \in A$. ☐

3.4.2 Proposition. Let $a \in \mathbb{R}$ and $A \subseteq (-\infty,b)$ with the property that if $y \in A$ and $y < x < b$ then $x \in A$. Then A is either $(-\infty,b)$ or $[a,b)$ or (a,b) for some $a \in \mathbb{R}$ and $b > a$.

3.4.3 Proposition. (Characterization of an interval) A subset A of $\mathbb{R}$ is an interval (bounded or unbounded, possibly degenerate) iff it has the property that if $x,y \in A$ and $x < z < y$, then $z \in A$, i.e. $(x,y) \subseteq A$.

Proof. It is trivial that an interval has this property. It remains to prove that a set A with this property is an interval. If A is empty, then $A = (c,c)$ for any real c. We may therefore suppose that A has at least one element. Let $a \in A$ be a fixed element of A, and let

$$A_1 = A \cap (a,\infty), A_2 = A \cap (-\infty,a).$$

If A_1 is non-empty, then A_1 is either (a,∞) or $(a,b]$ or (a,b) where $b > a$. Similarly, if A_2 is non-empty, then A_2 is either $(-\infty,a)$ or one of $[c,a),(c,a)$ where $c < a$, and since $A = A_1 \cup \{a\} \cup A_2$, the result follows. ☐

3.4.4 Definition. A subset A of $\mathbb{R}$ has the **intermediate value property,** if for any $x,y \in A$ and any $t \in \mathbb{R}$,

$$x < t < y \Rightarrow t \in A.$$

Thus any interval has the intermediate value property. Conversely, any subset of $\mathbb{R}$ that has the intermediate value property is an interval:

Let $p \in S$, recall that S is a **nbhd.** of p iff $\exists\, \epsilon > 0$ such that $B(p;\epsilon) \subseteq S$ where $B(p;\epsilon) = \{x; |x-p| < \epsilon\}$. i.e. $(p-\epsilon, p+\epsilon) \subseteq S$. Hence the set $B[p;\epsilon] = \{x; |x-p| \leq \epsilon\}$ is also a nbhd. of p.

3.4.5 Theorem. Let $p \in \mathbb{R}$, and let $U \subseteq \mathbb{R}$. Then the following are equivalent:

1. U is a nbhd. of p. i.e. $\exists\, \eta > 0$ such that $(p-\eta, p+\eta) \subseteq U$.

2. $\exists\, \eta' > 0$ such that $[p-\eta', p+\eta'] \subseteq U$.

3. $\exists\, \alpha, \beta$ satisfying $\alpha < p < \beta$ such that $(\alpha, \beta) \subseteq U$.

4. $\exists\, \alpha', \beta'$ satisfying $\alpha' < p < \beta'$ such that $[\alpha', \beta'] \subseteq U$.

3.4.6 Remark. In some contexts it is natural to restrict our work to a particular subset of $\mathbb{R}$, and ignore what is outside. For example, notions like "open," "closed," "neighborhood," "dense," etc., may be considered just in the ambience of, say, the interval [0,1], a natural thing to do in case we are dealing, e.g., with a function defined just there. In this circumstance, it will be not only inconvenient, but maybe even wrong, to consider points out of the given set. Our "universe" will be just this given set. To be precise, let us consider the following definition.

3.4.7 Definition. Let $X \subseteq \mathbb{R}$ then a subset $Y \subseteq \mathbb{R}$ is **open in** X or **open relative to** X iff $Y = U \cap X$ for some open set in $\mathbb{R}$.

Once we have this notion, all the other related notions are defined accordingly: for example, a subset F of S is **closed relative to** S if $S \backslash F$ is open relative to S; a subset U of S is a neighborhood of a point $x \in S$ relative to S if U contains a set O open relative to S and $x \in O$; a subset D of S is dense in S relative to S if every nonempty open relative to S subset of S intersects D. As a particular instance, observe that the set D of rational points in [0,1] is dense in [0,1] relatively to [0,1]. Sometimes, if it would not cause any misunderstanding, we will just say that D is dense in [0,1]. The same for other notions. It is simple to show that, if S and T are subsets of $\mathbb{R}$, and $S \subseteq T$, the set S is open (closed) relative to T, and T is open (closed) in $\mathbb{R}$, then S is open (closed) in $\mathbb{R}$. In general, a set can be open relative to a certain superset and not open in $\mathbb{R}$.

3.4.8 Example. Let $X = (0,1] \cup \{3\}$, considered as a subset of $\mathbb{R}$, which is neither open nor closed in $\mathbb{R}$. Then, $X = (0,1] \cup \{3\}$, is open in X for $X = (0,4) \cap X$ and X is closed in X for $X = [0,4] \cap X$ and $\{3\}$ is both open and closed in X for $\{3\} = (2,4) \cap X$ and $\{3\} = [2,4] \cap X$.

3.4.9 Example. The set [0,1] is open relative to [0,1], since $[0,1] = (-1,2) \cap [0,1]$, and $(-1,2)$ is open in $\mathbb{R}$. Certainly, [0,1] is not open in $\mathbb{R}$. Analogously, the set [0,1/2) is open relative to [0,1], since $[0,1/2) = (-1,1/2) \cap [0,1]$.

3.5 Locally Finite Family

3.5.1 Definition. A family $\mathcal{A}$ of subsets of $\mathbb{R}$ is said to be **locally finite** if, given any $x \in \mathbb{R}$, there exists $\delta_x > 0$ such that the open ball $B(x, \delta_x)$ has a nonempty intersection with (at most) a finite number of the sets $A \in \mathcal{A}$.

3.5.2 Example. The family $\mathcal{A} = \{A_n; A_n = (n, n+1), n \in \mathbb{N}\}$ is locally finite.

3.6 Problems and Solutions on Chapter 3

Warning! *Never Never Ever read this solution unless you tried problems quite long time and gave up. Doing so may impair your ability to think and solve problems.*

3.6.1 Problem. Let $p \in \mathbb{R}$, and let $U \subseteq \mathbb{R}$. Then the following are equivalent:

1. U is a nbhd. of p. i.e. $\exists\, \eta > 0$ such that $(p-\eta, p+\eta) \subseteq U$.

2. $\exists\, \eta' > 0$ such that $[p-\eta', p+\eta'] \subseteq U$.
3. $\exists\, \alpha, \beta$ sayisfying $\alpha < p < \beta$ such that $(\alpha, \beta) \subseteq U$.
4. $\exists\, \alpha', \beta'$ sayisfying $\alpha' < p < \beta'$ such that $[\alpha', \beta'] \subseteq U$.

3.6.1.1 Solution.

1. (2) $\Rightarrow$ (1) with $\eta = \eta'$.
2. (1) $\Rightarrow$ (2) for any η' satisfying $0 < \eta' < \eta$,
3. (4) $\Rightarrow$ (3) with $\alpha = \alpha', \beta = \beta'$ and
4. (3) $\Rightarrow$ (4) for any α', β' such that $\alpha < \alpha' < p < \beta' < \beta$.
   Next (1) implies $\exists\, \eta > 0$ such that $(p-\eta, p+\eta) \subseteq U$. Now taking $\alpha = p-\eta, \beta = p+\eta$, we get $(\alpha, \beta) \subseteq U$. i.e. (1) implies (3).
   Again, if $U \supseteq (\alpha, \beta)$ where $\alpha < p < \beta$ then U contains the interval $(p-\eta, p+\eta)$ with $\eta = \min\{p-\alpha, \beta-p\}$

$$(\text{Since } \alpha = p-(p-\alpha) \leq p-\eta < p+\eta \leq p+(\beta-p) = \beta).$$

Hence (1) and (3) are equivalent. □

3.6.2 Problem. Let $a, b > 0$, (a, b) be an interval and $c < b-a$, then $\exists\, k \in \mathbb{N}$ such that $a < kc < b$.

3.6.2.1 Solution. Suppose that, there exists no $n \in \mathbb{N}$ satisfying the condition, then for all $n \in \mathbb{N}$ either $cn < a$ or $cn > b$. Let $m = \max\{n \in \mathbb{N};\ nc < a\}$ then $mc < a$ and hence $(m+1)c > b \Rightarrow mc > b-c > a$, a contradiction. Thus $\exists\, k \in \mathbb{N}$ such that $a < kc < b$. □

3.6.3 Problem.

1. If ξ is an irrational number and $0 < r < 1$, then prove that, there are integers $m \in \mathbb{Z}$ and $n \in \mathbb{Z} \setminus \{0\}$ such that $0 < m + n\xi < r$.
2. If ξ is irrational and $a < b$, then $\exists\, m, n \in \mathbb{Z}$ such that $a < m + n\xi < b$. In other words, the set $D = \{m + n\xi,\ \xi \text{ irrational and } m, n \in \mathbb{Z}\}$ is dense in $\mathbb{R}$.

3.6.3.1 Solution.

1. Hint: For $k \in \mathbb{N}$, consider $x_k = k\xi - [k\xi]$. Hence $0 \leq x_k < 1$. But from the irrationality of ξ, $0 < x_k < 1$ and also if $i \neq k$ then $x_i \neq x_k$, for $x_i - x_k = i\xi - [i\xi] - k\xi + [k\xi] = (i-k)\xi - ([i\xi] - [k\xi]) = m\xi + n$ for some $m, n \in \mathbb{Z} \setminus \{0\}$. Since $0 < r < 1$, so $\left[\frac{1}{r}\right] \leq \frac{1}{r} < \left[\frac{1}{r}\right] + 1$, let $N = \left[\frac{1}{r}\right] + 1$, divide (0,1) into N equal parts. Then among the numbers $x_1, x_2, , ..., x_{N+1}$ there are at least two which differ by less than $\frac{1}{N} < r$.
2. Hint: Choose $r, s \in \mathbb{Z}$ such that $0 < r + s\xi < b - a$. Then by the above problem (3.6.2) $\exists\, N \in \mathbb{N}$ such that $a < Nr + Ns\xi < b$. Thus any open interval contains a point of D. Hence D is dense in $\mathbb{R}$. □

3.6.4 Problem. (Applications to Dirichlet's Principle). Let $\mathcal{F}$ be the family of all triplets of integer numbers, not all zero, whose absolute values are less than 10^6.

1. Prove that there exists $(a, b, c) \in \mathcal{F}$ such that $|a + b\sqrt{2} + c\sqrt{3}| < 10^{-11}$.

2. Prove that $|a + b\sqrt{2} + c\sqrt{3}| > 10^{-21}$ for every $(a, b, c) \in \mathcal{F}$.

3.6.4.1 Solution.

1. Let S be the set of the 10^{18} real numbers $r + s\sqrt{2} + t\sqrt{3}$ with each of $r, s, t \in \{0, 1, ..., 10^6 - 1\}$ and let $d = (1 + \sqrt{2} + \sqrt{3})10^6$. Then each $x \in S$ is in the interval $0 \leq x < d$. This interval is partitioned into $10^{18} - 1$ "small" intervals $(k-1)t \leq x < kt$ with $t = d/(10^{18} - 1)$ and k taking on the values $1, 2, ..., 10^{18} - 1$. By the pigeonhole principle, two of the 10^{18} numbers of S must be in the same small interval and their difference $a + b\sqrt{2} + c\sqrt{3}$ gives the desired a, b, c since $c < 10^{-11}$.

2. Let $F_1 = a + b\sqrt{2} + c\sqrt{3}$ and F_2, F_3, F_4 be the other numbers of the form $a \pm b\sqrt{2} \pm c\sqrt{3}$. Using the irrationality of $\sqrt{2}$ and $\sqrt{3}$ and the fact that a, b, c are not all zero, one easily shows that no F_i, is zero. One also sees that the product $P = F_1F_2F_3F_4$ is an integer. Hence $|P| \geq 1$. Then $|F_1| \geq 1/|F_2F_3F_4| > 10^{-21}$ since $|F_1| \leq 10^7$ and thus $1/|F_i| \geq 10^{-7}$ for each i. □

3.6.5 Problem. An infinite closed subset $S \subseteq \mathbb{R}$ is a closure of a countable set.

3.6.5.1 Solution. Since every point of a closed set is either an accumulation point or an isolated point. So, we can write $S = A \cup B$ where A is an isolated set which is countable and $B = S'$. Consider the set $C = B \cap \mathbb{Q}$, it is easy to show that $\overline{C} = B$. Since $A \cup C$ is countable, thus $\overline{A \cup C} = S$. □

3.6.6 Problem. Let E be the set of all $x \in [0, 1]$ whose decimal expansion contains only the digits 4 and 7.

1. Is E countable?

2. Is E dense in [0,1]?

3. Is E compact?

4. Is E perfect?

3.6.6.1 Solution.

1. Suppose that E is countable then we can list the elements as

$$\begin{aligned} a_1 &= .a_{11}a_{12}a_{13}.... \\ a_2 &= .a_{21}a_{22}a_{23}.... \\ &........................... \\ a_n &= .a_{n1}a_{n2}a_{n3}.... \\ &........................... \end{aligned}$$

where $a_{ij} = 4$ or 7, then consider the element whose the decimal expansion

$$x = .x_1x_2x_3....$$

where $x_n = \begin{cases} 4, & \text{if } a_{nn} = 7 \\ 7, & \text{if } a_{nn} = 4. \end{cases}$

Then no term of the sequence (a_n) can be equal to x, since x differs from a_1, in the first decimal place, differs from a_2 in the second decimal place,....., from a_n in the n-th decimal place. Thus E is not countable.

2. E is not dense in [0,1], because $E \subseteq [.4, .8]$.

3. We show that E is closed. Let $x \in [0,1] \setminus E$, i.e. the decimal expansion of x contains a digit different from 4 and 7. Let the first such digit occur in the n-th place x_n. Let $y \in E$, and let the first digit in which x and y differ be the m-th digit ($m \leq n, x_m \neq y_m$). Then

$$|x - y| \geq 10^{-m} - \epsilon, \epsilon \leq \sum_{k=m+1}^{\infty} |x_k - y_k|.$$

Since $y_k \in \{4, 7\}$ and $x_k \in \{0,1,2,3,4,5,6,7,8,9\}$. Hence $\epsilon \leq \frac{7}{9}10^{-m}$, and it follows that $|x-y| \geq \frac{2}{9.10^m} \geq \frac{1}{9.10^n}$. Thus x is an interior pont of $[0,1] \setminus E$ and so E is closed and bounded. Hence E is compact.

4. Now, let $x \in E$ then for $\epsilon > 0$ we can find a point $y \in E$ by changing the n-th digit of x from 4 to 7 or from 7 to 4 in the n-th place for any $n > 1 - \log_{10} \epsilon$. Hence $x \in E'$ i.e., $E \subseteq E'$. Since E is closed, it follows that $E = E'$. □

3.6.7 Problem. A real number $\frac{m}{2^n}$, where $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ is known as **dyadic rational number**. Prove that there is a dyadic rational number between any two real numbers. In other words, the set $D = \left\{\frac{m}{2^n}; m \in \mathbb{Z}, n \in \mathbb{N}\right\}$ is dense in $\mathbb{R}$.

3.6.7.1 Solution. Let (a, b) be any open interval. By Archimedean Principle $\exists\, N \in \mathbb{N}$ such that $0 < \frac{1}{N} < b - a$. Since $2^N > N$, so $0 < \frac{1}{2^N} < \frac{1}{N} < b - a$ implies $2^N(b-a) > 1 \Rightarrow \exists\, m \in \mathbb{N}$ such that $2^N a < m < 2^N b \Rightarrow a < \frac{m}{2^N} < b \Rightarrow (a,b) \cap D \neq \emptyset$. Thus any open interval contains a point of D. Hence D is dense in $\mathbb{R}$. □

3.6.8 Problem. Let $a, b \in \mathbb{R}$ such that $b - a > 1$. Prove that $\exists\, n \in \mathbb{Z}$ such that $a < n < b$.

3.6.8.1 Solution. Suppose that $a, b > 0$, then $b - a > 1 \Rightarrow b > a$ and by Archimedean principle $\exists\, n \in \mathbb{N}$ such that $n \cdot 1 > a$. Let $T = \{n \in \mathbb{N}; n \cdot 1 > a\}$ and $m = \min T$. We claim that $a < m < b$. If possible, let $m > b > a$. Then $m - a > b - a > 1 \Rightarrow m > a + 1 \Rightarrow m - 1 > a \Rightarrow m - 1 \in T$, this contradicts the minimality of m. Hence $a < m < b$. □

3.6.9 Problem. Let $x \in \mathbb{R}$. Prove that, for each $\epsilon > 0\ \exists\, r \in \mathbb{Q}$ such that $0 < |x - r| < \epsilon$.

3.6.9.1 Solution. By density principle $\exists\, r \in \mathbb{Q}$ such that $x < r < x + \epsilon \Rightarrow 0 < |x - r| < \epsilon$. □

3.6.10 Problem. If $0 < a < b$, then $\exists\, m, n \in \mathbb{Z}$ such that $a < 2^m 3^n < b$.

3.6.10.1 Solution. Suppose that $\frac{\ln 3}{\ln 2} = \frac{p}{q}, (p,q) = 1$ is rational. Then $q \ln 3 = p \ln 2 \Rightarrow 3^q = 2^p$, which is impossible. Thus $\frac{\ln 3}{\ln 2}$ is irrational, and by the previous problem, choose $m, n \in \mathbb{Z}$ such that $\frac{\ln a}{\ln 2} < m + n\frac{\ln 3}{\ln 2} < \frac{\ln b}{\ln 2} \Rightarrow a < 2^m 3^n < b$. □

3.6.11 Problem. Lef G be a nonempty subset of $\mathbb{R}$, which is a group under addition. Show that, either G is dense in $\mathbb{R}$ or G is a cyclic group, i.e. there exists $a \in \mathbb{R}$ such that $G = \{na; n = 0, \pm1, \pm2, ...\}$.

3.6.11.1 Solution. Assume $G \neq \{0\}$. Let $a = \inf G \cap (0, \infty)$. We distinguish two cases.

1. Case: $a > 0$. In this case, we shall show that $G = \{na; n = 0, \pm1, \pm2, ...\}$. Note first that $a \in G$. Indeed, if $a \in G$, then for $\epsilon = a/2$ there exist $x, y \in G$ with $a < x < y < \frac{3a}{2}$. Then, the element $z = y - x \in G$ satisfies $0 < z < \frac{a}{2} < a$, contradicting the definition of a. Now, if $x \in G$, and $na < x < (n+1)a$ for some integer n, then the element $x - na \in G$ satisfies $0 < x - na < a$, which is again a contradiction.

2. Case: $a = 0$. In this case, we claim that between any two distinct real numbers there is an element of G. To see this, we only need to consider $0 < x < y$. Let $d = \min\{x, y - x\} > 0$. Choose some element $z \in G$ with $0 < z < d$. By the Archimedean property, the set $A = \{n \in \mathbb{N}; nz > y\}$ is nonempty, and by the Well Ordering Principle the element $k = \min A$ exists. Now, note that the element $b = (k-1)z \in G$ satisfies $x < b < y$. □

3.6.12 Problem. Let α and β be real numbers such that the subgroup Γ of $\mathbb{R}$ generated by α and β is closed. Prove that α and β are linearly dependent over $\mathbb{Q}$.

3.6.12.1 Solution. If the origin were a limit point of Γ then Γ would be dense in $\mathbb{R}$, since Γ contains all integer multiples of its elements. But then Γ would equal $\mathbb{R}$, since it is closed, which is impossible because Γ is countable as a set $\Gamma = \{m\alpha + n\beta; m, n \in \mathbb{Z}\}$. Hence the origin is not a limit point of Γ. Therefore Γ contains a smallest positive number γ. If x is in Γ and n is the largest integer such that $n\gamma \leq x$, then $x - n\gamma$ is in Γ, and $0 \leq x - n\gamma < \gamma$, implying that $x - n\gamma = 0$. Therefore $\Gamma = \gamma\mathbb{Z}$, from which the desired conclusion is immediate. □

3.6.13 Problem. Let $\alpha \in \mathbb{Q}^C$ and define $x_n = n\alpha - [n\alpha]$, where $[x]$ denotes the greatest integer in x. Determine the cluster points of (x_n) by prodeeding as follows:

1. Show that the terms of the sequence are distinct, that is, $x_m = x_n$ implies $m = n$.
2. Prove that for each $\epsilon > 0$ and $N \in \mathbb{N}$, $\exists\, n > N$ such that $0 < x_n < \epsilon$.
3. Let $x \in [0, 1)$. Prove that for each $\epsilon > 0$ and $N \in \mathbb{N}$, $\exists\, n > N$ such that $|x_n - x| < \epsilon$.
4. Obtain the cluster points of (x_n).

3.6.13.1 Solution.

1. If possible, let $m \neq n$, then

$$\begin{aligned} & x_m = x_n \\ \Rightarrow & m\alpha - [m\alpha] = n\alpha - [n\alpha] \\ \Rightarrow & m\alpha - n\alpha = [m\alpha] - [n\alpha] \\ \Rightarrow & \alpha = \frac{[m\alpha] - [n\alpha]}{m - n} \in \mathbb{Q}, \end{aligned}$$

which is impossible. Hence $m = n$.

2. Since $\epsilon > 0$ $\exists\, m \in \mathbb{N}$ such that $\frac{1}{m} < \epsilon$. For $1 \leq k \leq m$, let $I_k = \left(\frac{k-1}{m}, \frac{k}{m}\right)$ and note that the I_k's are disjoint and their union is (0,1). Now we consider the set $S = \{x_j; j = N+1, 3N+1, .., (2m-1)N+1, (2m+1)N+1\}$ and observe that the set consists of $m+1$ points in (0,1). Thus by Pigeonhole principle $\exists\, x_p, x_q \in S$ such that they lie in one of the intervals, then

$$\begin{aligned} & |x_p - x_q| < \frac{1}{m} < \epsilon \\ \Rightarrow & |(pN+1)\alpha - [(pN+1)\alpha] - ((qN+1)\alpha + [(qN+1)\alpha])| < \epsilon \\ \Rightarrow & |(p-q)N\alpha - [(pN+1)\alpha] + [(qN+1)\alpha]| < \epsilon \\ \Rightarrow & |(p-q)N\alpha - [(p-q)N\alpha] + [(p-q)N\alpha] - [(pN+1)\alpha] + [(qN+1)\alpha]| < \epsilon \\ \Rightarrow & (p-q)N\alpha - [(p-q)N\alpha] < \epsilon \\ \Rightarrow & x_{(p-q)N} < \epsilon, \text{ where } (p-q)N > N. \end{aligned}$$

3. Choose an $m \in \mathbb{N}$ such that $\frac{1}{m} < x + \epsilon$. Apply part (2) to choose $n > N$ such that $0 < x_n < \frac{1}{m} < x + \epsilon$. Thus $|x_n - x| < \epsilon$.

4. Hence every point of [0,1] is a cluster point of (x_n). □

3.6.14 Problem. Let G be an open subset of $\mathbb{R}$.

1. If $0 \notin G$, then show that $H = \{xy; x, y \in G\}$ is an open subset of $\mathbb{R}$.

2. If $0 \in G$, and $x + y \in G \; \forall \; x, y \in G$, then show that $G = \mathbb{R}$.

3.6.14.1 Solution.

1. Is is clear that
$$H = GG = \bigcup_{a \in G} aG.$$
We show that aG is open. Let $x \in aG$, so $x = ag$ for some $g \in G$. As G is open, $\exists \; \epsilon > 0$ such that $(g - \epsilon, g + \epsilon) \subseteq G$, then $a(g - \epsilon, g + \epsilon) \subseteq aG$ implies $(ag - a\epsilon, ag + a\epsilon) \subseteq aG$ implies $(x - a\epsilon, x + a\epsilon) \subseteq aG$ shows that aG is open.

2. Let $x \in \mathbb{R}$, and $\epsilon > 0$. Since $0 \in G \; \exists \; n \in \mathbb{N}$ such that $\frac{x}{n} \in B(0; \epsilon)$ and $x = \frac{x}{n} + \frac{x}{n} + .. + \frac{x}{n}(n \; times) \in G$ implies $\mathbb{R} \subseteq G$. Hence $G = \mathbb{R}$. □

3.6.15 Problem. Let $\mathcal{F}$ be a family of (nondegenerate) intervals; that is, each member of $\mathcal{F}$ is an interval (open, closed or neither) but is not a single point. Suppose that any two intervals I and J in the family have no point in common. Show that the family $\mathcal{F}$ can be arranged in a sequence $I_1, I_2, ...$

3.6.15.1 Solution. Hint: Select a rational number from each member of the family and use that to place them in an order. □

3.6.16 Problem. Show that every uncountable set E of real numbers has a point of accumulation.

3.6.16.1 Solution. This uncountable set E might be unbounded. We prove that an uncountable set would have to contain an infinite bounded subset. Consider
$$E = \bigcup_{n=1}^{\infty} E_n \text{ where } E_n = E \cap [-n, n].$$
If each E_n is countable, then E would be countable, so at least one of E_n is uncountable. Hence by Bolzano-Weierstrass theorem E_n has an accumulation point. □

3.6.17 Problem. Is it true that a set, all of whose points are isolated, must be closed?

3.6.17.1 Solution. Yes. Since no point of the set is an accumulation point, so the derived set is empty. Thus the set is closed. □

3.6.18 Problem. If a set has no isolated points must it be closed? Must it be open?

3.6.18.1 Solution. No. The set $[0, 1)$ has no isolated points but the set is neither open nor closed. □

3.6.19 Problem. A careless student, when asked, incorrectly answers that a set is closed " if all its points are points of accumulation." Must such a set be closed?

3.6.19.1 Solution. False. Note that interior points are accumulation points. Consider the set $(2,3), [2,3)$ in which all its points are points of accumulation, but the sets are not closed. $\square$

3.6.20 Problem. A careless student, when asked, incorrectly answers that a set is open " if it contains all of its interior points." Is there an example of a set that fails to have this property? Is there an example of a non-open set that has this property?

3.6.20.1 Solution. False. Note that for every set $E \subseteq \mathbb{R}$, E is open iff $E^\circ = E$. Consider the sets $[1,2]$, and $[2,3)$, in these cases they contain all of its interior points, but $E^\circ \neq E$. $\square$

3.6.21 Problem. Give examples of closed sets that are countable and closed sets that are uncountable.

3.6.21.1 Solution. $\mathbb{Z}$ and $[0,1]$. $\square$

3.6.22 Problem. Is there a non-empty open subset of $\mathbb{R}$ that is countable?

3.6.22.1 Solution. Since non-empty open set contains an interval and each interval is uncountable, so non-empty open set is not countable. $\square$.

3.6.23 Problem. If a set is countable, what can you say about its complement?

3.6.23.1 Solution. For any set S, $S \cup S^C = \mathbb{R}$, so S^C is uncountable. $\square$

3.6.24 Problem. Is the intersection of two uncountable sets uncountable?

3.6.24.1 Solution. Consider the set $S = \bigcup_{i=1}^{\infty}[2i-1, 2i]$ and $T = \bigcup_{i=1}^{\infty}[2i, 2i+1]$, then $S \cap T = (\bigcup_{i=1}^{\infty}[2i-1, 2i]) \bigcap (\bigcup_{i=1}^{\infty}[2i, 2i+1]) = \{2, 3, 4, ...\}$ which is countable. Again, $(1,3) \cap (2,4) = (2,3)$ which is uncountable. $\square$

3.6.25 Problem. Give (if possible) an example of a set with

1. Countably many points of accumulation.
2. Uncountably many points of accumulation.
3. Countably many boundary points.
4. Uncountably many boundary points.
5. Countably many interior points.
6. Uncountably many interior points.

3.6.25.1 Solution.

1. Consider the set $S = \left\{\frac{1}{m} + \frac{1}{n}; m, n \in \mathbb{N}\right\}$.
2. Consider the set $S = [1,2]$.
3. Consider the set $\mathbb{N}$.
4. Consider the set $\mathbb{Q}$.
5. Impossible. If $a \in E \subseteq \mathbb{R}$ is an interior point, then it must contain an interval $(a-r, a+r)$ for some $r > 0$. Since $(a-r, a+r)$ is uncountable, so the result follows.

6. Consider the set $\mathbb{R}$. $\square$

3.6.26 Problem. A subset of $\mathbb{R}$ is said to be **co-countable** if it has a countable complement. Show that the intersection of finitely many co-countable sets is itself co-countable.

3.6.26.1 Solution. Let $A_1, A_2, ..., A_n$ be co-countable sets. Then

$$\left(\bigcap_{i=1}^{n} A_i\right)^C = \bigcup_{i=1}^{n} A_i^C$$

is the union of countable sets which is countable. Thus the intersection of finitely many co-countable sets is itself co-countable. $\square$

3.6.27 Problem. Show that there is no set with uncountably many isolated points.

3.6.27.1 Solution. Since the set of isolated points is countable, so there cannot exist a set with uncountably many isolated points. $\square$

3.6.28 Problem. Show that if $\alpha < x < \beta$ and $\alpha < y < \beta$, then $|x - y| < \beta - \alpha$ and interpret this geometrically as a statement about the interval (α, β).

3.6.28.1 Solution. Here,

$$\begin{aligned} \alpha < x &< \beta \\ \text{and } \alpha < y < \beta \Rightarrow -\beta &< -y < -\alpha \\ \text{by addition, } \alpha - \beta < x &- y < \beta - \alpha \\ \Rightarrow |x - y| &< \beta - \alpha \end{aligned}$$

The intervals (x, y) or (y, x) is contained in the interval (α, β). $\square$

3.6.29 Problem. Prove that every subset of $\mathbb{R}$ can be written as the intersection of open sets.

3.6.29.1 Solution. Let $A \subseteq \mathbb{R}$, then

$$\begin{aligned} A^C &= \bigcup_{x \in A^C} \{x\} \\ \Rightarrow A &= \bigcap_{x \in A^C} \{x\}^C \end{aligned}$$

Since $\{x\}$ is closed, so $\{x\}^C$ is open. $\square$

3.6.30 Problem (A sequential criterion for open sets). A set $E \subseteq \mathbb{R}$ is open iff $\forall$ (x_n) in $\mathbb{R}$ that converges to $x \in E$, the set $\{n; x_n \notin E\}$ is finite.

3.6.30.1 Solution. Let $E \subseteq \mathbb{R}$ be open and (x_n) be a sequence in $\mathbb{R}$ that converges to $x \in E$. So, $\exists$ $m \in \mathbb{N}$ such that $n \geq m \Rightarrow x_n \in E$ implies that at most m members lie outside the set E. Hence the set $\{n; x_n \notin E\}$ is finite.
If E is not open, then there is an $x \in E$ such that $B(x; \epsilon) \cap E^C \neq \emptyset$ $\forall$ $\epsilon > 0$. In particular, for each n there is some $x_n \in B\left(x; \frac{1}{n}\right) \cap E^C$. But then $\{x_n\} \subseteq E^C$ and $x_n \to x$. Thus, the condition fails. $\square$

3.6.31 Problem. For a subset $A \subseteq \mathbb{R}$, the following conditions are equivalent:

1. A is open;
2. if $x_n \to x$ and $x \in A$, then $x_n \in A$ ultimately,
3. if $x_n \to x$ and $x \in A$, then $x_n \in A$ frequently.

3.6.31.1 Solution. Use the above solution. □

3.6.32 Problem. For a subset $A \subseteq \mathbb{R}$, the following conditions are equivalent:

1. A is closed;
2. if $x_n \to x$ and $x_n \in A$ frequently, then $x \in A$;
3. if $x_n \to x$ and $x_n \in A$ ultimately, then $x \in A$.

3.6.32.1 Solution. Left to the reader. □

3.6.33 Problem. The following are equivalent:

1. $A' \subseteq A$.
2. $\mathbb{R} \setminus A = A^C$ is open.

3.6.33.1 Solution.

1. (1) $\Rightarrow$ (2) : Let $A' \subseteq A$ and $p \in A^C$. Then $p \notin A \Rightarrow p$ is not a limit point of A. Hence $\exists\ \epsilon > 0$ such that $B(p;\epsilon) \cap A = \emptyset \Rightarrow B(p;\epsilon) \subseteq \mathbb{R} \setminus A \Rightarrow p$ is an interior point of A^C. Thus every point of A^C is an interior point of it. So A^C is open.
2. (2) $\Rightarrow$ (1) : Let A^C be open. Suppose $p \in A'$ but $p \notin A$. Then $p \in A^C$. Now, A^C is open implies p is an interior point of $A^C \Rightarrow\ \exists\ \epsilon > 0$ such that $B(p;\epsilon) \subseteq A^C \Rightarrow B(p;\epsilon) \cap A = \emptyset \Rightarrow p$ is not a limit point of A, a contradiction. Thus $p \in A$. □

3.6.34 Problem. Any finite subset of $\mathbb{R}$ is closed.

3.6.34.1 Solution. 1. Suppose that $S = \{x_1, x_2, ..., x_n\}$ be any finite subset of $\mathbb{R}$. Let $x \in \mathbb{R}$, with $x \neq x_i\ \forall\ i = 1, 2, .., n$, then taking $\epsilon < \min_{1 \leq i \leq n} |x - x_i|$, we get $B(x;\epsilon) \cap S = \emptyset$. For, if $x_k \in B(x;\epsilon)$ then $|x - x_k| < \epsilon$, a contradiction. If $x = x_i$ for some i, then take $\epsilon < \min_{1 \leq j \leq n}\{|x_i - x_j|; j \neq i\}$, then again $\hat{B}(x_i;\epsilon) \cap S = \emptyset$. Thus no member of $\mathbb{R}$ can be a limit point of S and hence $S' = \emptyset \subseteq S$ shows that S is closed.

3.6.34.2 Solution. 2. Since $S = \{x_1, x_2, ..., x_n\}$ is finite, without loss of generality, we assume that $x_1 < x_2 < ..... < x_n$. Then $S^C = (-\infty, x_1) \cup (x_1, x_2) \cup (x_2, x_3) \cup .. \cup (x_n, \infty)$ is the union of open intervals is an open set. Thus S is closed. □

3.6.35 Problem. If A is a finite subset of $\mathbb{R}$, prove that any point in A is an accumulation point of $\mathbb{R} \setminus A$.

3.6.35.1 Solution. Suppose that $A = \{x_1, x_2, ..., x_n\}$. Then, for every $\epsilon > 0$, the ball $B(x_i;\epsilon) \cap (\mathbb{R} \setminus A) \neq \emptyset$, $\forall\ i = 1, 2, .., n$. □

Let $A \subseteq \mathbb{R}$. For the following problems, we denote the interior of A by A° or $int(A)$, Derived set of A by A' or $d(A)$, and closure of A by $\overline{A}$ or $cl(A)$, boundary of A by ∂A or by $bd(A)$. Exterior of a set A by $Ext(A)$.

3.6.36 Problem. For every subset $A \subseteq \mathbb{R}$, define $\alpha(A) = \left(\overline{A}\right)^{\circ}$ and $\beta(A) = \overline{(A^{\circ})}$.

1. $\alpha\alpha(A) \subseteq \alpha(A), \beta\beta(A) \supseteq \beta(A)$.

2. Show that if A is open, then $A \subseteq \alpha(A)$ and that if A is closed, then $\beta(A) \subseteq A$.

3. Using (1), show that we always have $\alpha(\alpha(A)) = \alpha(A)$ and $\beta(\beta(A)) = \beta(A)$.

4. Give an example $A \subseteq \mathbb{R}$ such that $A, A^{\circ}, \overline{A}, \alpha(A), \beta(A), \alpha(A^{\circ})$, and $\beta(\overline{A})$, are all distinct.

3.6.36.1 Solution.

1.

$$\begin{aligned}
&int(\overline{A}) \subseteq \overline{A} \\
\Rightarrow &\alpha(A) \subseteq \overline{A} \\
\Rightarrow &\overline{\alpha(A)} \subseteq \overline{A} \\
\Rightarrow &int(\overline{\alpha(A)}) \subseteq int\overline{A} \\
\Rightarrow &\alpha\alpha(A) \subseteq \alpha(A).
\end{aligned}$$

Again,

$$\begin{aligned}
&intA \subseteq \overline{intA} = \beta(A) \\
\Rightarrow &intA \subseteq int\beta(A) \\
\Rightarrow &\overline{intA} \subseteq \overline{int\beta(A)} \\
\Rightarrow &\beta(A) \subseteq \beta\beta(A).
\end{aligned}$$

2. Let A be open, then $A^{\circ} = A$, so

$$\begin{aligned}
&A \subseteq \overline{A} \\
\Rightarrow &A^{\circ} \subseteq \left(\overline{A}\right)^{\circ} = \alpha(A) \\
\Rightarrow &A \subseteq \alpha(A).
\end{aligned}$$

Again, let A be closed, then $\overline{A} = A$, so

$$\begin{aligned}
&A^{\circ} \subseteq A \\
\Rightarrow &\overline{A^{\circ}} \subseteq \overline{A} = A \\
\Rightarrow &\beta(A) \subseteq A.
\end{aligned}$$

3. By definition, $\alpha(A)$ is open, so $\alpha(A) \subseteq \alpha(\alpha(A))$ by (2). Using (1), we get $\alpha(A) = \alpha\alpha(A)$. Similarly the other follows.

4. Let $A = \left\{\frac{1}{n}; n \in \mathbb{N}\right\} \cup [2,3) \cup (3,4) \cup \left([5,6] \cap \mathbb{Q}^C\right)$. Then

$$\begin{aligned}
&A^\circ = (2,3) \cup (3,4) \subset A,\\
&\overline{A} = \{0\} \cup [2,4] \cup [5,6] \supset A,\\
&\alpha(A) = \left(\overline{A}\right)^\circ = (2,4) \cup (5,6) \neq A,\\
&\beta(A) = \overline{(A^\circ)} = [2,4] \neq A,\\
&\alpha(A^\circ) = \left(\overline{(A^\circ)}\right)^\circ = (2,4) \neq A,\\
&\beta\left(\overline{A}\right) = \overline{\left(\left(\overline{A}\right)^\circ\right)} = [2,4] \cup [5,6] \neq A. \quad \square
\end{aligned}$$

3.6.37 Problem. A point $a \in A$ is called **isolated** whenever $a \in A \setminus A'$. A set is called **perfect** if it is closed and has no isolated points. Prove:

1. If A has no isolated points, then $\overline{A}$ is perfect.
2. Every open set and every dense set in $\mathbb{R}$ also have no isolated points.

3.6.37.1 Solution.

1. If A has no isolated points, so $A \setminus A' = \emptyset \Rightarrow A \subseteq A'$ then $\overline{A}$ is closed and $\left(\overline{A}\right)' = (A \cup A')' = A' \cup A'' = A' = A \cup A' = \overline{A}$ implies $\overline{A}$ ontains no isolated points and hence perfect.
2. Every point in an open set A is a limit point of A, thus an open set cannot contain isolated points.
   Since every point in a dense set in $\mathbb{R}$ is a limit point of $\mathbb{R}$, so dense set has no isolated points. $\square$

3.6.38 Problem. A subset $S \subseteq \mathbb{R}$ is said to be a **discrete** set if $S' = \emptyset$. A subset $S \subseteq \mathbb{R}$ is said to be an **isolated** set if $S \cap S' = \emptyset$.

1. Every discrete set is an isolated set, but not conversely.
2. Give an example of an infinite discrete subset of $\mathbb{R}$.
3. Give an example of a bounded discrete subset of $\mathbb{R}$.
4. Can there be an infinite bounded discrete subset of $\mathbb{R}$?

3.6.38.1 Solution.

1. Let S be a discrete set $\Rightarrow S' = \emptyset \Rightarrow S \cap S' = \emptyset \Rightarrow S$ is isolated.
2. Consider $S = \mathbb{N}$.
3. Consider any finite subset of $\mathbb{R}$.
4. No. Because Bolzano-Weierstrass theorem "every bounded infinite subset of $\mathbb{R}$ has a limit point" implies $S' \neq \emptyset$. $\square$

3.6.39 Problem.

1. Give an example of a family $\mathcal{A} = \{I_n \subseteq \mathbb{R}, I_n \text{ is a closed interval}; n \in \mathbb{N}\}$ such that $I_1 \supseteq I_2 \supseteq I_3 \supseteq ...$, and $\bigcap_{n=1}^{\infty} I_n = \emptyset$.

2. Give an example of a family $\mathcal{A} = \{A_n \subseteq \mathbb{R}; n \in \mathbb{N}\}$ where A_n is a bounded open interval such that $I_1 \supseteq I_2 \supseteq I_3....$ and $\bigcap_{n=1}^{\infty} I_n = \emptyset$.

3.6.39.1 Solution.

1. Consider $A_n = [n, \infty)$
2. Consider $A_n = (0, \frac{1}{n})$

3.6.40 Problem. Prove the following and give examples of each where equality does not occur.

1. For any set A, $A^\circ \subseteq A$.
2. $A \subseteq B \Rightarrow A^\circ \subseteq B^\circ$.
3. $(A \cup B)^\circ \supseteq A^\circ \cup B^\circ$.
4. $(A \cap B)^\circ = A^\circ \cap B^\circ$.
5. A is open iff $A^\circ = A$.
6. $(A^\circ)^\circ = A^\circ$.
7. A° is the largest open set contained in A.
8. $(A \setminus B)^\circ \subseteq A^\circ \setminus B^\circ$.

3.6.40.1 Solution.

1. $x \in A^\circ \Rightarrow \exists\, r > 0$ such that $B(x; r) \subseteq A \Rightarrow x \in A$.
2. $x \in A^\circ \Rightarrow \exists\, r > 0$ such that $B(x; r) \subseteq A \subseteq B \Rightarrow x \in B^\circ$.
3. $A \subseteq A \cup B$ and $B \subseteq A \cup B \Rightarrow A^\circ \cup B^\circ \subseteq (A \cup B)^\circ$.
   Let $A = [1, 2], B = [2, 3]$. Then $A^\circ \cup B^\circ \subset (A \cup B)^\circ$.
4. $A \cap B \subseteq A$ and $A \cap B \subseteq B \Rightarrow (A \cap B)^\circ \subseteq A^\circ \cap B^\circ$.
   On the other hand, $x \in A^\circ \cap B^\circ \Rightarrow \exists\, r_1, r_2 > 0$ such that $B(x; r_1) \subseteq A$ and $B(x; r_2) \subseteq B$ then $B(x; r) \subseteq A \cap B$ where $r = \min\{r_1, r_2\}$, which shows that $x \in (A \cap B)^\circ$. Thus $(A \cap B)^\circ = A^\circ \cap B^\circ$.
5. Now, A is open if and only if every point of A is an interior point of A if and only if $A = \cup_{x \in A}\{N_x; N_x \text{ is a nbhd. of } x \text{ with } N_x \subseteq A\} = A^\circ$.
6. Since A° is open, so $(A^\circ)^\circ = A^\circ$.
7. Let B be any open subset in A, then $B = B^\circ \subseteq A^\circ$. Thus A° is the largest open set contained in A.
8. Let $x \in (A \setminus B)^\circ \Rightarrow \exists\, r > 0$ such that $B(x; r) \subseteq A \setminus B$, so $B(x; r) \subseteq A$ and $B(x; r) \subseteq B^C$ implies $x \in A^\circ$ and $x \notin B^\circ \Rightarrow x \in A^\circ \setminus B^\circ$. □

3.6.41 Problem. Prove the following and give examples of each where equality does not occur.

1. $A \subseteq B \Rightarrow A' \subseteq B'$. Give an example that $A' \subseteq B'$ does not imply $A \subseteq B$.
2. $(A \cup B)' = A' \cup B'$.

3. $(A \cap B)' \subseteq A' \cap B'$.
4. $(A')' \subseteq A'$.
5. $A' \setminus B' \subseteq (A \setminus B)'$.
6. If $A \subseteq B$ and $B \setminus A$ is finite, then $A' = B'$.

3.6.41.1 Solution.

1. Let $A \subseteq B$ and $x \in A'$, then $\forall\, r > 0$ $\hat{B}(x;r) \cap A \neq \emptyset$ implies $\hat{B}(x;r) \cap B \neq \emptyset$. So $x \in B'$. Hence $A' \subseteq B'$. Note that, for the sets $A = (1,3) \cup \{4\}$ and $B = (1,2) \cup (2,3)$; $A' \subseteq B'$ but $A \nsubseteq B$.
2. $A \subseteq A \cup B$ and $B \subseteq A \cup B \Rightarrow A' \cup B' \subseteq (A \cup B)'$.
Again, let $x \in (A \cup B)'$. Then $\forall\, r > 0$ $\hat{B}(x;r) \cap (A \cup B) \neq \emptyset$ implies $\hat{B}(x;r) \cap A \neq \emptyset$ or $\hat{B}(x;r) \cap B \neq \emptyset$. So $x \in A'$ or $x \in B'$. Hence $(A \cup B)' \subseteq A' \cup B'$. Thus $(A \cup B)' = A' \cup B'$.
3. $A \cap B \subseteq A$ and $A \cap B \subseteq B \Rightarrow (A \cap B)' \subseteq A' \cap B'$.
Let $A = (1,2), B = (2,3)$. Then $(A \cap B)' \subset A' \cap B'$.
4. Let $x \in (A')'$. Then $\forall\, r > 0$ $\hat{B}(x;r) \cap A' \neq \emptyset$. So $\exists\, y \in \hat{B}(x;r)$ and $y \in A'$. As y is an interior point of $\hat{B}(x;r)$ and y is a limit point of A, so $\exists\, r_1 > 0$ such that $\hat{B}(y;r_1) \subseteq \hat{B}(x;r)$ and $\hat{B}(y;r_1) \cap A \neq \emptyset$. Hence $\forall\, r > 0$ $\hat{B}(x;r) \cap A \neq \emptyset \Rightarrow\ x \in A'$. Thus $(A')' \subseteq A'$.
5. For the sets $A, B \subseteq \mathbb{R}$ two cases can arise
(i) $A \cap B = \emptyset$
(ii) $A \cap B \neq \emptyset$. Now, case (i) implies $A \setminus B = A$. So, if $x \in A' \setminus B'$, then $\forall\, \epsilon > 0,\ \hat{B}(x;\epsilon) \cap A \neq \emptyset$ implies $\hat{B}(x;\epsilon) \cap (A \setminus B) \neq \emptyset$. Thus $A' \setminus B' \subseteq (A \setminus B)'$.
Case (ii): Let
$$\begin{aligned} & x \in A' \setminus B' \\ \Rightarrow & \exists\, \epsilon_1, \epsilon_2 > 0 \text{ such that } \hat{B}(x;\epsilon_1) \cap A \neq \emptyset \text{ and } \hat{B}(x;\epsilon_2) \cap B = \emptyset \\ \Rightarrow & \hat{B}(x;\epsilon) \cap A \neq \emptyset \text{ and } \hat{B}(x;\epsilon) \cap B = \emptyset \text{ where } \epsilon = \min\{\epsilon_1, \epsilon_2\} \\ \Rightarrow & \hat{B}(x;\epsilon) \cap (A \setminus B) \neq \emptyset \Rightarrow A' \setminus B' \subseteq (A \setminus B)'. \end{aligned}$$
6. $B \setminus A$ is finite implies $(B \setminus A)' = \emptyset$. Thus $B = A \cup (B \setminus A) \Rightarrow B' = A' \cup (B \setminus A)' \Rightarrow B' = A'$. □

3.6.42 Problem. Give an example of each of the following or explain why you think such a set could not exist.

1. A nonempty set E such that $E' = E$.
2. A nonempty set E such that $E' = \emptyset$.
3. A nonempty set E such that $E' \neq \emptyset$ but $E'' = \emptyset$.
4. A nonempty set E such that $E', E'' \neq \emptyset$ but $E''' = \emptyset$.
5. A nonempty set E such that E', E'', E''' are all different.
6. A nonempty set E such that $(E \cup E')' \neq E \cup E'$.

7. Is it possible for a non-empty perfect set to be nowhere dense?

3.6.42.1 Solution.

1. $E = [a, b] \Rightarrow E' = E$.
2. $E = \{1, 2, 3, 4\} \Rightarrow E' = \emptyset$
3. $E = \left\{1, \frac{1}{2}, \frac{1}{3}, .., \frac{1}{n}..\right\} \Rightarrow E' = \{0\} \neq \emptyset$ but implies $E'' = \emptyset$.
4. $E = \left\{\frac{1}{m} + \frac{1}{n}; m, n \in \mathbb{N}\right\}$ then $E' = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, .., \frac{1}{n}..\right\}$ so, $E'' = \{0\} \neq \emptyset$ but $E''' = \emptyset$.
5. $E = \left\{\frac{1}{m} + \frac{1}{n} + \frac{1}{p}; m, n, p \in \mathbb{N}\right\}$ then $E' = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, .., \frac{1}{n}..\right\} \cup \left\{\frac{1}{m} + \frac{1}{n}; m, n \in \mathbb{N}\right\}$ and $E'' = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, .., \frac{1}{n}..\right\}$ so, $E''' = \{0\}$.
6. (a) Let $E = \{1\} \cup (2, 3)$ then $E' = [2, 3]$, Thus $E \cup E' = \{1\} \cup [2, 3]$ and $(E \cup E')' = [2, 3] \neq E \cup E'$.
   (b) $E = \left\{\frac{1}{n}; n \in \mathbb{N}\right\}$ Then $E' = \{0\}$. So $E \cup E' = \{0\} \cup E$, and $(E \cup E')' = \{0\}$. Hence $(E \cup E')' \neq E \cup E'$.
7. The answer is affirmative. Cantor set is a perfect nowhere dense set.(see Cantor set in the special topics.) □

3.6.43 Problem. Consider the set $A = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{2} + \frac{1}{3}, \frac{1}{4}, \frac{1}{2} + \frac{1}{4}, \frac{1}{5}, \frac{1}{2} + \frac{1}{5}, ...\right\}$. Find E', E'', E'''.

3.6.43.1 Solution. For the subset $B = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, ...\right\}$, the set $B' = \{0\}$ and for the subset $C = \left\{\frac{1}{2}, \frac{1}{2} + \frac{1}{3}, , \frac{1}{2} + \frac{1}{4}, \frac{1}{2} + \frac{1}{5}, ...\right\}$, $C' = \left\{\frac{1}{2}\right\}$. Then $A' = (B \cup C)' = B' \cup C' = \left\{0, \frac{1}{2}\right\}$.
Again, for the set

$$A = \left\{1, x, x^2, x + x^2, x^3, x + x^3, x^2 + x^3, x^4, x + x^4, x^2 + x^4, x^3 + x^4, x^5.....\right\}$$

where $x = \frac{1}{2}$, then $A' = \left\{1, x, x^2, x^3, x^4, x^5.....\right\}$, $A'' = \{0\}$ and $A''' = \emptyset$.

3.6.1 Note. If a finite number of derived sets exist, then A is said to be of the **first species.** A set may have an infinite number of derived sets, and in this case it is said to be set of the **second species.**
The set of all rational numbers in the interval (0,1) is an illustration of a set of the second species. In this case, the first derived set contains all the real numbers in the given interval and each succeeding derived set is identical with the first.

3.6.44 Problem. Give an example of a set that has the set $\mathbb{N}$ as its set of accumulation points.

3.6.44.1 Solution. Let $E = \left\{m + \frac{1}{n}; m, n \in \mathbb{N}\right\}$ then $E' = \mathbb{N}$. □

3.6.45 Problem.

1. Show that there is no set which has the interval (0,1) as its set of accumulation points.
2. Show that there is no set which has the set $\mathbb{Q}$ as its set of accumulation points.

3.6.45.1 Solution. Since the set of accumulation points of a set is a closed set but the sets $(0, 1)$ and $\mathbb{Q}$ are not closed, hence they cannot be the set of accumulation points. □

3.6.46 Problem. Show that every accumulation point of a set that does not itself belong to the set must be a boundary point of that set.

3.6.46.1 Solution. Let $a \in A'$ but $a \notin A \subseteq \mathbb{R}$ implies $a \in A^C$ thus every nbhd. of a intersects both A and A^C. Hence a is a boundary point. $\square$

3.6.47 Problem. Prove the following and give examples of each where equality does not occur.

1. $A \subseteq B \Rightarrow \overline{A} \subseteq \overline{B}$.
2. $\overline{A} \setminus \overline{B} \subseteq \overline{A \setminus B}$. Give an example where the first set properly contains the second.
3. $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
4. $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$. Give examples of open sets A and B in the real line such that the four sets $A \cap \overline{B}, \overline{A} \cap B, \overline{A \cap B}, \overline{A} \cap \overline{B}$ are all different.
5. $\overline{\overline{A}} = \overline{A}$.
6. A is closed iff $A = \overline{A}$.
7. $\overline{A} = \bigcap\{B; B \text{ is closed and } B \supseteq A\}$. In other words, $\overline{A}$ is the smallest closed set containing A.

3.6.47.1 Solution.

1. Since $A \subseteq B \Rightarrow A' \subseteq B'$, so $A \cup A' \subseteq B \cup B' \Rightarrow \overline{A} \subseteq \overline{B}$. Again, for the sets $A = [1, 2], B = (1, 2) \Rightarrow \overline{A} \subseteq \overline{B}$ but $A \nsubseteq B$.
2. Let $x \in \overline{A} \setminus \overline{B}$, then $x \in \overline{A}$ and $x \notin \overline{B}$, so for all $r > 0$, $B(x; r) \setminus \overline{B}$ is an open set containing x, since $x \in \overline{A}$, so $\big(B(x; r) \setminus \overline{B}\big) \cap A \neq \emptyset$ and since $B(x; r) \setminus \overline{B} \subseteq B(x; r) \setminus B$, it follows that
$$B(x; r) \cap (A \setminus B) = (B(x; r) \setminus B) \cap A \neq \emptyset.$$
Hence $x \in \overline{A \setminus B}$. Thus $\overline{A} \setminus \overline{B} \subseteq \overline{A \setminus B}$. For the example: Take $A = [-1, 1]$ and $B = \{0\}$.
3. Now,
$$\begin{aligned}\overline{A \cup B} &= (A \cup B) \cup (A \cup B)' \\ &= (A \cup B) \cup (A' \cup B') \\ &= (A \cup A') \cup (B \cup B') \\ &= \overline{A} \cup \overline{B}.\end{aligned}$$
4. We get, $A \cap B \subseteq A$ and $A \cap B \subseteq B \Rightarrow \overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$.
Hint: $A = (1, 3) \cup (3, 5)$ and $B = (0, 1) \cup (4, 6)$
5. Here
$$\begin{aligned}\overline{\overline{A}} &= \overline{A} \cup (\overline{A})' \\ &= A \cup A' \cup (A \cup A')' \\ &= A \cup A' \cup A' \cup (A')' \\ &\subseteq A \cup A' = \overline{A}, \quad \text{as } (A')' \subseteq A'.\end{aligned}$$
$$\text{Again,} \quad A \subseteq \overline{A} \Rightarrow \overline{A} \subseteq \overline{\overline{A}}.$$
Thus,, $\overline{\overline{A}} = \overline{A}$.

6. A is closed $\Leftrightarrow A' \subseteq A \Leftrightarrow \overline{A} = A \cup A' = A$.

7. Let $\mathfrak{B} = \{B; A \subseteq B, \text{ and } B \text{ is closed}\}$. Since $\overline{A}$ is closed and $A \subseteq \overline{A}$, and $A \subseteq B \Rightarrow \overline{A} \subseteq \overline{B} = B$, so $\overline{A} \in \mathfrak{B}$. Thus $\overline{A} \subseteq B\ \forall\ B \in \mathfrak{B} \Rightarrow \overline{A} \subseteq \bigcap\{B \in \mathfrak{B}\}$ and $\overline{A} \in \mathfrak{B}$ implies $\bigcap\{B \in \mathfrak{B}\} \subseteq \overline{A}$. Thus $\overline{A} = \bigcap\{B \in \mathfrak{B}\}$. Hence the result follows and we can conclude that $\overline{A}$ is the smallest closed set containing A. □

3.6.48 Problem. Show that A is open if and only if the following holds for every set $B \subseteq \mathbb{R}$:

$$A \cap \overline{B} \subseteq \overline{A \cap B}.$$

3.6.48.1 Solution. Suppose that A is open. Let

$$\begin{aligned}
&x \in A \cap \overline{B} \text{ and } r > 0 \\
\Rightarrow &\exists\ s > 0 \text{ such that } 0 < s < r, \text{ with} \\
&B(x; s) \subseteq A \text{ and } B(x; r) \cap B \neq \emptyset \\
\Rightarrow &\forall\ r > 0, B(x; r) \cap A \cap B \neq \emptyset \\
\Rightarrow &x \in \overline{A \cap B}.
\end{aligned}$$

For the converse, the realation holds also, in particular, for $B = A^C$. Then

$$\begin{aligned}
&A \cap \overline{A^C} \subseteq \overline{A \cap A^C} \\
\Rightarrow &A \cap \overline{A^C} \subseteq \emptyset \\
\Rightarrow &\overline{A^C} \subseteq A^C \Rightarrow A^C = \overline{A^C}
\end{aligned}$$

Thus A^C is closed, hence A is open. □

3.6.49 Problem. Prove: A is open in $\mathbb{R}$ if and only if

$$\overline{A \cap \overline{B}} = \overline{A \cap B}$$

for every $B \subseteq \mathbb{R}$.

3.6.49.1 Solution. By the previous problem, A is open implies

$$\begin{aligned}
&A \cap \overline{B} \subseteq \overline{A \cap B} \\
\Rightarrow &\overline{A \cap \overline{B}} \subseteq \overline{A \cap B} \qquad (3.1) \\
\text{and for the other inclusion, } &A \cap B \subseteq A \cap \overline{B} \Rightarrow \overline{A \cap B} \subseteq \overline{A \cap \overline{B}}. \qquad (3.2)
\end{aligned}$$

Thus (3.1) and (3.2) yield the result. □

3.6.50 Problem. Show that A is open if and only if the following holds for every set $B \subseteq \mathbb{R}$,

$$A \cap B = \emptyset \Rightarrow A \cap \overline{B} = \emptyset.$$

3.6.50.1 Solution. A is open implies A^C is closed. Thus

$$\begin{aligned}
&A \cap B = \emptyset \\
\Rightarrow &B \subseteq A^C \\
\Rightarrow &\overline{B} \subseteq A^C \\
\Rightarrow &A \cap \overline{B} \subseteq A \cap A^C = \emptyset \\
\Rightarrow &A \cap \overline{B} = \emptyset.
\end{aligned}$$

For the converse, the relation holds also, in particular, for $B = A^C$. Then

$$\begin{aligned}
&A \cap \overline{A^C} = A \cap A^C = \emptyset \\
\Rightarrow &\overline{A^C} \subseteq A^C \Rightarrow A^C = \overline{A^C}.
\end{aligned}$$

Thus A^C is closed, hence A is open. $\square$

3.6.51 Problem. Find two intervals $A, B \subseteq \mathbb{R}$ such that $A \cap \overline{B} \subsetneq \overline{A \cap B}$.

3.6.51.1 Solution. Take $A = (1, 3), B = (2, 4)$. Then $A \cap \overline{B} = (2, 3)$ and $\overline{A \cap B} = [2, 3]$. $\square$

3.6.52 Problem. Let $D \subseteq \mathbb{R}$ be a dense subset. Show that, for any open set $G \subseteq \mathbb{R}$; we have $G \subseteq \overline{D \cap G}$.

3.6.52.1 Solution. If possible, let $\exists\ g \in G$ such that $g \notin \overline{G \cap D}$, which implies $\exists\ r > 0$ such that $B(g; r) \cap G \cap D = \emptyset$. Again, since $B(g; r) \cap G$ is open and D is dense, so $B(g; r) \cap G \cap D \neq \emptyset$, a contradiction. $\square$

3.6.53 Problem. Prove that for every closed set F in $\mathbb{R}$ and every $A \subseteq \mathbb{R}$, we have $(F \cup A^\circ)^\circ = (F \cup A)^\circ$.

3.6.53.1 Solution. For every closed set F and every $A \subseteq \mathbb{R}$ we get,

$$\begin{aligned}
&A^\circ \subseteq A \\
&\Rightarrow F \cup A^\circ \subseteq F \cup A \\
&\Rightarrow (F \cup A^\circ)^\circ \subseteq F \cup A^\circ \subseteq F \cup A \\
&\Rightarrow (F \cup A^\circ)^\circ \subseteq (F \cup A)^\circ. \qquad (1) \\
\text{Again, let } &x \in (F \cup A)^\circ \\
&\Rightarrow \exists\ \epsilon > 0 \text{ such that } B(x; \epsilon) \subseteq F \cup A \\
&\Rightarrow B(x; \epsilon) \setminus F \subseteq A.
\end{aligned}$$

Then, either $x \in F$ or $x \in A$. If $x \in F$ then choose $\epsilon_1 < \inf\{|x - a|; a \in A\}$ and $\epsilon_2 = \min\{\epsilon, \epsilon_1\}$. Thus $B(x; \epsilon_2) \subseteq F$ implies $x \in F^\circ$. If $x \in A$ then $B(x; \epsilon) \setminus F \subseteq A$ implies $x \in A^\circ$, as F is closed and $B(x; \epsilon) \setminus F$ is open containing x. Thus $(F \cup A)^\circ \subseteq F^\circ \cup A^\circ \subseteq F \cup A^\circ$ Hence $(F \cup A)^\circ \subseteq (F \cup A^\circ)^\circ$ and using (1), we get the required result. $\square$

3.6.54 Problem. Prove that $[a, b]^\circ = (a, b) = \bigcup_{n=1}^{\infty} \left[a + \frac{1}{n}, b - \frac{1}{n}\right]$.

3.6.54.1 Solution.

$$\left[a+\frac{1}{n}, b-\frac{1}{n}\right] \subseteq (a,b) \Rightarrow \bigcup_{n=1}^{\infty}\left[a+\frac{1}{n}, b-\frac{1}{n}\right] \subseteq (a,b).$$

Again, let

$$\begin{aligned}
&x \in (a,b)\\
\Rightarrow &x-a>0, b-x>0\\
&\text{by Archimedian principle } \exists\, n_1, n_2 \in \mathbb{N} \text{ such that } n_1(x-a)>1\\
&\text{and } n_2(b-x)>1\\
\Rightarrow &x > a+\frac{1}{n_1} \text{ and } x < b-\frac{1}{n_2} \Rightarrow\ x \in \left(a+\frac{1}{n_1}, b-\frac{1}{n_2}\right).
\end{aligned}$$

Let $n_3 = \max\{n_1, n_2\}$. Hence

$$\begin{aligned}
a+\frac{1}{n_3} \le a+\frac{1}{n_1} < x < b-\frac{1}{n_2} \le b-\frac{1}{n_3}\\
\Rightarrow x \in \left[a+\frac{1}{n_3}, b-\frac{1}{n_3}\right] \subseteq \bigcup_{n=1}^{\infty}\left[a+\frac{1}{n}, b-\frac{1}{n}\right].
\end{aligned}$$

Thus $[a,b]^\circ = (a,b) = \bigcup_{n=1}^{\infty}\left[a+\frac{1}{n}, b-\frac{1}{n}\right]$. □

3.6.55 Problem. Prove that $\overline{(a,b)} = [a,b] = \bigcap_{n=1}^{\infty}\left(a-\frac{1}{n}, b+\frac{1}{n}\right)$.

3.6.55.1 Solution. Left to the reader. □

3.6.56 Problem. Prove that

$$\overline{A} = \bigcap_{n=1}^{\infty}\left(\bigcup_{x\in A} B\left(x; \frac{1}{n}\right)\right).$$

3.6.56.1 Solution. For each $n \in \mathbb{N}$, denote $\bigcup_{x\in A} B\left(x; \frac{1}{n}\right)$ by $B\left(A; \frac{1}{n}\right) = C_n$ (say), then $C_n \supseteq C_{n+1}$ is a decreasing sequence and each C_n contains A. We show that

$$\overline{A} = \bigcap_{n=1}^{\infty} C_n$$

Let $y \in \overline{A}$, then $\forall\ n \in \mathbb{N}\ \exists\ y_n \in B\left(y; \frac{1}{n}\right) \cap A \neq \emptyset$ and $y \in B\left(y_n; \frac{1}{n}\right) \subseteq B\left(A; \frac{1}{n}\right) = C_n$ Thus $\overline{A} \subseteq \bigcap_{n=1}^{\infty}\left(\bigcup_{x\in A} B\left(x; \frac{1}{n}\right)\right)$.
Again, let $x \in \bigcap_{n=1}^{\infty} B\left(A; \frac{1}{n}\right)$ and $\epsilon > 0$, so $\exists\ n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$, then $x \in B\left(x; \frac{1}{n}\right) \subseteq B(x,\epsilon)$. Now,

$$\begin{aligned}
x \in B\left(A; \frac{1}{n}\right) &\Rightarrow \exists\ y_n \in A \text{ such that } y_n \in B\left(x; \frac{1}{n}\right)\\
\Rightarrow y_n \in A \cap B\left(x; \frac{1}{n}\right) &\subseteq A \cap B(x,\epsilon) \Rightarrow x \in \overline{A} \text{ and } \bigcap_{n=1}^{\infty} B\left(A; \frac{1}{n}\right) \subseteq \overline{A}.
\end{aligned}$$

Hence the result follows. □

3.6.57 Problem. Prove the following and give examples of each where equality does not occur.

1. If A is open, then $A \subseteq (\overline{A})^\circ$.
2. If A is closed, then $\overline{(A^\circ)} \subseteq A$.
3. $\bigcup A_\alpha^\circ \subseteq \left(\bigcup A_\alpha\right)^\circ$;
4. $\bigcap A_\alpha^\circ \supseteq \left(\bigcap A_\alpha\right)^\circ$;
5. $\bigcup_\alpha \overline{A_\alpha} \subseteq \overline{\bigcup_\alpha A_\alpha}$, equality holds if the family $\mathcal{A} = \{A_\alpha; \alpha \in \Lambda\}$ is locally finite.
6. $\bigcap_\alpha \overline{A_\alpha} \supseteq \overline{\bigcap_\alpha A_\alpha}$.
7. $A^\circ \subseteq A \subseteq \overline{A}$.
8. $A^\circ \subseteq \overline{A^\circ} \subseteq \overline{A}$;
9. $A^\circ \subseteq (\overline{A})^\circ \subseteq \overline{A}$;

3.6.57.1 Solution.

1. A is open implies $A = A^\circ$. Since $A \subseteq \overline{A}$, so $A = A^\circ \subseteq \overline{A}^\circ$.
   Example: Let $A = (1,2) \cup (2,3)$. Then $A \subset \overline{A}^\circ$.
2. A is closed implies $A = \overline{A}$. Since $A^\circ \subseteq A$, so $\overline{A^\circ} \subseteq \overline{A} = A$.
   Example: Let $A = \{1\} \cup [2,3]$, then $\overline{(A^\circ)} \subset A$.
3. left to the reader.
   Example: $A_n = [n, n+1)$.
4. left to the reader.
   Example: $A_n = \left[-\frac{1}{n}, \frac{1}{n}\right]$.
5. We have

$$\begin{aligned} & A_\alpha \subseteq \bigcup_{\alpha\in\Lambda} A_\alpha \\ \Rightarrow & \overline{A_\alpha} \subseteq \overline{\bigcup_{\alpha\in\Lambda} A_\alpha} \\ \Rightarrow & \bigcup_{\alpha\in\Lambda} \overline{A_\alpha} \subseteq \overline{\bigcup_{\alpha\in\Lambda} A_\alpha}. \end{aligned}$$

   To prove the reverse inclusion, suppose that $x \in \overline{\bigcup_{\alpha\in\Lambda} A_\alpha}$. We can choose $\delta > 0$ such that $B(x;\delta)$ has non-empty intersection with a finite number of the $A \in \mathcal{A}$, say $A_1, ..., A_m$. and we have $B(x;\delta) \cap \overline{A} = \emptyset$; unless $A \in \{A_1, ..., A_m\}$. In particular, note that the family $\{\overline{A}; A \in \mathcal{A}\}$ of all closures of the elements of A is also locally finite. It now follows that

$$x \in \overline{\left(\bigcup_{n=1}^{m} A_n\right)} = \bigcup_{n=1}^{m} \overline{A_n} \subseteq \bigcup_{A\in\mathcal{A}} \overline{A}.$$

3.6.2 Note. Note that, if $\{r_1, r_2, ...\}$ be the enumeration $\mathbb{Q}$, then if $A_n = \{r_n\}$, we get $\overline{A_n} = A_n$, hence

$$\mathbb{Q} = \bigcup_{n=1}^{\infty} \overline{A_n} \text{ and } \overline{\bigcup_{n=1}^{\infty} A_n} = \overline{\mathbb{Q}} = \mathbb{R}.$$

$$\text{Thus } \bigcup_{n=1}^{\infty} \overline{A_n} \subset \overline{\bigcup_{n=1}^{\infty} A_n}.$$

6. left to the reader.$A_n = \left(0, \frac{1}{n}\right)$; $n = 1, 2, ...$
7. left to the reader. $A = [1, 2)$.
8. left to the reader. $A = \{0\} \cup [1, 2)$.
9. left to the reader. $A = \{-1\} \cup (0, 1) \cup (1, 2]$. Let $A_n = \left(\frac{1}{n}, 1\right)$; $n = 1, 2, ...$ □

3.6.58 Problem. Let $A \subseteq \mathbb{R}$. Show that

1. $(\overline{A})^C = (A^C)^\circ$, i.e. complement of the closure is the interior of the complement.
2. $(A^\circ)^C = \overline{A^C}$, i.e. complement of the interior is the closure of the complement.

In other words, show that $\overline{A} = \left(\left(A^C\right)^\circ\right)^C$ and $A^\circ = \left(\overline{A^C}\right)^C$.

3.6.58.1 Solution.

1. We show that $\left(\overline{A}\right)^C = \left(A^C\right)^\circ$. Let

$$\begin{aligned} & x \in \left(A^C\right)^\circ \\ \Leftrightarrow & \exists\, r > 0 \text{ such that } B(x; r) \subseteq A^C \\ \Leftrightarrow & \exists\, r > 0 \text{ such that } B(x; r) \cap A = \emptyset \\ \Leftrightarrow & x \notin \overline{A} \\ \Leftrightarrow & x \in \left(\overline{A}\right)^C. \end{aligned}$$

2. Again, let

$$\begin{aligned} & x \in (A^\circ)^C \\ \Leftrightarrow & x \notin A^\circ \\ \Leftrightarrow & \forall\, r > 0 \;\; B(x; r) \cap A^C \neq \emptyset \\ \Leftrightarrow & x \in \overline{A^C}. \end{aligned}$$

Hence the result follows. □

3.6.59 Problem. For any subsets $A, B \subseteq \mathbb{R}$, then prove the following and give examples of each where equality does not occur.

1. $\partial A = \overline{A} \setminus A^\circ = \overline{A} \cap \overline{A^C}$.

2. $\partial A = \partial A^C$.

3. $\partial(A \cup B) \subseteq \partial A \cup \partial B$.

4. If $\partial A \subseteq B \subseteq A$ then $\partial A \subseteq \partial B$.

5. $\partial \overline{A} \subseteq \partial A$ and $\partial A^\circ \subseteq \partial A$.

6. ∂A is a closed set.

7. A is closed iff A contains ∂A.

8. A is open iff $A \cap \partial A = \emptyset$, and $\partial A = \overline{A} \setminus A$.

3.6.59.1 Solution.

1. Let

$$\begin{aligned}
& x \in \partial A \\
& \Leftrightarrow \forall\ \epsilon > 0\ B(x;\epsilon) \cap A \neq \emptyset \text{ and } B(x;\epsilon) \cap A^C \neq \emptyset \\
& \Leftrightarrow x \in \overline{A} \text{ and } x \notin A^\circ \\
& \Leftrightarrow x \in \overline{A} \setminus A^\circ. \\
\text{Again, } & x \in \partial A \\
& \Leftrightarrow \forall\ \epsilon > 0\ B(x;\epsilon) \cap A \neq \emptyset \text{ and } B(x;\epsilon) \cap A^C \neq \emptyset \\
& \Leftrightarrow x \in \overline{A} \text{ and } x \in \overline{A^C} \\
& \Leftrightarrow x \in \overline{A} \cap \overline{A^C}.
\end{aligned}$$

2. By (1), $\partial A^C = \overline{A^C} \cap \overline{(A^C)^C} = \overline{A^C} \cap \overline{A} = \partial A$.

3.

$$\begin{aligned}
\partial(A \cup B) &= \overline{(A \cup B)} \cap \overline{(A \cup B)^C} \\
&= \left(\overline{A} \cup \overline{B}\right) \cap \left(\overline{A^C \cap B^C}\right) \\
&\subseteq \left(\overline{A} \cup \overline{B}\right) \cap \overline{A^C} \cap \overline{B^C} \\
&\subseteq \left(\overline{A} \cap \overline{A^C} \cap \overline{B^C}\right) \cup \left(\overline{B} \cap \overline{A^C} \cap \overline{B^C}\right) \\
&\subseteq \left(\overline{A} \cap \overline{A^C}\right) \cup \left(\overline{B} \cap \overline{B^C}\right) = \partial A \cup \partial B.
\end{aligned}$$

Let $A = [2, 3)$ and $B = [3, 4)$. Then $\partial(A \cup B) \subset \partial A \cup \partial B$.

4. Let $x \in \partial A$, so $\forall\ \epsilon > 0\ B(x;\epsilon) \cap A \neq \emptyset$ and $B(x;\epsilon) \cap A^C \neq \emptyset$. Since $A^C \subseteq B^C$ so $B(x;\epsilon) \cap B^C \neq \emptyset$ and $x \in B$ implies $B(x;\epsilon) \cap B \neq \emptyset$ thus $x \in \partial B$. Example: Let $A = [1, \infty), \partial A = \{1\}, B = [1, 3]$ then $\partial A = \{1\} \subset \partial B = \{1, 3\}$.

5. Suppose that

$$\begin{aligned}
&x \in \partial(\overline{A})\\
\Rightarrow&\forall\, r > 0\ B(x;r) \cap \overline{A} \neq \emptyset \text{ and } B(x;r) \cap \left(\overline{A}\right)^C \neq \emptyset\\
\Rightarrow&\forall\, r > 0\ B(x;r) \cap A \neq \emptyset \text{ and } B(x;r) \cap A^C \neq \emptyset\\
\Rightarrow&x \in \partial A.
\end{aligned}$$

Hence $\partial\overline{A} \subseteq \partial A$.

We show that $(A^\circ)^C = \overline{A^C}$. Let

$$\begin{aligned}
&x \in (A^\circ)^C\\
\Leftrightarrow&x \notin A^\circ\\
\Leftrightarrow&\forall\, r > 0\ B(x;r) \nsubseteq A\\
\Leftrightarrow&\forall\, r > 0\ B(x;r) \cap A^C \neq \emptyset\\
\Leftrightarrow&x \in \overline{A^C}.
\end{aligned}$$

$$\begin{aligned}
\text{Now, } \partial A^\circ &= \overline{A^\circ} \cap \overline{(A^\circ)^C}\\
&= \overline{A^\circ} \cap (A^\circ)^C,\ \text{as } (A^\circ)^C \text{ is closed}\\
&= \overline{A^\circ} \cap \overline{A^C} \subseteq \overline{A} \cap \overline{A^C} = \partial A.
\end{aligned}$$

Let $A = \{2\} \cup (3,4) \cup (4,5)$. Then $\partial\overline{A} \subset \partial A$ and $\partial A^\circ \subset \partial A$.

6. $\partial A = \overline{A} \cap \overline{A^C} \Rightarrow\ \partial A$ is closed. (As it is the intersection of two closed sets.)

7. Suppose that A is closed, then $\overline{A} = A$, hence $\partial A = \overline{A} \setminus A^\circ = A \setminus A^\circ \subseteq A$.
Again, suppose that $\partial A \subseteq A$ and A is not closed, then $\exists$ a point $p \in A'$ such that $p \notin A$, hence $\exists\ r > 0$ such that $\hat{B}(x;r) \cap A = \emptyset$, which shows that $p \notin \partial(A)$, a contradiction. Hence A is closed.

8. Suppose that A is open, then A^C is closed and $\partial A = \overline{A} \cap \overline{A^C} = \overline{A} \cap A^C$. Hence $A \cap \partial A = A \cap \overline{A} \cap A^C = \emptyset$.
Again, suppose that $A \cap \partial A = \emptyset$ and A is not open, then $\exists$ a point $p \in A$ and $\forall\ r > 0$ such that $B(p;r) \cap A^C \neq \emptyset$, which shows that $p \in \partial A \cap A$, a contradiction. Hence A is open. Other part follows from (1). □

3.6.60 Problem.

1. Prove that, $(\partial A)^\circ = \emptyset$, if A is open or if A is closed in $\mathbb{R}$.

2. Give an example in which $(\partial A)^\circ = \mathbb{R}$.

3.6.60.1 Solution.

1. Suppose that A is open and $(\partial A)^\circ \neq \emptyset$, so let

$$\begin{aligned} & x \in (\partial A)^\circ \\ \Rightarrow & x \in \left(\overline{A} \cap \overline{A^C}\right)^\circ \\ \Rightarrow & x \in \left(\overline{A} \cap A^C\right)^\circ \\ \Rightarrow & x \in \left(\overline{A}\right)^\circ \cap \left(A^C\right)^\circ . \end{aligned}$$

So $\exists\, \epsilon_1, \epsilon_2 > 0$ such that $B(x;\epsilon_1) \subseteq \overline{A} = A \cup A'$ and $B(x;\epsilon_2) \subseteq A^C$. Now, let $\epsilon = \min\{\epsilon_1, \epsilon_2\}$ then $B(x;\epsilon) \subseteq (A \cup A') \cap A^C = A' \cap A^C$. Thus

$$\begin{aligned} & B(x;\epsilon) \subseteq A' \Rightarrow B(x;\epsilon) \cap A \neq \emptyset \text{ and} \\ & B(x;\epsilon) \subseteq A^C \Rightarrow B(x;\epsilon) \cap A = \emptyset, \end{aligned}$$

a contradiction. Hence $(\partial A)^\circ = \emptyset$.
Suppose that A is closed, then we have A^C is open. By the above, we have $(\partial A^C)^\circ = \emptyset$. Since $\partial A = \partial A^C$, hence again $(\partial A)^\circ = \emptyset$.

2. Let $A = \mathbb{Q}$, then $\partial A = \mathbb{R} \Rightarrow (\partial A)^\circ = \mathbb{R}$. □

3.6.61 Problem.

1. If $intA = intB = \emptyset$ and if A is closed in $\mathbb{R}$, then $int(A \cup B) = \emptyset$.

2. Give an example in which $intA = intB = \emptyset$ but $int(A \cup B) = \mathbb{R}$.

3.6.61.1 Solution.

1. Assume that $int(A \cup B) \neq \emptyset$. Then $x \in int(A \cup B)$ implies there exists $r > 0$ such that $B(x;r) \subseteq A \cup B$, since $intA = \emptyset$ so $B(x;r) \nsubseteq A$. Hence $B(x;r) \cap (B \setminus A) \neq \emptyset$ implies $B(x;r) \cap (\mathbb{R} \setminus A) \neq \emptyset$. Now,

$$\begin{aligned} & y \in B(x;r) \cap (\mathbb{R} \setminus A) \\ \Rightarrow & B(y;r_1) \subseteq B(x;r), \text{ where } 0 < r_1 < r \text{ and} \\ & y \in \mathbb{R} \setminus A \Rightarrow \exists\, r_2 > 0 \text{ such that } B(y;r_2) \subseteq \mathbb{R} \setminus A. \end{aligned}$$

Let $\epsilon = \min\{r_1, r_2\}$, then we have,

$$\begin{aligned} B(y;\epsilon) & \subseteq B(x;r) \cap (\mathbb{R} \setminus A) \\ & \subseteq (A \cup B) \cap A^C \\ & \subseteq B. \end{aligned}$$

That is, $intB \neq \emptyset$, which is absurd. Hence, we have $int(A \cup B) = \emptyset$.

2. Let $A = \mathbb{Q}$ and $B = \mathbb{Q}^C$. □

3.6.62 Problem. Show that $\mathbb{R} = S^\circ \cup \partial(S) \cup Ext(S)$.

3.6.62.1 Solution. Now $x \in \mathbb{R} \Rightarrow$ either $x \in S$ or $x \in S^C$. Suppose $x \in S$. Then either $\exists\, r > 0$ such that $B(x;r) \subseteq S$ or $\forall\, s > 0$ $B(x;s) \cap S^C \neq \emptyset$. Hence $x \in S^\circ$ or $x \in \partial S$. Similarly, if $x \in S^C$ then either $\exists\, p > 0$ such that $B(x;r) \subseteq S^C$ or $\forall\, s > 0$ $B(x;s) \cap S \neq \emptyset$. Hence $x \in \left(S^C\right)^\circ$ or $x \in \partial S^C$. Since $\partial S = \partial S^C$, the desired result follows. □

3.6.63 Problem.

1. If A and B are open subsets in $\mathbb{R}$, show that
$$(A\cap\partial B)\cup(B\cap\partial A)\subseteq\partial(A\cap B)\subseteq(A\cap\partial B)\cup(B\cap\partial A)\cup(\partial A\cap\partial B)$$
and give an example (in the real line) in which these three sets are distinct.

2. Let A and B be subsets of $\mathbb{R}$. Show that
   (a) $\partial(A\cap B)\subseteq\left[\overline{A}\cap\partial B\right]\cup\left[\partial A\cap\overline{B}\right]$.
   (b) $\partial A\cup\partial B=\partial(A\cup B)\cup\partial(A\cap B)\cup[\partial A\cap\partial B]$.

3. Verify that if the sets A and B satisfy the condition $A\cap\overline{B}=\emptyset=\overline{A}\cap B$, then $\partial(A\cup B)=\partial A\cup\partial B$.

4. Let A and B be subsets of $\mathbb{R}$, if $\overline{A}\cap\overline{B}=\emptyset$ then show that
   (a) $\partial(A\cup B)=\partial A\cup\partial B$.
   (b) $\partial(A\cap B)=\left[\overline{A}\cap\partial B\right]\cup\left[\partial A\cap\overline{B}\right]$.
   (c) $(A\cup B)^{\circ}=A^{\circ}\cup B^{\circ}$.

3.6.63.1 Solution.

1. Suppose that
$$\begin{aligned}
&x\in(A\cap\partial B)\cup(B\cap\partial A)\\
\Rightarrow&x\in(A\cap\partial B)\text{ or }x\in(B\cap\partial A)\\
\Rightarrow&\forall\ \epsilon>0, B(x;\epsilon)\cap A\neq\emptyset\text{ and }B(x;\epsilon)\cap B\neq\emptyset\text{ and }B(x;\epsilon)\cap B^{C}\neq\emptyset\\
\text{or }&\forall\ \epsilon>0, B(x;\epsilon)\cap B\neq\emptyset\text{ and }B(x;\epsilon)\cap A\neq\emptyset\text{ and }B(x;\epsilon)\cap A^{C}\neq\emptyset\\
\Rightarrow&B(x;\epsilon)\cap(A\cap B)\neq\emptyset\text{ and }B(x;\epsilon)\cap\left(A^{C}\cup B^{C}\right)\neq\emptyset\\
\Rightarrow&x\in\partial(A\cap B).
\end{aligned}$$
Thus $(A\cap\partial B)\cup(B\cap\partial A)\subseteq\partial(A\cap B)$, and
$$\begin{aligned}
\partial(A\cap B)&=\overline{(A\cap B)}\cap(A\cap B)^{C}\\
&\subseteq\left(\overline{A}\cap\overline{B}\right)\cap\left(A^{C}\cup B^{C}\right)\\
&\subseteq\left(\overline{A}\cap\overline{B}\cap A^{C}\right)\cap\left(\overline{A}\cap\overline{B}\cap B^{C}\right)\\
&\subseteq\left(\partial A\cap\overline{B}\right)\cap\left(\partial B\cap\overline{A}\right)\\
&\subseteq(\partial A\cap(B\cup\partial B))\cap(\partial B\cap(A\cup\partial A))\\
&\subseteq(A\cap\partial B)\cup(B\cap\partial A)\cup(\partial A\cap\partial B).
\end{aligned}$$
Hence
$$(A\cap\partial B)\cup(B\cap\partial A)\subseteq\partial(A\cap B)\subseteq(A\cap\partial B)\cup(B\cap\partial A)\cup(\partial A\cap\partial B).$$
Example: Let $A=(1,2)\cup(2,4)\cup(4,5)\cup(5,6)\cup(6,8);\ B=(2,3)\cup(3,4)\cup(4,6)\cup(6,7)\cup(7,8)$ then
$$\begin{aligned}
&\partial A=\{1,2,4,5,6,8\},\ \partial B=\{2,3,4,6,7,8\}\\
&A\cap\partial B=\{3,7\},\ B\cap\partial A=\{5\}\\
&A\cap B=(2,3)\cup(3,4)\cup(4,5)\cup(5,6)\cup(6,7)\\
&\text{and }\partial(A\cap B)=\{2,3,4,5,6,7\}\text{ and }\partial A\cap\partial B=\{2,4,6,8\}.
\end{aligned}$$

Thus, $(A \cap \partial B) \cup (B \cap \partial A) = \{2,5,6\}$, $\partial(A \cap B) = \{2,3,4,5,6,7\}$ and $(A \cap \partial B) \cup (B \cap \partial A) \cup (\partial A \cap \partial B) = \{2,3,4,5,6,7,8\}$ are all distinct.

2. Left to the reader.

3. Left to the reader.

4. (a) $\overline{A} \cap \overline{B} = \emptyset \Rightarrow A \cap B$ and $\partial A \cap \partial B$ are both empty and apply (2).
Alternative way:
Let $x \in \partial(A \cup B)$, then for all $r > 0$,

$$B(x;r) \cap (A \cup B) \neq \emptyset \Rightarrow B(x;r) \cap A \neq \emptyset \text{ or } B(x;r) \cap B \neq \emptyset \text{ and}$$
$$B(x;r) \cap [(A \cup B)^C] \neq \emptyset \Rightarrow B(x;r) \cap A^C \neq \emptyset \text{ and } B(x;r) \cap B^C \neq \emptyset,$$

which shows that $x \in \partial A$ or $x \in \partial B$ and thus $x \in \partial A \cup \partial B$. Therefore $\partial(A \cup B) \subseteq \partial A \cup \partial B$.
Again, let $x \in \partial A \cup \partial B$. Suppose that $x \in \partial A$, then $\forall\ r > 0$, we have

$$B(x;r) \cap A \neq \emptyset \text{ and } B(x;r) \cap A^C \neq \emptyset.$$

Since $B(x;r) \cap A \neq \emptyset$, we have

$$B(x;r) \cap (A \cup B) \neq \emptyset$$
$$\Rightarrow (B(x;r) \cap A) \cup (B(x;r) \cap B) \neq \emptyset$$

We claim that $B(x;r) \cap [(A \cup B)^C] = B(x;r) \cap A^C \cap B^C \neq \emptyset$. Suppose that $B(x;r) \cap A^C \cap B^C = \emptyset$. Then we have,

$$B(x;r) \subseteq A \Rightarrow B(x;r) \subseteq \overline{A} \text{ and } B(x;r) \subseteq B \Rightarrow B(x;r) \subseteq \overline{B}.$$

It implies that by hypothesis, $B(x;r) \subseteq \overline{A} \cap \overline{B} = \emptyset$ which is absurd. Hence, we have proved the claim. Thus

$$B(x;r) \cap (A \cup B) \neq \emptyset \text{ and } B(x;r) \cap [(A \cup B)^C] \neq \emptyset.$$

Hence $\partial(A) \cup \partial(B) \subseteq \partial(A \cup B)$. Thus $\partial(A) \cup \partial(B) = \partial(A \cup B)$.

(b) Left to the reader.

(c) We can prove the result $A^\circ \cup B^\circ \subseteq (A \cup B)^\circ$ easily. To prove the other side, let $x \in (A \cup B)^\circ$ then $\exists\ r > 0$ such that $B(x,r) \subseteq A \cup B$. If $B(x,r) \subseteq A$ or $B(x,r) \subseteq B$, then we are done. Otherwise, $\forall\ r > 0$ $B(x,r) \nsubseteq A$ and $B(x,r) \nsubseteq B$, i.e. $B(x,r) \cap A^C \neq \emptyset$ and $B(x,r) \cap B^C \neq \emptyset$. Again, $B(x,r) \cap A \neq \emptyset$ and $B(x,r) \cap B \neq \emptyset$. For, if $\exists\ r > 0$ with $B(x;r) \cap A = \emptyset$ we get $B(x;r) \subseteq B$, a contradiction. Thus we get $x \in \partial A$ and $x \in \partial B$ i.e. $x \in \partial A \cap \partial B$ contradicts $\partial A \cap \partial B = \emptyset$. Hence $B(x,r) \subseteq A$ or $B(x,r) \subseteq B$, i.e. $(A \cup B)^\circ \subseteq A^\circ \cup B^\circ$ and the result follows. □

3.6.64 Problem. Show that the boundary of a closed or open set in $\mathbb{R}$ is nowhere dense. Is this statement true for an arbitrary subset?

3.6.64.1 Solution. Let A be closed. Since $\partial A = \partial A^C = \overline{A} \cap \overline{A^C}$, so

$$\begin{aligned}(\partial A)^\circ &= \left(\overline{A} \cap \overline{A^C}\right)^\circ = \left(A \cap \overline{A^C}\right)^\circ = A^\circ \cap \left(\overline{A^C}\right)^\circ \\ &\subseteq A^\circ \cap \overline{A^C} = A^\circ \cap (A^\circ)^C = \emptyset.\end{aligned}$$

This shows that ∂A is nowhere dense.
Let A be open. So $A^\circ = A \Rightarrow A^C$ is closed and $\overline{A^C} = A^C$. So,

$$(\partial A)^\circ = \left(\overline{A} \cap \overline{A^C}\right)^\circ = \left(\overline{A} \cap A^C\right)^\circ = \emptyset.$$

The boundary of an arbitrary set need not be nowhere dense. An example: Let $A = \mathbb{Q}$ then $\partial \mathbb{Q} = \mathbb{R}$. □

3.6.65 Problem.

1. Show that $Ext(\overline{A}) = Ext(A)$ and $Ext(A \cup B) = Ext(A) \cap Ext(B)$.

2. Show that $Ext(A) = \left(\overline{A}\right)^C$. Also, show that $\overline{A^C} = (A^\circ)^C$; and deduce that $\partial A = \overline{A} \setminus A^\circ$.

3. Show that, if A is open, then $A \cup Ext(A)$ is dense in $\mathbb{R}$.

4. Let $A \subseteq \mathbb{R}$. Show that A is nowhere dense $\Leftrightarrow$ $\overline{A}$ is nowhere dense $\Leftrightarrow$ $Ext(A)$ is dense in $\mathbb{R}$.

3.6.65.1 Solution.

1. By definition, we get

$$\begin{aligned} x \in Ext(\overline{A}) &\Leftrightarrow \exists\, r > 0 \text{ such that } B(x;r) \subseteq \left(\overline{A}\right)^C \\ &\Leftrightarrow B(x;r) \cap \overline{A} = \emptyset \\ &\Leftrightarrow B(x;r) \cap (A \cup A') = \emptyset \\ &\Leftrightarrow B(x;r) \cap (A \cup A') = \emptyset \\ &\Leftrightarrow B(x;r) \cap A = \emptyset \text{ or } B(x;r) \cap A' = \emptyset \\ &\Leftrightarrow B(x;r) \subseteq A^C \Leftrightarrow x \in Ext(A). \\ \text{Again, } x \in Ext(A \cup B) &\Leftrightarrow \exists\, r > 0 \text{ such that } B(x;r) \subseteq (A \cup B)^C \\ &\Leftrightarrow B(x;r) \subseteq A^C \cap B^C \\ &\Leftrightarrow B(x;r) \subseteq A^C \text{ and } B(x;r) \subseteq B^C \\ &\Leftrightarrow x \in Ext(A) \text{ and } x \in Ext(B) \\ &\Leftrightarrow x \in Ext(A) \cap Ext(B) \end{aligned}$$

Thus $Ext(A \cup B) = Ext(A) \cap Ext(B)$.

2. We get,

$$
\begin{aligned}
x \in Ext(A) &\Leftrightarrow \ \exists\, r > 0 \text{ such that } B(x;r) \subseteq A^C \\
&\Leftrightarrow \ B(x;r) \cap A = \emptyset \text{ and } \ B(x;r) \cap A' = \emptyset. \\
&\Leftrightarrow \ B(x;r) \cap (A \cup A') = \emptyset. \\
&\Leftrightarrow B(x;r) \subseteq \left(\overline{A}\right)^C \Leftrightarrow x \in (\overline{A})^C.
\end{aligned}
$$

$$
\begin{aligned}
&\text{Let } \ x \in (A^\circ)^C \\
&\Leftrightarrow x \notin A^\circ \\
&\Leftrightarrow \forall\, r > 0 \ B(x;r) \nsubseteq A \\
&\Leftrightarrow \forall\, r > 0 \ B(x;r) \cap A^C \neq \emptyset \\
&\Leftrightarrow x \in \overline{A^C}. \\
&\text{Hence } \overline{A^C} = (A^\circ)^C \\
&\Rightarrow \overline{A} \cap \overline{A^C} = \overline{A} \cap (A^\circ)^C \\
&\Rightarrow \partial A = \overline{A} \setminus A^\circ.
\end{aligned}
$$

3. Let $x \in \mathbb{R}$ and

$$
\begin{aligned}
&x \notin A \cup Ext(A) \\
\Rightarrow\, &x \in (A \cup Ext(A))^C = A^C \cap (Ext(A))^C \\
\Rightarrow\, &x \in A^C \cap \overline{A} = \partial A \text{ as } A \text{ is open.}
\end{aligned}
$$

Thus $\forall\, \epsilon > 0 \ B(x;\epsilon) \cap (A \cup Ext(A)) \neq \emptyset$ which implies $x \in \overline{A \cup Ext(A)}$ i.e. $\mathbb{R} = \overline{A \cup Ext(A)}$. Hence $A \cup Ext(A)$ is dense in $\mathbb{R}$.

4. By definition, we get

$$
\begin{aligned}
A \text{ is nowhere dense} &\Leftrightarrow \left(\overline{A}\right)^\circ = \emptyset \\
&\Leftrightarrow \left(\overline{\overline{A}}\right)^\circ = \emptyset \\
&\Leftrightarrow \overline{A} \text{ is nowhere dense} \\
&\Leftrightarrow \left(\left(\overline{A}\right)^\circ\right)^C = \mathbb{R} \\
&\Leftrightarrow \overline{\left(\left(\overline{A}\right)^C\right)} = \mathbb{R} \\
&\Leftrightarrow \overline{Ext(A)} = \mathbb{R} \\
&\Leftrightarrow Ext(A) \text{ is dense in } \mathbb{R}. \quad \square
\end{aligned}
$$

3.6.66 Problem. Show that, for each $x \in \mathbb{R}$, we have

$$\bigcap_{\epsilon \in \mathbb{R}^+} B(x;\epsilon) = \{x\}.$$

3.6.66.1 Solution. Suppose that $y > x$ and $y \in \bigcap_{\epsilon \in \mathbb{R}^+} B(x;\epsilon)$. Choose $0 < \epsilon < y - x \Rightarrow y \notin (x-\epsilon, x+\epsilon) \Rightarrow y \notin \bigcap_{\epsilon \in \mathbb{R}^+} B(x;\epsilon)$, a contradiction.
Again, if $y < x$ and $y \in \bigcap_{\epsilon \in \mathbb{R}^+} B(x;\epsilon)$. Then choose $0 < \epsilon < x - y \Rightarrow y \notin (x-\epsilon, x+\epsilon) \Rightarrow y \notin \bigcap_{\epsilon \in \mathbb{R}^+} B(x;\epsilon)$, a contradiction. Hence no member other than x can belong to the intersection. □

3.6.67 Problem. Let F be a collection of sets in $\mathbb{R}$, and let $S = \bigcup_{A \in F} A$ and $T = \bigcap_{A \in F} A$. For each of the following statements, either give a proof or exhibit a counterexample.

1. If x is an accumulation point of T, then x is an accumulation point of each set A in F.
2. If x is an accumulation point of S, then x is an accumulation point of at least one set A in F.

3.6.67.1 Solution.

1. Let x be an accumulation point of T, then $\hat{B}(x;r) \cap T \neq \emptyset$, for any $r > 0$. Note that for any $A \in F$, we have $T \subseteq A$. Hence $\hat{B}(x;r) \cap A \neq \emptyset$, for any $r > 0$. That is, x is an accumulation point of A for any $A \in F$.
2. No. Let F be the collection of sets consisting of a single point $x \in \mathbb{R}$. Then it is trivially seen that $S = \cup_{x \in S}\{x\}$. And if x is an accumulation point of S, then x is not an accumulation point of each set $\{x\}$ in F. □

3.6.68 Problem. Let S be a set and let P be an equivalence relation on S. That is, $\forall\ x, y, z \in S, xPx; xPy \Rightarrow yPx$ and xPy and $yPz \Rightarrow xPz$. For each subset $A \subseteq S$, define $A = \{x \in S; \exists\ y \in A$ such that $yPx\}$.

1. $A \subseteq S$ and $B \subseteq S$ implies
   (a) $A \subseteq \overline{A}$.
   (b) $\overline{\overline{A}} = \overline{A}$.
   (c) $\overline{\overline{A \cup B}} = \overline{A} \cup \overline{B}$.
2. Let S be the set of points in the Cartesian plane. Define

$$R : (x_1, y_1)R(x_2, y_2) \text{ iff } y_1 - y_2 = 3(x_1 - x_2)$$

   Granted that R is an equivalence relation on S, describe or sketch A, where A is the unit circle with center at (0,0). That is $A = \{(x, y) \in S; x^2 + y^2 = 1\}$.

3.6.68.1 Solution.

1. (a) $x \in A \Rightarrow x \in S \Rightarrow xRx$ (with $x \in A$) $\Rightarrow x \in A$, hence $A \subseteq \overline{A}$.
   (b) By (1), $\overline{A} \subseteq \overline{\overline{A}}$. Now, $x \in \overline{\overline{A}} \Rightarrow \exists y \in \overline{A}$ such that xPy. $y \in \overline{A} \Rightarrow \exists\ z \in A$ such that yPz. Now, $xPy \wedge yPz \Rightarrow xPz$ (with $z \in A$), so $x \in \overline{A}$. It follows that $\overline{\overline{A}} \subseteq \overline{A}$, hence $\overline{\overline{A}} = \overline{A}$.
   (c) $x \in \overline{A \cup B} \Rightarrow \exists\ y \in A \cup B$ such that xPy. If $y \in A$, then $x \in \overline{A}$ and if $y \in B$, then $x \in \overline{B}$. Therefore $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$. Also, $x \in \overline{A} \cup \overline{B} \Rightarrow \underline{(x \in \overline{A}) \vee (x \in \overline{B})}$. So one of A or B contains a y such that xPy. It follows that $x \in \overline{A \cup B}$ and hence $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
2. 2nd part: Left to the reader. □

3.6.69 Problem. Give an example of a subspace E of the plane $\mathbb{R}^2$, such that there is an open ball in E which is a closed set but not a closed ball, and a closed ball which is an open set but not an open ball.

3.6.69.1 Solution. Take E consisting of the two points (0,1) and $(0,-1)$ and of a suitable subset of the x-axis. $\square$

3.6.70 Problem. Show that a nonempty perfect subset P of $\mathbb{R}$ is uncountable. This gives yet another proof that the Cantor set is uncountable.

3.6.70.1 Solution. Hint: First show that P is infinite, and assume that P is countable, say $P = \{x_1, x_2, ...\}$. Construct a decreasing sequence of nested closed intervals $[a_n, b_n]$ such that $(a_n, b_n) \cap P \neq \emptyset$ but $x_n \notin [a_n, b_n]$. Use the nested interval theorem to get a contradiction. $\square$

3.6.71 Problem. Every isolated set of real numbers is countable.

3.6.71.1 Solution. Let A be an isolated set, then for each $a \in A$, $\exists$ a real number $\delta_a > 0$ such that $(a - \delta_a, a + \delta_a) \cap A = \{a\}$. Suppose the enumeration of $\mathbb{Q}$ be $\{q_1, q_2, q_3, ...\}$. Since $(a - \delta_a, a + \delta_a)$ contains many rationals, choose the smallest index $m(a)$ and $a \mapsto m(a) \in \mathbb{N}$ is an injection. Hence A is countable. $\square$

3.6.72 Problem. Give an example of:

1. A set with no accumulation points.
2. A set with infinitely many accumulation points, none of which belong to the set.
3. A set that contains some, but not all, of its accumulation points.

3.6.72.1 Solution.

1. Consider the set $\mathbb{N}$.
2. $S = \left\{ \frac{1}{m} + \frac{1}{n}; m, n \in \mathbb{N} \right\} \setminus \left\{ 1, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{n}, ... \right\}$.
3. Consider the set $(2, 3)$. $\square$

3.6.73 Problem. Give an example of a nonempty set with the following properties or explain why no such set can exist:

1. A set with no accumulation points and no isolated points.
2. A set with no interior points and no isolated points.
3. A set with no boundary points and no isolated points.

3.6.73.1 Solution.

1. Such a set cannot exist.
2. $\mathbb{Q}$.
3. $\mathbb{R}$. $\square$

3.6.74 Problem. Let $S \subseteq \mathbb{R}$, $a \notin S$. Then a is limit point of S iff a is a boundary point of S.

3.6.74.1 Solution. Suppose that a is limit point of S then for every $\epsilon > 0 \Rightarrow (a-\epsilon.a+\epsilon)\cap S \neq \emptyset$ and $a \in S^C \Rightarrow (a-\epsilon, a+\epsilon)\cap S^C \neq \emptyset$. Hence a is a boundary point of S. Again, a is a boundary point of $S \Rightarrow$ for every $\epsilon > 0, (a-\epsilon, a+\epsilon)\cap S \neq \emptyset$ and $(a-\epsilon, a+\epsilon)\cap S^C \neq \emptyset$. Hence a is limit point of S. □

3.6.75 Problem. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ such that the set $\{x \in \mathbb{R}; f(x) = 0\}$ is neither open nor closed.

3.6.75.1 Solution. Let $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } 0 < x \leq 1 \\ x & \text{otherwise.} \end{cases}$$

Thus $f^{-1}(0) = 0 < x \leq 1$, which neither open nor closed. □

3.6.76 Problem. Define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x, & \text{if } x < 0 \\ 1, & \text{if } x \geq 0. \end{cases}$$

Find an open set G and a closed set E such that the set $\{x \in \mathbb{R}; f(x) \in G\}$ is not open and the set $\{x \in \mathbb{R}; f(x) \in E\}$ is not closed.

3.6.76.1 Solution. $G = (0, 2)$ then $f^{-1}(G) = [0, \infty)$ which is not open and $E = (-\infty, 1/2]$ then $f^{-1}(E) = (0, \infty)$ which is not closed.

3.6.77 Problem.

1. Find a sequence (G_n) of open sets such that $\mathbb{Z} = \bigcap_{n=1}^{\infty} G_n$.
2. Find a sequence (K_n) of closed sets such that $(0,1)\cup(2,3) = \bigcup_{n=1}^{\infty} K_n$.

3.6.77.1 Solution.

1. For each $k \in \mathbb{Z}$, we let $G_n = \bigcup_{k\in\mathbb{Z}} \left(k - \frac{1}{n}, k + \frac{1}{n}\right)$
2. $K_n = \left[\dfrac{1}{n}, 1 - \dfrac{1}{n}\right] \cup \left[2 - \dfrac{1}{n}, 3 - \dfrac{1}{n}\right]$ □

3.6.78 Problem. Show that the intersection of two dense sets may not be a dense set.

3.6.78.1 Solution. $\mathbb{Q}$, $\mathbb{Q}^C$ are dense subsets in $\mathbb{R}$, but $\mathbb{Q}\cap\mathbb{Q}^C = \emptyset$ is not dense. □

Sierpiński's lemma: There is a family $\mathcal{F}$ formed by infinite subsets of $\mathbb{N}$ such that $card\mathcal{F} = \mathfrak{c}$ and, if F_1 and F_2 are two different elements of $\mathcal{F}$, then $card(F_1 \cap F_2) < \infty$.

3.6.79 Problem. Prove Sierpiński's lemma. In other words, show that $\mathbb{N}$ contains uncountably many infinite subsets $(N_\alpha)_{\alpha\in\mathbb{R}}$, where $N_\alpha \subseteq \mathbb{N}$, such that $N_\alpha \cap N_\beta$ is finite if $\alpha \neq \beta$.

3.6.79.1 Solution. Let $S = \{r_1, r_2, ..., r_n, ...\}$ be the fixed enumeration of rationals in $[0, 1]$ and let $t \in [0, 1]$. Now consider a fixed sequence of rationals converging to t. Again define

$$N_t = \{i \in \mathbb{N}; r_i \to t\}.$$

It is clear that N_t is infinite. Again, if $t_1 \neq t_2$ then $N_{t_1} \cap N_{t_2}$ is a finite set, since the corresponding sequences converge to distinct real numbers. □

3.6.80 Problem. Using Schróder-Bernstein theorem, prove that $|\mathbb{R}| = |\mathcal{P}(\mathbb{N})|$.

3.6.80.1 Solution. Let $\mathbb{Q} = \{r_1, r_2, ..., r_n, ...\}$ be the fixed enumeration of rationals in $\mathbb{R}$ and let $t \in \mathbb{R}$. Now consider a fixed sequence[1] of rationals converging to t. Again define

$$N_t = \{i \in \mathbb{N}; r_i \to t\}.$$

We see that $N_t \in \mathcal{P}(\mathbb{N})$. Define a function $\mu : \mathbb{R} \to \mathcal{P}(\mathbb{N})$ by $\mu : t \mapsto N_t$. Again, if $t_1 \neq t_2$ then $N_{t_1} \cap N_{t_2}$ is a finite set, since the corresponding sequences converge to distinct real numbers. Thus $\mu(t_1) \neq \mu(t_2)$ showing that μ is injective. Hence $|\mathbb{R}| \leq |\mathcal{P}(\mathbb{N})|$.
Now, we use continued fractions[2] to show the other part. The set $\mathcal{P}(\mathbb{N})$, can be mapped one-to-one and onto the set of all real numbers in [0,1]. This can be proved by using the concept of continued fractions; the required map ϕ is given by

$$\phi(A) = \begin{cases} \frac{1}{n_1 + \frac{1}{n_2 + \frac{1}{n_3} + \cdots}}, & \text{if } A = \{n_1, n_2, ...\} \in \mathcal{P}(\mathbb{N}). \\ 0 & \text{if } A = \emptyset. \end{cases}$$

Thus $|\mathcal{P}(\mathbb{N})| \leq |\mathbb{R}|$ implies $|\mathbb{R}| = |\mathcal{P}(\mathbb{N})|$. □

3.6.81 Problem. Let f be a function on $\mathbb{R}$ such that

1. $f(x) \geq 0$ for all x, and
2. There exists $M < \infty$ such that for all finite $F \subset \mathbb{R}$,

$$\sum_F f(x) \leq M.$$

Prove that the set $\{x; f(x) > 0\}$ is countable.

3.6.81.1 Solution. Define $S_n = \left\{x; f(x) > \frac{1}{n}\right\}$. Fix n and suppose for the sake of contradiction that $|S_n| > Mn$. Then there exists a subset F of S_n of cardinality $Mn + 1$. By property (ii),

$$\sum_F f(x) \leq M.$$

$$\text{However, } \sum_F f(x) \geq \sum_F \frac{1}{n} = \frac{Mn+1}{n} > M.$$

a contradiction. Thus S_n is finite for each n. We have

$$\{x; f(x) > 0\} = \bigcup_{n \in \mathbb{N}} \left\{x; f(x) > \frac{1}{n}\right\} = \bigcup_{n \in \mathbb{N}} S_n$$

which is a countable union of finite sets, and thus countable. □

[1]Let C_r be the class of all rational sequences converging to r. Thus $\mathcal{C} = \{C_r; r \in \mathbb{R}\}$ is a disjoint family of sets, and by Axiom of Choice there exists a set C such that $C \cap C_r$ is singleton. This assures that such a fixed choice for each real is possible. Here $N_t \in \mathcal{C}$.
[2]See Higher Algebra by Bernard and Child.

3.6.82 Problem. Prove that the set of real numbers can be written as the union of uncountably many pairwise disjoint subsets, each of which is uncountable.

3.6.82.1 Solution. Define the map $f : (0,1) \times (0,1) \to (0,1)$ so that

$$f(x, y) = 0.x_0 y_0 x_1 y_1..$$

where $x = 0.x_0x_1...$ and $y = 0.y_0y_1...$ Here we replace any infinite chain of 9's in x or y by incrementing the digit preceding the chain and replacing the chain of 9's by a chain of 0's, then evaluate f. Then this map is well-defined and actually an injection. Consider the set S of all vertical lines in $(0,1) \times (0,1)$. There are uncountably many, and their union is $(0,1) \times (0,1)$. Also, each vertical line is an uncountable set of points. Now consider $f(S)$. Note the only decimals that f misses form a countable set. Since f is an injection, for each line $L \in S$, $f(S)$ is uncountable. Also, the images of distinct lines of S under f are disjoint. Thus

$$(0,1) = f(S) = \bigcup_{L \in S} f(L) \cup \bigcup (\text{countable collection of points})$$

can be written as a union of uncountably many pairwise disjoint subsets, each of which is uncountable.
Let g be a bijection between (0,1) and $\mathbb{R}$ ($\tan(\pi(x - 1/2))$, for instance). Composing g with f above, we can write the set of real numbers as the desired union. □

3.6.83 Problem. Prove that $\mathbb{R}^2$ is not a (countable) union of sets $S_i, i = 1, 2, ...$ with each S_i being a subset of some straight line $L_i \in \mathbb{R}^2$.

3.6.83.1 Solution. Suppose that $\mathbb{R}^2$ is a countable union of sets $S_i, i = 1, 2, ...$ with each S_i being a subset of some straight line $L_i \in \mathbb{R}^2$. Since $\mathbb{R}$ is uncountable, there exists some $x \in \mathbb{R}$ such that the line $L_x = \{(x, y); y \in \mathbb{R}\}$ is not equal to any S_i. (Otherwise $\{(x, 0); x \in \mathbb{R}\}$ would be countable, implying that $\mathbb{R}$ was countable.) Now each line L_x intersects S_i at either zero or one point. Thus the set

$$\bigcup_{i \in \mathbb{N}} L_x \cap S_i$$

is countable. Since $\mathbb{R}^2$ is a countable union of the S_i however,

$$\bigcup_{i \in \mathbb{N}} L_x \cap S_i = L_x$$

so L_x is countable. But this implies that $\mathbb{R}$ is countable, a contradiction. □

3.6.84 Problem. Can a countably infinite set have an uncountable collection of nonempty subsets such that the intersection of any two of them is finite?

3.6.84.1 Solution. The set $\mathbb{Q}$ of rational numbers is countably infinite. For each real number α, choose a sequence of distinct rational numbers tending to α, and let S_α be the set of terms. If α, β are distinct real numbers, then $S_\alpha \cap S_\beta$ is finite, since otherwise a sequence obtained by listing its elements would converge to both S_α and S_β. In particular, $S_\alpha \neq S_\beta$. Thus $\{S_\alpha; \alpha \in \mathbb{R}\}$ is an uncountable collection of nonempty subsets of $\mathbb{Q}$ with the desired property. □

3.6.85 Problem. There exists an uncountable collection of distinct subsets of $\mathbb{N}$ that are totally ordered by the inclusion relation.

3.6.85.1 Solution. This example is from Golomb and Gilmer [1974]. Let $f : \mathbb{N} \to \mathbb{Q}$ be one-to-one and, for each $t \in \mathbb{R}$, let $N_t = \{n \in \mathbb{R}; f(n) < t\}$. Note that the collection $\{N_t; t \in \mathbb{R}\}$ contains uncountably many distinct sets, since if $t \neq s$, then $N_t \neq N_s$. Further, if $s < t$, then $N_s \subseteq N_t$. That is, the elements in $\{N_t; t \in \mathbb{R}\}$ are totally ordered by the inclusion relation. $\square$

3.6.86 Problem. [3] There exists a subset B of real numbers such that B and B^C each have at ieast one point in common with every uncountable closed subset of $\mathbb{R}$.

3.6.86.1 Solution. Note that there are at most $\mathfrak{c}$ open subsets of $\mathbb{R}$, since any open set may be expressed as a countable union of open intervals with rational endpoints and hence that there are at most $\mathfrak{c}$ closed subsets of $\mathbb{R}$. Also, there are at least $\mathfrak{c}$ closed uncountable sets, since there are $\mathfrak{c}$ closed uncountable intervals. Thus, there are exactly $\mathfrak{c}$ uncountable closed subsets of $\mathbb{R}$. Let $\mathfrak{F} = \{F_\alpha; \alpha < \mathfrak{c}\}$ denote a well ordering of the family of closed uncountable intervals. Note from the result[4] that each element of $\mathfrak{F}$ has cardinality $\mathfrak{c}$ since each closed subset of $\mathbb{R}$ is a G_δ subset of $\mathbb{R}$. Consider a well ordering of $\mathbb{R}$ and let p_1 and q_1 denote the first two elements of F_1 with respect to that ordering. Let p_2 and q_2 denote the first two elements of F_2 that are different from p_1 and q_1. In general, if $1 < \alpha < \mathfrak{c}$ and if p_β and q_α have been defined in this way for all $\beta < \alpha$, then let p_α and q_α be the first two elements from H_α, where $H_\alpha = F_\alpha \setminus \bigcup_{\beta<\alpha}\{p_\beta, q_\alpha\}$. Note that H_α has cardinality $\mathfrak{c}$ for each $\alpha < \mathfrak{c}$, since F_α has cardinality $\mathfrak{c}$. In this way, p_α and q_α are defined for all $\alpha < \mathfrak{c}$. Let $B = \{p_\alpha; \alpha < \mathfrak{c}\}$ and the desired result follows. Such a set B is called a Bernstein set. $\square$

3.6.87 Problem. It is impossible to express [0,1] as a countable union of disjoint closed sets.

3.6.87.1 Solution. Suppose $[0, 1] = \cup_{n=1}^{\infty} F_n$ with the F_n's are closed and pairwise disjoint. Since $F_1 \cap F_2 = \emptyset$, we can find a closed interval I_1 such that $I_1 \cap F_1 = \emptyset$, $I_1 \cap F_2 \neq \emptyset$, $I_1 \setminus F_2 \neq \emptyset$. We repeat the same procedure inside I_1 with $I_1 \cap F_2$ playing the role of F_1 and $I_1 \cap F_k$ playing the role of F_2, where F_k is the first set in the sequence $\{F_k\}_{n=3}^{\infty}$ intersecting I_1. We thereby construct a decreasing sequence of closed intervals $\{I_n\}_{n=1}^{\infty}$, such that $I_n \cap F_n = \emptyset$, a contradiction. $\square$

3.6.88 Problem. Prove that the set of irrational numbers in $\mathbb{R}$ is not a countable union of closed sets.

3.6.88.1 Solution. Suppose that the set of irrational numbers I_{rr} can be represented as $\bigcup_{n\in\mathbb{N}} F_n$ where the F_n are closed. Then

$$\mathbb{R} = \left(\bigcup_{n\in\mathbb{N}} F_n\right) \cup \left(\bigcup_{r\in\mathbb{Q}} \{r\}\right)$$

Since $\mathbb{R}$ has non-empty interior, the Baire Category Theorem implies that one of the sets in the union on the right hand side has non-empty interior. Clearly it is not $\{r\}$ for some rational r, so some F_n must have non-empty interior. Thus there exists $x \in F_n$ such that $B(x; \epsilon) \subseteq F_n \subset I_{rr}$. But the rationals are dense in $\mathbb{R}$, so some rational number is an element of $B(x; \epsilon)$ and thus an element of I_{rr}, a contradiction. $\square$

[3]This example is from Oxtoby [1980]

[4]Any uncountable G_δ subset of $\mathbb{R}$ contains a nowhere dense ciosed subset of Lebesgue measure zero that can be mapped continuously onto [0,1]. [Oxtoby, 1980]

3.6.89 Problem. Show that the set $\mathbb{Q}$ of rational numbers in $\mathbb{R}$ is not expressible as the intersection of a countable collection of open subsets of $\mathbb{R}$.

3.6.89.1 Solution. Suppose that $\mathbb{Q} = \bigcap_{n\in\mathbb{N}} U_n$, where U_n is open for each n. Clearly $\mathbb{Q} \subseteq U_n$ for each n, and since the rational numbers are dense in $\mathbb{R}$, each U_n is dense in $\mathbb{R}$. For each rational number $r, \mathbb{R} \setminus \{r\}$ is open and dense in $\mathbb{R}$. Thus

$$\emptyset = I_{rr} \cap \mathbb{Q} = \left(\bigcap_{r\in\mathbb{Q}} \mathbb{R} \setminus \{r\} \right) \cap \bigcap_{n\in\mathbb{N}} U_n$$

is a countable intersection of dense open sets. Applying the Baire Category Theorem, we get that $\emptyset$ is dense in $\mathbb{R}$, a contradiction. □

3.6.90 Problem. Find a subset S of the real numbers $\mathbb{R}$ such that both (1) and (2) hold for S:

1. S is not the countable union of closed sets.
2. S is not the countable intersection of open sets.

3.6.90.1 Solution. Let A be a subset of [0,1] that is not a countable union of closed sets, and let B be a subset of [2,3] that is not a countable intersection of open sets. (Irrationals and rationals for instance, respectively.) We show that $S = A \cup B$ satisfies (1) and (2). Suppose that S is the countable union of closed sets $\{F_n\}$. Then

$$A = S \cap [0,1] = \left(\bigcup_{n\in\mathbb{N}} F_n \right) \cap [0,1] = \bigcup_{n\in\mathbb{N}} (F_n \cap [0,1])$$

Note that $F_n \cap [0,1]$ is closed for each n, so A is a countable union of closed sets, a contradiction. Likewise, if A is the countable intersection of open sets, it follows that B is the countable intersection of open sets, a contradiction. Hence S satisfies (1) and (2). □

3.6.91 Problem. Let S be a set. Prove that S is infinite if and only if $|A| = |S|$ for some proper subset A of S.

3.6.91.1 Solution. Assume $|A| = |S|$ where A is a proper subset of S. If S is finite, say $|S| = |\{1, 2, .., n\}|$ then $|A| = |\{1, 2, .., n-k\}| < |\{1, 2, .., n\}|$, where k is the number of elements in $S \setminus A$, contradiction. So S is infinite as desired. Conversely assume S is infinite. Choose $x_1 \in S$, then $x_2 \in S \setminus \{x_1\}$, then $x_3 \in S \setminus \{x_1, x_2\}$ in this way we have (by induction) distinct points $x_n \in S$ for all $n \in \mathbb{N}$. Let $T = \{x_n; n \in \mathbb{N}\}$ and define $f : T \to T$ by $f(x_n) = x_{2n}$. Since the x_n's are distinct f is injective, so $f : T \to f(T)$ is bijective. Now define $g : S \to S$ by $g(x) = f(x)$ if $x \in T$, and $g(x) = x$ if $x \in S \setminus T$. Since f is injective so is g, so $g : S \to g(S)$ is bijective. Now $A \equiv g(S)$ has the same cardinality as S, yet since $f(T)$ is a proper subset of T, $A = (S \setminus T) \cup f(T)$ is a proper subset of S, as desired. □

3.6.92 Problem. Let G be an open set in $\mathbb{R}$ and $S \subseteq \mathbb{R}$ such that $G \cap S = \emptyset$. Prove that $G \cap S' = \emptyset$. (where S' denote the derived set of S)

3.6.92.1 Solution. Assume that $G \cap S = \emptyset$ and $G \cap S' \neq \emptyset \Rightarrow\ \exists\, x \in G \cap S' \Rightarrow \exists\, r > 0$ such that $N(x; r) \subseteq G$ as x is an interior point of G and $\dot{N}(x; r) \cap S \neq \emptyset$ as x is a limit point of S. Thus $\emptyset \neq \dot{N}(x; r) \cap S \subseteq G \cap S \Rightarrow G \cap S \neq \emptyset$, a contradiction. Hence $G \cap S' = \emptyset$. □

3.6.93 Problem. Find the derived set A' of the following sets:

1. $A = \left\{\frac{1}{2^m} + \frac{1}{3^n} + \frac{1}{5^p};\ m, n, p \in \mathbb{N}\right\}$.

2. $B = \left\{\frac{1}{m} + \frac{1}{n};\ m, n \in \mathbb{N}\right\}$.

3. $C = \left\{\frac{1}{m} + \frac{1}{n} + \frac{1}{p};\ m, n, p \in \mathbb{N}\right\}$.

3.6.93.1 Solution.

1. We claim that the derived set is

$$\{0\} \bigcup \left\{\frac{1}{2^n};\ n \in \mathbb{N}\right\} \bigcup \left\{\frac{1}{3^n};\ n \in \mathbb{N}\right\} \bigcup \left\{\frac{1}{5^n};\ n \in \mathbb{N}\right\}$$
$$\bigcup \left\{\frac{1}{2^m} + \frac{1}{3^n};\ m, n \in \mathbb{N}\right\} \bigcup \left\{\frac{1}{3^m} + \frac{1}{5^n};\ m, n \in \mathbb{N}\right\} \bigcup \left\{\frac{1}{5^m} + \frac{1}{2^n};\ m, n \in \mathbb{N}\right\}.$$

We prove only three of them, and other parts are similar.
Claim 1: 0 is the limit point of A.
Let $\epsilon > 0$. Then $\exists\, u, v, w \in \mathbb{N}$ such that $\frac{1}{2^u} < \frac{\epsilon}{3};\ \frac{1}{3^v} < \frac{\epsilon}{3};\ \frac{1}{5^w} < \frac{\epsilon}{3}$.
Now observe that $0 < \frac{1}{2^u} + \frac{1}{3^v} + \frac{1}{5^w} < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$. Which shows that $\frac{1}{2^u} + \frac{1}{3^v} + \frac{1}{5^w} \in (0, \epsilon)$.
Thus 0 is the limit point of A.
Claim 2: $\frac{1}{2^n}$ is the limit point of A.
Again, there exists $v, w \in \mathbb{N}$ such that $\frac{1}{3^v} < \frac{\epsilon}{2};\ \frac{1}{5^w} < \frac{\epsilon}{2}$.
So, $0 < \frac{1}{2^n} + \frac{1}{3^v} + \frac{1}{5^w} < \frac{1}{2^n} + \frac{\epsilon}{2} + \frac{\epsilon}{2} < \frac{1}{2^n} + \epsilon$. Which shows that $\frac{1}{2^n} + \frac{1}{3^v} + \frac{1}{5^w} \in \left(0, \frac{1}{2^n} + \epsilon\right)$ $\forall\, n \in \mathbb{N}$. Thus $\frac{1}{2^n}$ is the limit point of A.
Claim 3: $\frac{1}{2^m} + \frac{1}{3^n}$ is the limit point of A.
Again, there exists $w \in \mathbb{N}$ such that $\frac{1}{5^w} < \epsilon$.
So, $0 < \frac{1}{2^m} + \frac{1}{3^n} + \frac{1}{5^w} < \frac{1}{2^n} + \frac{1}{3^n} + \frac{\epsilon}{3} < \frac{1}{2^n} + \frac{1}{3^n} + \epsilon$.
Which shows that $\frac{1}{2^m} + \frac{1}{3^n} + \frac{1}{5^w} \in \left(0, \frac{1}{2^m} + \frac{1}{3^n} + \epsilon\right)$ $\forall\, m, n \in \mathbb{N}$. Thus $\frac{1}{2^m} + \frac{1}{3^n}$ is the limit point of A. Hence the result.

2. $B' = \{0\} \cup \left\{\frac{1}{n};\ n \in \mathbb{N}\right\}$.

3. $C' = \{0\} \cup \left\{\frac{1}{n};\ n \in \mathbb{N}\right\} \bigcup \left\{\frac{1}{m} + \frac{1}{n};\ m, n \in \mathbb{N}\right\}$. $\square$

3.6.94 Problem. As the statement $(A')' \subseteq A'$ is true, using the notation $A^{(2)} = (A')', A^{(n)} = (A^{(n-1)})', n = 3, 4, ...$ Give an example, for every positive integer n, of a set A for which A^n is a proper subset of $A^{(n-1)}$.

3.6.94.1 Solution. Let $P = \{2, 3, 5, ..., p_n, ...\}$ be the sequence of primes, then consider the set

$$A = \left\{ \frac{1}{2^{n_1}} + \frac{1}{3^{n_2}} + \frac{1}{5^{n_3}} + ... + \frac{1}{p_n^{n_k}} + ...; n_k \in \mathbb{N}, k = 1, 2, ... \right\}.$$

3.6.95 Problem. Let $E \subseteq \mathbb{R}$, then E' is countable implies that E is countable.

3.6.95.1 Solution. The set $E \cap E' \subseteq E'$ is countable, so $A = E \setminus (E \cap E')$ is the set of isolated points and hence is countable. The set $E = A \cup (E \cap E')$ is the union of two countable sets and is therefore countable. $\square$

3.6.96 Problem. For each of the following statements, determine whether it is true or false and justify your answer.

1. Every sequence in the interval (0,1) has a convergent subsequence.
2. Every sequence in the interval (0,1) has a subsequence that converges to a point in (0,1).
3. Every sequence of rational numbers in $\mathbb{Q}$ has a convergent subsequence in $\mathbb{Q}$.
4. If a sequence of non-negative numbers converges, its limit also is non-negative.
5. Every sequence of non-negative numbers has a convergent subsequence.

3.6.96.1 Solution.

1. False. Consider the subsequence $(1/2n)$ of the sequence $(1/n)$ in (0,1).
2. Same as (1).
3. False. Consider the sequence $(1 + 1/n)^n$ in $\mathbb{Q}$, that converges to an irrational number $e \in \mathbb{Q}^C$.
4. True. If possible, let $(x_n); x_n \geq 0$ be a sequence of non-negative numbers converging to a number x with $x < 0$. Then the nbhd. $\left(x - \frac{|x|}{2}, x + \frac{|x|}{2}\right)$ of x contains infinitely many x_n's which are not non-negative, a contradiction.
5. Same as (1). $\square$

3.6.97 Problem. A **condensation point** x of a subset $A \subseteq \mathbb{R}$ is a point $x \in \mathbb{R}$ such that in every neighborhood of x, there is a nondenumerable set of points of A. Assume that $S \subseteq \mathbb{R}$ and assume that S is not countable. Let T denote the set of condensation points of S. Prove that:

1. $S \setminus T$ is countable,
2. $S \cap T$ is not countable,
3. T is a closed set,
4. T contains no isolated points.

3.6.97.1 Solution.

1. If $S \setminus T$ is uncountable, then there exists a point $x \in S \setminus T$ such that x is a condensation point of S. So $x \in T$. Thus we have $x \in S \cap T$ which is absurd as $x \in S \setminus T$.

2. Suppose $S \cap T$ is countable, then $S = (S \cap T) \cup (S \setminus T)$ is countable by (a) which is impossible. Hence $S \cap T$ is not countable.

3. Let x be an accumulation point of T, then $\hat{B}(x; r) \cap T \neq \emptyset \ \forall \ r > 0$. We show that $x \in T$. i.e. x is a condensation point of S. Suppose not, then there exists $s > 0$ such that $B(x; s) \cap T$ is a countable set. Let $y \in B(x; s) \cap T$ then there exists a $t > 0$ such that $B(y; t) \subseteq B(x; s)$ and $B(y; t) \cap S$ is countable, which shows that an uncountable is a subset of a countable set, a contradiction. Hence $x \in T$. Thus T is closed.

4. Suppose that $x \in T$ is an isolated point of T, then there exists $s > 0$ such that $B(x; s) \cap T = \{x\}$. Again $x \in T \Rightarrow \hat{B}(x; r) \cap S$ is uncountable.i.e. $y \in \hat{B}(x; r) \cap S$ is a condensation point of S. i.e. $y \in T$, which is impossible. Hence x is not an isolated point of T. □

3.6.98 Problem. Show that if A has no condensation point, then it is denumerable.

3.6.98.1 Solution. Left to the reader. □

3.6.99 Problem (Cantor-Bendixon theorem). A set $S \subseteq \mathbb{R}$ is called **perfect** if $S = S'$, that is, if S is a closed set which contains no isolated points. Prove that every uncountable closed set $S \subseteq \mathbb{R}$ can be expressed in the form $S = A \cup B$, where A is perfect and B is countable.

3.6.99.1 Solution. Let S be an uncountable closed set. Then $S = (S \cap T) \cup (S \setminus T)$ where T denote the set of condensation points of S. Note that condensation points of S are accumulation points of S. Thus $T \subseteq S \Rightarrow S$ is closed. Define $A = S \cap T$ and $B = F \setminus T$, then $S = A \cup B$. □

3.6.100 Problem (Lindelöf). Let $\{A_\lambda; \lambda \in \Lambda\}$ be a collection of open subsets of $\mathbb{R}$. Then there is a countable subset $\{\lambda_1, \lambda_2, ...\} \subset \Lambda$ such that

$$\bigcup_{\alpha \in \Lambda} A_\lambda = \bigcup_{k=1}^{\infty} A_{\lambda_k}.$$

3.6.100.1 Solution. Let $\mathcal{A} = \bigcup_{\lambda \in \Lambda} A_\lambda$. Then, $\forall \ x \in \mathcal{A}$; we have $x \in A_{\lambda_x}$ for some $\lambda_x \in \Lambda$ and, since A_{λ_x} is open, we can find $\epsilon_x > 0$ with $x \in B(x; \epsilon_x) \subseteq A_{\lambda_x}$. Using the fact that the set $\mathbb{Q}$ of rational numbers is dense in $\mathbb{R}$; we can find a rational number $r_x > 0$ such that $x \in B(x; r_x) \subseteq B(x; \epsilon_x)$. Now the set $\{r_x; x \in \mathcal{A}\} \subseteq \mathbb{Q}$ is countable and hence can be written as $\{r_x; x \in \mathcal{A}\} = \{r_1, r_2, ...\}$ where $r_k = r_{x_k}$ for some $x_k \in \mathcal{A}$. If for each $k \in \mathbb{N}$ we pick $\lambda_k \in \Lambda$ such that $B(x_k; r_k) \subseteq A_{\lambda_k}$; then we have a countable subcollection $\{A_{\lambda_k}; k \in \mathbb{N}\} \subseteq \{A_\lambda; \lambda \in \Lambda\}$ which satisfies $\mathcal{A} = \bigcup_{k=1}^{\infty} A_{\lambda_k}$. □

3.6.101 Problem. A set $O \subseteq \mathbb{R}$ is open if and only if it is a countable union of pairwise disjoint open intervals.

3.6.101.1 Solution. If $O \subseteq \mathbb{R}$ is a disjoint union of open intervals, then it is obviously open. To prove the converse, define an equivalence relation on O as follows. For any $a, b \in O$; let us say that a is equivalent to b, and write $a \sim b$; if the (possibly empty) open interval with endpoints a and b is contained in O: Now, this is obviously reflexive and symmetric. (Why?) To prove the transitivity property, let a, b, c be the three (distinct) elements of O such that $a \sim b$ and $b \sim c$: Then, assuming

(without loss of generality) that $a < b$; we have the three possible cases $a < c < b; c < a < b$; and $a < b < c$; and it follows at once from $a \sim b$ and $b \sim c$ that we have $(a, c) \subseteq O$ in all these cases. Now, for each $x \in O$; let $[x]$ denote its equivalence class. Since $x \in [x]$ and since two equivalence classes are either identical or disjoint, $\{[x]; x \in O\}$ is a partition of O; so we need only show that each $[x]$ is an open interval; because then, by the density of $\mathbb{Q}$; each $[x]$ contains a (necessarily different) rational number and hence $\{[x]; x \in O\}$ must be countable. First, to prove that $[x]$ is an interval, let $y, z \in [x]$ be any pair of distinct elements with, say, $y < z$. Then, if $y < u < z$; we have $y \sim u \sim z$ (why?) and hence $(y, z) \subseteq [x]$. Finally, to show that $[x]$ is open, note that, if $y \in [x]$; then (since O is open) there exists $\epsilon > 0$ such that $(y - \epsilon, y + \epsilon) \subseteq O$. But then $(y - \epsilon, y + \epsilon) \subseteq [y] = [x]$ and the proof is complete. □

3.6.102 Problem. Show that the set S of rational numbers in the interval (0,1) cannot be expressed as the intersection of a countable collection of open sets.

3.6.102.1 Solution. Write $S = \{x_1, x_2, .., x_n, ..\}$. Assume $S = \cap_{k=1}^{\infty} S_k$, where each S_k is open, construct a sequence (Q_n) of closed intervals such that $Q_{n+1} \subseteq Q_n \subseteq S_n$ and such that $x_n \notin Q_n$, then obtain a contradiction. □

3.6.103 Problem. Show that for each $\epsilon > 0$ there exists a sequence of intervals (I_n) with the properties

$$\mathbb{Q} \subset \bigcup_{n=1}^{\infty} I_n \text{ and } \sum_{n=1}^{\infty} |I_n| < \epsilon.$$

3.6.103.1 Solution. Enumerate the rationals with the sequence (r_n). Define

$$I_n = [r_n, r_n + \epsilon/2^{n+1}].$$

Then $\mathbb{Q} \subset \bigcup_{n=1}^{\infty} I_n$ and

$$\sum_{n=1}^{\infty} |I_n| = \sum_{n=1}^{\infty} \epsilon/2^{n+1} < \epsilon/2 < \epsilon,$$

as desired. □

3.6.104 Problem. Is it possible to express [0,1] as the union of nondegenerate disjoint closed intervals each of them of length less than 1?

3.6.104.1 Solution. No. If yes, consider the set S of all endpoints of such intervals. Show that S is perfect and thus uncountable, but any such interval contains rational points. □

3.6.105 Problem. Prove (1) ⇒ (2)

1. (Baire Category Theorem) Let F be a nonempty closed subset of $\mathbb{R}$. Let (G_n) be a sequence of open dense subsets of F. Then $\bigcap_{n=1}^{\infty} G_n$ is dense in F.

2. (Baire) Let F be a nonempty closed subset of $\mathbb{R}$. Then, every nonempty open subset O of F is of second category in F.

3.6.105.1 Solution. Let O be a nonempty open subset of F. Assume that O is of first category in F, i.e., $O = \bigcup_{n=1}^{\infty} A_n$, where each A_n is nowhere dense in F. It follows easily that $G_n = (\overline{A_n})^C$ is open and dense in F. By Baire Category Theorem, $\bigcap_{n=1}^{\infty} G_n \left(= \left(\bigcup_{n=1}^{\infty} \overline{A_n}\right)^C\right)$ is dense. However, $\left(\bigcup_{n=1}^{\infty} \overline{A_n}\right)^C \cap O = \emptyset$, a contradiction. □

3.6.106 Problem. Let K be a nonempty finite set in the real line $\mathbb{R}$ and let (a_n) be a sequence in K that converges to $a \in \mathbb{R}$. Show that $a \in K$.

3.6.106.1 Solution. Hint. If $a \notin K$ find $\delta > 0$ such that $(a-\delta, a+\delta) \cap K = \emptyset$. Then, eventually $a_n \in (a-\delta, a+\delta)$, a contradiction. Observe that this exercise shows that every finite subset of $\mathbb{R}$ is closed. □

3.6.107 Problem. Let f be a real valued function defined on [0,1]. Suppose that there is a positive number M having the following condition: for every choice of a finite number of points $x_1, x_2..., x_n \in [0,1]$, we have

$$|f(x_1) + f(x_2) + ... + f(x_n)| \leq M$$

Prove that $S = \{x; x \in [0,1], f(x) \neq 0\}$ is countable.

3.6.107.1 Solution. Hint: Let $S_n = \{x \in [0,1]; |f(x)| \geq 1/n\}$, then S_n is a finite set by hypothesis. In addition, $S = \cup_{n=1}^{\infty} S_n$. So, S is countable. □

3.6.108 Problem. Prove that a nonempty, bounded closed set S in $\mathbb{R}$ is either a closed interval, or that S can be obtained from a closed interval by removing a countable disjoint collection of open intervals whose endpoints belong to S.

3.6.108.1 Solution. If S is an interval, then it is clear that S is a closed interval. Suppose that S is not an interval. Since $S(\neq \emptyset)$ is bounded and closed, both $\sup S$ and $\inf S$ are in S. So, $\mathbb{R} \setminus S = [\inf S, \sup S] \setminus S$. Denote $[\inf S, \sup S]$ by I. So $\mathbb{R} \setminus S$ is open, then by Representation Theorem for Open Sets on The Real Line, we have

$$\mathbb{R} \setminus S = \bigcup_{i=1}^{\infty} I_m = I \setminus S$$

which implies that

$$S = I \setminus \bigcup_{i=1}^{\infty} I_m.$$

That is, S can be obtained from a closed interval by removing a countable collection of disjoint open intervals whose endpoints belong to S. □

3.6.109 Problem. Prove the uncountability of $I = [0,1]$ by arguing as follows: Assume that $I = \{x_1, x_2, x_3, ...\}$ is denumerable and choose a closed interval $I_1 \subseteq I$ such that $x_1 \notin I_1$ and a closed interval $I_2 \subseteq I_1$ such that $x_2 \notin I_2$ and so on. Now apply Nested Intervals Theorem to obtain a contradiction.

3.6.109.1 Solution. Left as an exercise.

3.6.110 Problem. Suppose that $-\infty < a < b < \infty$ and $(a,b) \subseteq \cup_{i=1}^{\infty} E_i$. Prove that the closure of at least one of the sets E_n has an interior point.

3.6.110.1 Solution. Let $\overline{E_n}$ be the closure of E_n. Arguing by contradiction, construct a sequence of nested closed intervals $A_n \subseteq (a,b)$ such that $A_n \cap \overline{E_n} \neq \emptyset$ for any $n \in \mathbb{N}$. □

3.7 Additional Exercises on Chapter 3.

3.7.1 Exercise. Let $x \in \mathbb{R}$ and let $\mathcal{N}_x$ be the family of all neighborhoods of x. Then show that

1. If $U \in \mathcal{N}_x$, then $x \in U$.
2. If $U, V \in \mathcal{N}_x$, then $U \cap V \in \mathcal{N}_x$.
3. If $U \in \mathcal{N}_x$ and $V \subseteq U$, then $V \in \mathcal{N}_x$.
4. If $U \in \mathcal{N}_x$, then $\exists\, V \in \mathcal{N}_x$ such that $V \subseteq U$ and $V \in \mathcal{N}_y \;\forall\, y \in V$.

3.7.2 Exercise. Let D be dense in $\mathbb{R}$. Prove: $\overline{D \cap G} = \overline{G}$ for every open set $G \subseteq \mathbb{R}$.

3.7.3 Exercise. Let E and G be dense in $\mathbb{R}$. Prove: If E and G are open, then $E \cap G$ is also dense in $\mathbb{R}$.

3.7.4 Exercise. Let $A \subseteq \mathbb{R}$. Prove the following statements:

1. A is open, iff $\partial A \subseteq \mathbb{R} \setminus A$ iff $\partial A = \overline{A} \setminus A$.
2. A is closed, iff $\partial A \subseteq A$ iff $\partial A = A \setminus A^{\circ}$.
3. A is clopen (both open and closed) iff $\partial A = \emptyset$.

3.7.5 Exercise. Prove the following for subsets A, B of $\mathbb{R}$:

1. $\partial\,(\partial A) \subseteq \partial A$.
2. $\partial\,(\partial\,(\partial A)) = \partial\,(\partial A)$.
3. $\partial\,(A \cup B) \subseteq \partial A \cup \partial B \subseteq \partial\,(A \cup B) \cup A \cup B$
4. $A \cap B \cap \partial(A \cap B) = A \cap B \cap (\partial A \cup \partial B)$

where ∂A denotes the boundary of $A \subseteq \mathbb{R}$. Give examples where the equality does not hold.

3.7.6 Exercise. Let $E \subseteq \mathbb{R}$ and $\epsilon > 0$. For $x \in E$, define $E(x, \epsilon) = E \cap (x - \epsilon, x + \epsilon)$. If for some $\epsilon > 0, E(x, \epsilon)$, is an infinite set, but countable, then x is said to be a **sparse point** of E. By considering intervals with rational end points, prove that the set of isolated points of E and the set of sparse points of E are both countable.

3.7.7 Exercise. Prove or disprove:

1. Every point of an open set is an accumulation point.
2. Every point of a closed interval is an accumulation point.

3.7.8 Exercise. Show that the union of a finite number of nowhere dense subsets of $\mathbb{R}$ is itself nowhere dense.

3.7.9 Exercise. Show that, for any closed or open set S in $\mathbb{R}$, its boundary ∂S is nowhere dense. Is this still true if S is neither closed nor open?

3.7.10 Exercise. Let $S = \left\{ \frac{5}{n} + \frac{n}{\sqrt{5}};\ n \in \mathbb{N} \right\}$. Show that $S' = \emptyset$.

3.7.11 Exercise. A perfect set is uncountable.

3.7.12 Exercise. Prove that every countably infinite closed set has an isolated point.

3.7.13 Exercise. Determine which of the following statements are true?

1. $\left(\overline{A}\right)^{\circ} = A^{\circ}$
2. $\overline{A} \cap A = A$.
3. $\overline{A^{\circ}} = A$.
4. $\partial\left(\overline{A}\right) = \partial A$.

3.7.14 Exercise. Consider an uncountable subset A of $\mathbb{R}$. Prove that A contains points x (called condensation points) such that every neighborhood of x contains uncountably many points of A.

3.7.15 Exercise. Show that every point of a closed set is either an accumulation point or an isolated point.

3.7.16 Exercise. Every uncountable set E is the union of a countable set C and a set B which is a set of all condensation points of B.

3.7.17 Exercise. Answer the following questions.

1. Let $A = [0, 1]$, describe, if possible, sets that are open relative to A but not open as subsets of $\mathbb{R}$.
2. Let $A = [0, 1]$, describe, if possible, sets that are closed relative to A but not closed as subsets of $\mathbb{R}$.
3. Let $A = (0, 1)$, describe, if possible, sets that are open relative to A but not open as subsets of $\mathbb{R}$.
4. Let $A = (0, 1)$, describe, if possible, sets that are closed relative to A but not closed as subsets of $\mathbb{R}$.

3.7.18 Exercise. Under what conditions is it true that

1. $|x + y| = |x| + |y|$?
2. $|x - y| + |y - z| = |x - z|,\ x, y, z \in \mathbb{R}$?

3.7.19 Exercise. Use this definition of "dense in a set" to answer the following questions: A set E of real numbers is said to be **dense** in a set A if every interval (a, b) that contains a point of A also contains a point of E.

1. Show that dense in the set of all reals is the same as dense.
2. Give an example of a set E dense in $\mathbb{N}$ but with $E \cap \mathbb{N} = \emptyset$.
3. Show that the irrationals are dense in the rationals. (A real number is *irrational* if it is not rational, that is if it belongs to $\mathbb{R}$ but not to $\mathbb{Q}$.)
4. Show that the rationals are dense in the irrationals.

5. What property does a set E have that is equivalent to the assertion that $\mathbb{R} \setminus E$ is dense in E?

3.7.20 Exercise. Prove that A is dense in B iff $\overline{A} \supseteq B$.

3.7.21 Exercise. Prove that every set A is dense in its closure $\overline{A}$..

3.7.22 Exercise. Prove that if A is dense in B and $C \subseteq B$, then A is dense in C.

3.7.23 Exercise. Prove that if $A \subseteq B$, and A is dense in B, then $A = B$. Is the statement correct without the assumption that $A \subseteq B$?

3.7.24 Exercise. Give an example of a sequence of nowhere dense sets whose union is not nowhere dense.

3.7.25 Exercise. Which of the following statements are true?

1. Every subset of a nowhere dense set is nowhere dense.
2. If A is nowhere dense, then so too is $A + c = \{t + c; t \in A\}$ for every number c.
3. If A is nowhere dense, then so too is $cA = \{ct; t \in A\}$ for every positive number c.
4. If A is nowhere dense, then so too is A', the set of derived points of A.
5. A nowhere dense set can have no interior points.
6. A set that has no interior points must be nowhere dense.
7. Every point in a nowhere dense set must be isolated.
8. If every point in a set is isolated, then that set must be nowhere dense.

3.7.26 Exercise. If A is nowhere dense, what can you say about $\mathbb{R} \setminus A$? If A is dense, what can you say about $\mathbb{R} \setminus A$?

3.7.27 Exercise. Prove that a set $A \subseteq \mathbb{R}$ is nowhere dense iff A contains no intervals; equivalently, the interior of A is empty.

3.7.28 Exercise. What should the statement "A is nowhere dense in the interval I" mean? Give an example of a set that is nowhere dense in [0,1] but is not nowhere dense in $\mathbb{R}$.

3.7.29 Exercise. Let A and B be subsets of $\mathbb{R}$. What should the statement A is nowhere dense in the B" mean? Is $\mathbb{N}$ nowhere dense in [0,10]? Is $\mathbb{N}$ nowhere dense in $\mathbb{Z}$? Is $\{4\}$ nowhere dense in $\mathbb{N}$?

3.7.30 Exercise. Prove that the complement of a dense open subset of $\mathbb{R}$ is nowhere dense in $\mathbb{R}$.

3.7.31 Exercise. Let $f : \mathbb{R} \to \mathbb{R}$ be a strictly increasing continuous function. Show that f maps nowhere dense sets to nowhere dense sets; that is,

$$f(E) = \{f(x); x \in E\}$$

is nowhere dense if E is nowhere dense.

3.7.32 Exercise. Show that the union of any number of sets each of which is dense-in-itself is dense-in-itseIf.

3.7.33 Exercise. Is the intersection of a finite or denumerable number of sets each of which is dense-in-itself always dense-in-itself?

3.7.34 Exercise. Give an example of a set which is neither nowhere dense nor everywhere dense.

3.7.35 Exercise. Let F be a closed nonempty subset of $\mathbb{R}$. Assume that $F = \cup_{n=1}^{\infty} F_n$, where each F_n is a closed set. Then there exists $n \in \mathbb{N}$ such that F_n has a nonempty interior relatively to F.

3.7.36 Exercise.

1. Every interval is the union of an increasing sequence of closed intervals.
2. The union of an increasing sequence of intervals is an interval.

3.7.37 Exercise. How would you extend the operation $x \to -x$ from $\mathbb{R} \to \mathbb{R} \cup \{\infty, \infty\}$? Why?

3.7.38 Exercise. Let $f : \mathbb{R} \to \mathbb{R}$ be increasing, i.e., $x \leq y \Rightarrow f(x) \leq f(y)$. Fix a real number a. Let $A = \{x \in \mathbb{R}; a \leq f(x)\}$. Suppose A is nonempty. Prove A is an interval. (Hint: f need not be onto.)

3.7.39 Exercise. Show that $\mathbb{N}$ contains infinitely many pairwise disjoint infinite subsets.

3.7.40 Exercise. The boundary of a subset S of $\mathbb{R}$ is the set $\overline{S} \cap \overline{S^C}$. Show that the boundary of a closed set is nowhere dense.

3.7.41 Exercise. A subset of $\mathbb{R}$ is called **clairsemé** if it has no nonempty dense-in-itself subset. Show that every set is the union of a dense-in-itself set and a set which is clairsemé.

3.7.42 Exercise. Prove that any infinite set can be written as the countably infinite union of pairwise disjoint infinite subsets.

3.7.43 Exercise. Give an example of an infinite closed set in $\mathbb{R}$ containing only irrationals. Is there an open set consisting entirely of irrationals?

3.7.44 Exercise. Give an example of a function $f : [0, 1] \to \mathbb{R}$ for which $f([0, 1])$ is a countably infinite union of disjoint intervals.

3.7.45 Exercise.

1. Prove that, if $E(C)$ is the set of all condensation points of E, then $E \setminus E(C)$ is countable.
2. Prove that $E(C)$ is a closed set, and a perfect set.
3. Every uncountable set has a condensation point.
4. Every closed set can be written as the union of a perfect set and a countable set.

3.7.46 Exercise. (Lord Rayleigh). Let α and β be two positive irrational numbers such that $1/\alpha + 1/\beta = 1$. Consider the sets

$$A = \{[n\alpha]; n \in \mathbb{N}\} \text{ and } B = \{[n\beta] : n \in \mathbb{N}\}.$$

Prove that $A \cap B = \{0\}$ and $A \cup B = \mathbb{N}$.

3.7.47 Exercise. Show that the set $A = \{m + n\alpha;\ m, n \in \mathbb{Z} \text{ and } \alpha \text{ is irrational}\}$ is dense in $\mathbb{R}$.

3.7.48 Exercise. Call a set $A \subseteq \mathbb{R}$ **residual** if its complement is dense, and call it **nowhere dense** if its closure has empty interior. Example: every finite subset $\{x_1, x_2, ..., x_n\}$ is residual, and nowhere dense.

1. A nowhere dense set is a residual set.
2. A is nowhere dense if and only if $A \subseteq \overline{(\overline{A})^C}$.
3. The union of a residual and a nowhere dense set is a residual set.
4. The boundary of a closed (or open) set is nowhere dense.
5. For any set A, both $A \cap \overline{A^C}$ and $\overline{A} \cap A^C$ are residual.
6. The boundary of any set is the union of two residual sets.

3.7.49 Exercise. An open set $U \subseteq \mathbb{R}$ is called **regular open** set if $U = (\overline{U})^\circ$; a closed set $A \subseteq \mathbb{R}$ is called **regular closed** if $A = \overline{(A)^\circ}$. Example: any open interval and any closed interval is regular. But $(a, b) \cup (b, c)$ is not regular open and $[1, 2] \cup \{3\}$ is not regular closed. Prove that:

1. If A is closed, then A° is a regular open set.
2. If A is open, then $\overline{A}$ is a regular closed set.
3. The complement of a regular open (closed) set is a regular closed (open) set.
4. If U, V are regular open sets, then $U \subseteq V$ iff $\overline{U} \subseteq \overline{V}$.
5. If A, B are regular closed sets, then $A \subseteq B$ iff $A^\circ \subseteq B^\circ$.
6. If A, B are regular closed sets, so also is $A \cup B$.
7. If A, B are regular open sets, so also is $A \cap B$.

3.7.50 Exercise. Let A be a nonempty subset of the real line, B the set of points $x \in A$ such that there is an interval (x, y) with $y > x$ which has an empty intersection with A. Show that B is at most denumerable (prove that B is equipotent with a set of open intervals, no two of which have common points).

3.7.51 Exercise. Show that each of the following is equivalent to the statement that A is nowhere dense in $\mathbb{R}$:

1. A contains no nonempty open set.
2. Each nonempty open set in $\mathbb{R}$ contains a nonempty open subset that is disjoint from A.
3. Each nonempty open set in $\mathbb{R}$ contains an open ball that is disjoint from A.

3.7.52 Exercise. If A is nowhere dense in $\mathbb{R}$, and if G is a nonempty open set in $\mathbb{R}$, prove that A is nowhere dense in G.

3.7.53 Exercise. Is there a dense, open set in $\mathbb{R}$ with uncountable complement? Explain.

3.7.54 Exercise. If E and F are subsets of $\mathbb{R}$, define the distance δ from E to F by $\delta(E, F) = \inf\{d(x, F); x \in E\}$. Show that $\delta(E, F) = \delta(F, E)$.
(For $A \subseteq \mathbb{R}$, define $\delta(x, A) = \inf\{|x - a|; x \in A\}$.)

3.7.55 Exercise. Show that $diam(E \cup F) \leq diamE + diamF + \delta(E, F)$. Give an example in which strict inequality holds.
(For $A \subseteq \mathbb{R}$, define $diam(A) = \sup\{|a - b|; a, b \in A\}$.)

3.7.56 Exercise. Prove or disprove:

1. Every bounded infinite set of real numbers has an accumulation point.
2. Every infinite, countable set of real numbers has an accumulation point.
3. Every uncountable set of real numbers has an accumulation point.
4. Every nonempty open set of real numbers has an accumulation point.
5. Every nonempty closed set of real numbers has an accumulation point.

3.7.57 Exercise. Give an example of a nested sequence of bounded sets with empty intersection.

3.7.58 Exercise. Give an example of a nested sequence of closed sets with empty intersection.

3.7.59 Exercise. Let E be an uncountable set (bounded or not). Show that E has a point of accumulation.

3.7.60 Exercise. Let E be a closed set. Show that E is compact if and only if every infinite subset of E has at least one point of accumulation.

3.7.61 Exercise. Define what is meant by an interior point of a set and by an open set. Give examples of each of the following or else prove that such a set cannot exist.

1. A nonempty bounded, open set that is denumerable.
2. A nonempty bounded, closed set that is denumerable.
3. A nonempty bounded, open set with no accumulation points.
4. A nonempty bounded, closed set with no accumulation points.
5. Two sets A and B that are not open and yet $A \cup B$ is open.
6. Two sets A and B that are not open and yet $A \cap B$ is open.

3.7.62 Exercise. Let E be a bounded, nonempty open set. Show that $\sup E$ and $\inf E$ are points of accumulation of E neither of which belongs to E.

3.7.63 Exercise. For any set S we let S' denote the set of its accumulation points. Give an example that illustrates how each of the following can occur:

1. $S' = \emptyset$.
2. S' contains just one point.

3. S' contains exactly two points.
4. S' is countably infinite.
5. S' is uncountable.
6. S' is nonempty but $S'' = \emptyset$.
7. S' is nonempty but $S'' = S$.
8. S' is nonempty but $S = S' = S'' = ... = S^{(n)}$.

3.7.64 Exercise. A set E is said to be dense in $\mathbb{R}$ if $\overline{E} = \mathbb{R}$.

1. Find a set so that E and $\mathbb{R} \setminus E$ are both dense.
2. Find a countable dense set.
3. Find an uncountable dense set.
4. Show that E is dense if and only if $E \cap I \neq \emptyset$; for every open interval I.
5. Show that the intersection of two dense sets need not be dense.
6. Show that the intersection of two dense open sets is dense.
7. Show that the union of two dense sets is dense.

3.7.65 Exercise. Represent [0,1] as the sum of $\mathfrak{c}$ perfect sets which are pairwise disjoint. Hint: Let C be the Cantor set. Then $[0,1] = \cup_{x\in(0,1]} xC$.

3.7.66 Exercise. Give an example of a set E (other than $\emptyset$; and $\mathbb{R}$) that has the following property or else show that such a set cannot exist:

1. E has infinitely many points but no interior points.
2. E has infinitely many points but no points of accumulation.
3. E is open and unbounded.
4. E is closed and unbounded.
5. E has infinitely many points of accumulation but no interior points.
6. E is open but has no points of accumulation.
7. E is closed but has no points of accumulation.
8. E is compact and has no interior points.
9. E, E' and E'' are different.
10. E is countable and $E' = \{0, 1\}$.
11. E is countable and $E' = [0, 1]$.
12. E is countable and $E' = (0, 1)$.

3.7.67 Exercise. Can a perfect set be enumerable? isolated? consist of boundary points only? Give your reason in each case and illustrate your answer.

3.7.68 Exercise. If the derived set E' is enumerable, could one exhaust E by the continued seperation of isolated subsets? Illustrate your conclusion.

3.7.69 Exercise. In case a set E is dense in itself but not closed, show that E' cannot be enumerable but must have the cardinal number of the continuum.

3.7.70 Exercise. Could a closed set be enumerable? have a complementary set having the cardinal the cardinal number of the continuum. Illustrate your conclusion.

3.7.71 Exercise. Can a set of first species be everywhere dense in a given interval?

3.7.72 Exercise. Do the irrational points form a closed set? an open set? What points are the boundary points of the set? Does it have a closed subset? Does it contain points of condensation? Justify.

3.7.73 Exercise. Is the union of finite number of discrete sets necessarily discrete? Can such a union be everywhere dense in a given interval? Illustrate.

3.7.74 Exercise. If $S \subseteq \mathbb{R}$ is a nonempty bounded set, and $I_s = [\inf S, \sup S]$, show that $S \subseteq I_S$. Moreover, if J is any closed bounded interval containing S, show that $I_S \subseteq J$.

3.7.75 Exercise. A point a in a set $E \subseteq \mathbb{R}$ is said to be **semi-isolated** if there exists a number $\epsilon > 0$ such that at least one of the intervals $(a - \epsilon, a)$ and $(a, a + \epsilon)$ does not contain points of E. Prove that the set of semi-isolated points of any set $E \subseteq \mathbb{R}$ is at most countable.

3.7.76 Exercise. Let G be an open subset of $\mathbb{R}$ that is unbounded above. Does there exist a positive number x_0 such that the set G contains infinitely many points of the form $mx_0 (n \in \mathbb{N})$?

3.7.77 Exercise. Let $\{G_n\}$ be a sequence of open subset of $\mathbb{R}$ that are unbounded above. Prove that there is a number $x_0 > 0$ such that each of the sets G_n contains infinitely many points of the form mx_0 $(m \in \mathbb{N})$.

3.7.78 Exercise. Prove that a nonempty open interval cannot be represented as the union of a sequence of disjoint closed sets.

3.7.79 Exercise. Prove that the set of irrational numbers is not the union of a sequence of closed sets.

3.7.80 Exercise. Prove that there exists a closed set containing only irrational numbers. Hint: Enumerate the rationals in [0,1] by $\{r_1, r_2, r_3, ..., r_i, ...\}$ and consider the intervals $E_i = \left(r_i - \frac{1}{4^i}, r_i + \frac{1}{4^i}\right)$ then $E = \bigcup_{i=1}^{\infty} E_i$ is an open set and $F = [0, 1] \setminus E$ is a closed set containing no rational numbers. The sum of the lengths of all intervals in E is

$$\sum_{i=1}^{\infty} \frac{2}{4^i} = \frac{2}{3}.$$

Since the interval [0,1] has length 1, there must be quite few points in F containing no intervals.

3.7.81 Exercise. Define an equivalence relation on the circle $S^1 = \{z \in \mathbb{C}; |z| = 1\}$ by taking $z \sim w$ if $z/w = e^{2\pi i\theta}$, where $\theta \in \mathbb{Q}$. Prove that the set of limit points of any equivalence class coincides with S^1.

Chapter 4

Real-valued Functions on Subsets of $\mathbb{R}$

The essential quality of a proof is to compel belief.
- Pierre de Fermat

4.1 Real-valued functions

4.1.1 Definition. A **real-valued function** is a function whose domain is a subset of $\mathbb{R}$ and whose co-domain is $\mathbb{R}$.

4.1.2 Definition. (Monotone functions) Let $X \subseteq \mathbb{R}$. A function $f : X \to \mathbb{R}$ is called

1. **increasing** if $x \geq y \Rightarrow f(x) \geq f(y)$,
2. **strictly increasing** if $x > y \Rightarrow f(x) > f(y)$,
3. **decreasing** if $x \geq y \Rightarrow f(x) \leq f(y)$,
4. **strictly decreasing** if $x > y \Rightarrow f(x) < f(y)$, for all $x, y \in X$.

f is called **monotone** if f is either increasing or decreasing and f is called **strictly monotone** if f is either strictly increasing or strictly decreasing.

4.1.3 Note.

1. The sum of two monotonic functions need not be a monotonic function. Let $f(x) = x, g(x) = -x^2, x \geq 0$. Then $f + g$ is neither increasing nor decreasing on $[0, \infty)$. So, the set of monotonic functions does not form a vector space.
2. The product of two monotonic functions need not be a monotonic function. Indeed, consider $f, g : \mathbb{R} \to \mathbb{R}$ defined by $f(t) = t, g(t) = signt$. Then $(fg)(t) = |t|$, which is not monotonic on $\mathbb{R}$.

4.2 Periodic functions

4.2.1 Definition. Symmetric: The set $X \subseteq \mathbb{R}$ is said to be **symmetric** (to the origin) if for every point $x \in X \Rightarrow -x \in X$. (Notice that for $x \neq 0$ the points x and $-x$ are symmetric to the origin.) Suppose that $A, B \subseteq \mathbb{R}$ and $f : A \to B$ and A is symmetric. Then f is an **even function** if for every $x \in A \Rightarrow f(-x) = f(x)$ and f is an **odd function** if for every $x \in A \Rightarrow f(-x) = -f(x)$. Geometrically, the graph of an even function is symmetric to the y-axis, while the graph of an odd function is symmetric to the origin.
A function $f : \mathbb{R} \to \mathbb{R}$ is said to be periodic on $\mathbb{R}$ if there exists a number $p > 0$ such that $f(x+p) = f(x)\ \forall\ x \in \mathbb{R}$. In other words, a number $p \neq 0$ is called the **period** of the function $f : A \to B$ the points $x+p$ and $x-p$ are also in A and $f(x+p) = f(x)$. The smallest positive period, if it exists, is called the **basic period** or **fundamental period** of the function f. Clearly, if we know the basic period p of a function, then it is enough to draw its graph on any set $X \subseteq A$ of the length p.

Every real-valued function f defined on a bounded interval $[a, b]$ can be extended to a periodic function f defined on $\mathbb{R}$ provided that $f(a) = f(b)$. Indeed, its *extension by periodicity* is given by the formula

$$\overline{f} = f\left(x - \left[\frac{x-a}{b-a}\right](b-a)\right),\ x \in \mathbb{R}.$$

Signum function $sign : \mathbb{R} \to \mathbb{R}$ is defined by $sign(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x < 0. \end{cases}$

4.3 Convex functions

4.3.1 Definition. A function $f : I \to \mathbb{R}$ is called **convex** on I, if $\forall\ x, y \in I$,

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \tag{4.1}$$

whenever $0 < \lambda < 1$. A function $f : I \to \mathbb{R}$ is called **strictly convex** on I, if $\forall\ x, y \in I$,

$$f(\lambda x + (1-\lambda)y) < \lambda f(x) + (1-\lambda)f(y)$$

whenever $0 < \lambda < 1$.

4.3.2 Theorem. (Jensen's Inequality). If $f : I \to \mathbb{R}$ is convex on I; then it satisfies Jensen's inequality:

$$f\left(\sum_{k=1}^{n} \lambda_k x_k\right) \leq \sum_{k=1}^{n} \lambda_k f(x_k),$$

for any $x_1, ..., x_n \in I$ and $\lambda_1, ..., \lambda_n \in [0, 1]$, with $\sum_{k=1}^{n} \lambda_k = 1$.

4.3.3 Theorem. (Three Chords Lemma). Let $f : I \to \mathbb{R}$. Then, f is convex on I if and only if, for any points $a, b, c \in I$ with $a < b < c$, we have

$$\frac{f(b)-f(a)}{b-a} \leq \frac{f(c)-f(a)}{c-a} \leq \frac{f(c)-f(b)}{c-b}$$

which is equivalent to saying that for any fixed $x_0 \in I$ the function

$$\phi(x) = \frac{f(x) - f(x_0)}{x - x_0} \;\forall\, x \in I \setminus \{x_0\}$$

is increasing. It follows from the identity $a(b-c) + b(c-a) + c(a-b) = 0$

4.4 Locally Bounded functions

4.4.1 Definition. We say that a function $f : I \to \mathbb{R}$ is **locally bounded** at $x \in I$, if there exists a neighbourhood $U = B(x, \delta_x) \cap I$ of x, in which f is bounded, and f is **locally bounded** on a set I, if f is locally bounded at each $x \in I$.

Can we conclude that f is bounded on the whole of the set I?

4.4.2 Example. The function $f : (0,1] \to \mathbb{R}$ defined by $f(x) = 1/x$ is locally bounded at every $0 < c \leq 1$, but it is not bounded on (0,1). Since the statement

$$\frac{1}{t} \leq M \;\forall\, t \in (0,1)$$

cannot be true for any M. But this function is locally bounded at each point x here. Let $x \in (0,1)$. Take $\delta_x = \frac{x}{2}$ and $M_x = \frac{2}{x}$. Then

$$f(t) = \frac{1}{t} \leq \frac{2}{x} M_x$$

if $\frac{x}{2} = x - \delta_x < t < x + \delta_x$. What is wrong here? What is there about this set $E = (0,1)$ that does not allow the conclusion? The point 0 is a point of accumulation of (0,1) that does not belong to (0,1), and so there is no assumption that f is bounded at that point. We avoid this difficulty if we assume that E is closed.

4.4.3 Example. The function $f(x) = x$ is locally bounded at each point x in the set $[0,\infty)$ but is not bounded on the set $[0,\infty)$. It is clear that f cannot be bounded on $[0,\infty)$ since the statement

$$f(t) = t \leq M \;\forall\, t \in [0,\infty)$$

cannot be true for any M. But this function is locally bounded at each point x here. Let $x \in [0,\infty)$. Take $\delta_x = 1$ and $M_x = x + 1$. Then

$$f(t) = t \leq x + 1 = M_x$$

if $x - 1 < t < x + 1$. What is wrong here? What is there about this set $E = [0,\infty)$ that does not allow the conclusion? This set is closed and so contains all of its accumulation points so that the difficulty we saw in the preceding example does not arise. The difficulty is that the set is too big, allowing larger and larger bounds as we move to the right. We could avoid this difficulty if we assume that E is bounded.

Thus a function f is locally bounded at each point of a closed and bounded set E. Then f is bounded on the whole of the set E.

4.5 Locally Open, Closed, and Locally Increasing functions

4.5.1 Definition. We say that a function $f : I \to \mathbb{R}$ is **locally open** at $x \in I$, if there exists a neighbourhood $U = B(x, \delta_x) \cap I$ of x, such that $f|_U$ is open, and f is **locally open** on a set I, if f is locally open at each $x \in I$.

4.5.2 Definition. We say that a function $f : I \to \mathbb{R}$ is **locally closed** at $x \in I$, if there exists a neighborhood $U = B(x, \delta_x) \cap I$ of x, such that $f|_U$ is closed, and f is **locally closed** on a set I, if f is locally closed at each $x \in I$.

4.5.3 Definition. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be **locally increasing** or **increasing** at a point x_0, if there is a $\delta > 0$ so that

$$f(x) < f(x_0) < f(y)$$

whenever

$$x_0 - \delta < x < x_0 < y < x_0 + \delta.$$

4.6 Problems and Solutions on Chapter 4

Warning! *Never Never Ever read this solution unless you tried problems quite long time and gave up. Doing so may impair your ability to think and solve problems.*

4.6.1 Problem. Let f be a real-valued function defined for every $x \in [0, 1]$. Suppose there is a positive number M having the following property: for every choice of a finite number of points $x_1, x_2, ..., x_n \in [0, 1]$ in the interval, the sum

$$|f(x_1) + ... + f(x_n)| \leq M.$$

Let S be the set of those x in $0 \leq x \leq 1$ for which $f(x) \neq 0$. Prove that S is countable.

4.6.1.1 Solution. It is clear that $|f(x)| \leq M$, for all $x \in [0, 1]$. We have to show that the set $\{x; 0 < |f(x)| \leq M\} = S$ is countable. Now

$$S = \bigcup_{n=2}^{\infty} \left[\left\{ x \in [0,1]; -M \leq f(x) < -\frac{M}{n} \right\} \cup \left\{ x \in [0,1]; \frac{M}{n} < f(x) \leq M \right\} \right]$$

and the set $S_n^+ = \left\{ x \in [0,1]; \frac{M}{n} < f(x) \leq M \right\}$ contains at most $n-1$ elements. For, if it contains n elements $x_1, x_2, .., x_n \in [0, 1]$, then $\sum_{i=1}^{n} f(x_i) > M$ which contradicts our assumption. Hence S_n^+ is a finite set, similarly S_n^- is also a finite set. Thus $S = \bigcup_{n=2}^{\infty} [S_n^+ \cup S_n^-]$ is a countable union of finite sets and hence S is countable. □

4.6.2 Problem. Suppose that f and g are real-valued functions with common domain $D \subseteq \mathbb{R}$, and assume that f and g are bounded.

1. lf $f(x) \leq g(x)\ \forall\ x \in D$, then

$$\sup f = \sup_{x \in D} f(D) \leq \sup_{x \in D} g(D) = \sup g.$$

2. lf $f(x) \leq g(x)\ \forall\ x \in D$, then there may have no relation between $\sup_{x \in D} f(D)$ and $\inf_{x \in D} g(D)$. Give example.

3. If $f(x) \leq g(y)\ \forall\ x, y \in D$, then
$$\sup_{x\in D} f(D) \leq \inf_{y\in D} g(D).$$

4.6.2.1 Solution.

1. Since $f(x) \leq g(x) \leq \sup_{x\in D} g(D)$, i.e. $\sup_{x\in D} g(D)$ is an upper bound for $f(D)$, hence $\sup_{x\in D} f(D) \leq \sup_{x\in D} g(D)$.
2. Let $f(x) = x^2$ and $g(x) = x$ with $D = \{x; 0 \leq x \leq 1\}$, then $f(x) \leq g(x)\ \forall\ x \in D$. However, we see that $\sup(D) = 1$ and $\inf g(D) = 0$. Since $\sup g(D) = 1$, the conclusion of (1) holds.
3. If possible, let $p = \sup_{x\in D} f(D) > \inf_{y\in D} g(D) = q$ and $0 < \epsilon < \frac{1}{3}(p-q)$, then $\exists\ x_1, y_1 \in D$ such that $f(x_1) > p-\epsilon$ and $f(y_1) < q+\epsilon$, but $f(x_1)-f(y_1) > p-q-2\epsilon > \epsilon > 0$ implies $f(x_1) > f(y_1)$, a contradiction. (Note that the functions in (2) do not satisfy this hypothesis.) □

4.6.3 Problem. The non-empty functions $f,\ g$ have same domain D.

1. Prove that if $f,\ g$ are bounded above, then $f+g$ is bounded above and
$$\sup(f+g) \leq \sup f + \sup g.$$
2. Prove also that if f is bounded below and $g,\ f+g$ are bounded above, then
$$\inf f + \sup g \leq \sup(f+g).$$
3. And we get a chain
$$\inf f + \inf g \leq \inf(f+g) \leq \begin{Bmatrix} \inf f + \sup g \\ \sup f + \inf g \end{Bmatrix} \leq \sup(f+g) \leq \sup f + \sup g.$$

4.6.3.1 Solution.

1. f and g are bounded above implies $\sup f$ and $\sup g$ exists, then
$$\begin{aligned} \sup f &\geq f(x)\ \forall\ x \in D \text{ and} \\ \sup g &\geq g(y)\ \forall\ y \in D \\ \text{in particular, } \sup f + \sup g &\geq f(x) + g(x)\ \forall\ x \in D \\ \Rightarrow \sup f + \sup g &\geq (f+g)(x)\ \forall\ x \in D. \end{aligned}$$
Thus, $\sup f + \sup g$ is an upper bound of $\{(f+g)(x);\ x \in D\}$. Hence
$$\sup f + \sup g \geq \sup(f+g).$$
2. Since f is bounded below and $g,\ f+g$ are bounded above, so, $\inf f, \sup g$ and $\sup(f+g)$ exists. Thus
$$\begin{aligned} \sup(f+g) &\geq (f+g)(x)\ \forall\ x \in D. \\ &\geq f(x) + g(x) \geq \inf f + g(x) \\ \text{Thus, } \sup(f+g) - \inf f &\geq g(x)\ \forall\ x \in D. \\ \Rightarrow \sup(f+g) - \inf f &\text{ is an upper bound of } \{g(x);\ x \in D\} \\ \Rightarrow \sup(f+g) - \inf f &\geq \sup g \\ \Rightarrow \sup(f+g) &\geq \inf f + \sup g. \end{aligned}$$

3. Proof is similar to the above. □

4.6.4 Problem. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is bounded, then

$$\sup_{x\in\mathbb{R}} f(x) - \inf_{x\in\mathbb{R}} f(x) = \sup\{|f(x) - f(y)|; x, y \in \mathbb{R}\}$$

4.6.4.1 Solution. Let $S = \{|f(x) - f(y)|; x, y \in \mathbb{R}\}$ and $M = \sup_{x\in\mathbb{R}} f(x)$ and $m = \inf_{x\in\mathbb{R}} f(x)$, then $f(x) \leq M$, $\forall\, x \in \mathbb{R}$ and $m \leq f(y)$, $\forall\, y \in \mathbb{R}$. Thus $|f(x) - f(y)| \leq M - m\ \forall\, x, y \in \mathbb{R}$, and so $M - m$ is an upper bound for the set S. That is,

$$\sup S \leq M - m \tag{4.2}$$

Let $\epsilon > 0$, then $\exists\, x_1, y_1 \in \mathbb{R}$ such that

$$\begin{aligned} &f(x_1) > M - \frac{\epsilon}{2} \text{ and } f(y_1) < m + \frac{\epsilon}{2} \\ \Rightarrow\ &f(x_1) - f(y_1) > M - m - \epsilon \\ \Rightarrow\ &|f(x_1) - f(y_1)| > M - m - \epsilon. \end{aligned}$$

Hence $M - m = \sup S$. □

4.6.5 Problem. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is bounded, then

$$\sup |f| - \inf |f| \leq \sup f - \inf f.$$

Give an example where equality does not hold.

4.6.5.1 Solution. Use the inequality $||f(x)| - |f(y)|| \leq |f(x) - f(y)|$ and proceed as above.
Example: Let $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x, & \text{if } 0 \leq x \leq 1 \\ 2 - x, & \text{if } 1 \leq x \leq 4 \\ x - 6, & \text{if } 4 \leq x \leq 6 \\ 0, & \text{otherwise.} \end{cases}$$

So,

$$|f(x)| = \begin{cases} x, & \text{if } 0 \leq x \leq 1 \\ 2 - x, & \text{if } 1 \leq x \leq 2 \\ x - 2, & \text{if } 2 \leq x \leq 4 \\ 6 - x, & \text{if } 4 \leq x \leq 6 \\ 0, & \text{otherwise.} \end{cases}$$

Here $M = \sup_{x\in\mathbb{R}} f(x) = 1$ and $m = \inf_{x\in\mathbb{R}} f(x) = -2$. $P = \sup_{x\in\mathbb{R}} |f(x)| = 2$ and $p = \inf_{x\in\mathbb{R}} |f(x)| = 0$. Thus $P - p = 2 < 3 = M - m$. □

4.6.6 Problem. Let us recall that a real-valued function f defined on an open interval I is **increasing at a point** $a \in I$ if there is $\delta > 0$ such that $(a - \delta, a + \delta) \subseteq I$ and $f(x) < f(a) < f(y)$ whenever $a - \delta < x < a < y < a + \delta$. Let f be increasing at each point of an open interval $I \subseteq \mathbb{R}$. Show that f is increasing on I.

4.6.6.1 Solution. Let $a, b \in I$ such that $a < b$. We need to show that $f(a) < f(b)$. To this end let $M = \{x \in I; a < x \leq b, f(a) < f(x)\}$. This set is nonempty since f is increasing at a. We want to show that $b \in M$. First, we show that $\sup M = b$. If $\sup M < b$, then, as f is increasing in $\sup M$, there is $\delta > 0$ such that $(\sup M - \delta, \sup M + \delta) \subseteq (a, b)$ and $f(x) < f(\sup M) < f(y)$ $\forall$ $\sup M - \delta < x < \sup M < y < \sup M + \delta$. From the definition of $\sup M$, there is $x \in M$ such that $\sup M - \delta < x \leq \sup M$. Since $x \in M$, we have $f(a) < f(x)$, hence $f(a) < f(y)$ for all $y \in (\sup M, \sup M + \delta)$. This is a contradiction with the definition of $\sup M$, so $\sup M = b$.
We will now show that $b \in M$. Since f is increasing at b, there is a $\delta' > 0$ such that $(b-\delta', b+\delta') \subseteq I$ and $f(x) < f(b) < f(y)$ whenever $b - \delta' < x < b < y < b + \delta'$. From the definition of $\sup M (= b)$, we get that there is $x \in (b - \delta', b]$ such that $f(a) < f(x)$, hence $x \in M$. If $x = b$, then we are done. If $x \neq b$, then $f(x) < f(b)$ and $f(x) > f(a)$, giving together $f(a) < f(b)$. Thus $b \in M$. □

4.6.7 Problem. (**Schur inequality** for increasing functions) Consider $I \subseteq [0, \infty)$ an interval, $a, b, c \in I$, and f a positive and increasing function defined on I. Show that

$$f(a)(a-b)(a-c) + f(b)(b-a)(b-c) + f(c)(c-a)(c-b) \geq 0.$$

4.6.7.1 Solution. Without loss of generality, we assume that $a \leq b \leq c$. Then

$$\begin{aligned} & f(a) \leq f(b) \leq f(c) \\ \text{and } & b - a \leq c - a \Rightarrow (b-a)(c-b) \leq (c-a)(c-b). \\ & \Rightarrow f(b)(b-a)(c-b) \leq f(c)(c-a)(c-b). \\ & \Rightarrow 0 \leq f(b)(b-a)(b-c) + f(c)(c-a)(c-b). \end{aligned}$$

Again, we get $f(a)(a-b)(a-c) \geq 0$. Hence, by addition the result follows. □

4.6.8 Problem. Suppose that $f : A \to \mathbb{R}$ and c is an accumulation point of A. If $\lim_{x \to c} f(x)$ exists, then f is locally bounded at c.

4.6.8.1 Solution. Let $\lim_{x \to c} f(x) = L$. Taking $\epsilon = 1$ in the definition of the limit, we get that there exists a $\delta > 0$ such that $0 < |x - c| < \delta$ and $x \in A$ implies that $|f(x) - L| < 1$. Let $U = (c - \delta, c + \delta)$. If $x \in A \cap U$ and $x \neq c$, then $|f(x)| \leq |f(x) - L| + |L| < 1 + |L|$, so f is bounded on $A \cap U$. (If $c \in A$, then $|f| \leq \max\{1 + |L|, |f(c)|\}$ on $A \cap U$.) As for sequences, boundedness is a necessary but not sufficient condition for the existence of a limit. □

4.6.1 Example.

1. The limit $\lim_{x \to 0} \frac{1}{x}$, doesn't exist because the function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ given by $f(x) = \frac{1}{x}$ is not locally bounded at 0.

2. The function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ defined by $f(x) = \sin\left(\frac{1}{x}\right)$ is bounded, but $\lim_{x \to 0} f(x)$ doesn't exist.

4.6.9 Problem. Let $f : [a, b] \to \mathbb{R}$, be such that for every $x \in [a, b]$ $\exists$ $\delta_x > 0$ such that f is bounded on $N(x, \delta_x)$. Prove that f is bounded on $[a, b]$.

4.6.9.1 Solution. Let

$$\mathcal{A} = \{N(x, \delta_x);\ x \in [a, b] \text{ such that} f \text{ is bounded on } N(x, \delta_x)\}.$$

We see that $\mathcal{A}$ is an open cover of $[a,b]$. Since $[a,b]$ is compact, so by Heine-Borel theorem $\exists\, x_1, x_2, .., x_n$ such that

$$[a,b] \subseteq \bigcup_{i=1}^{n} N(x_i, \delta_{x_i}).$$

Now, let $|f(x)| \leq M_i$ for all $x \in N(x_i, \delta_{x_i})$ and let $\max_{1\leq i\leq n}\{M_i\} = M$, then $|f(x)| \leq M$ for all $x \in [a,b]$. Thus f is bounded. $\square$

4.6.9.2 Solution. Suppose that f is not bounded in $[a,b]$. So, for each $n \in \mathbb{N}\ \exists\ x_n \in [a,b]$ such that $f(x_n) \geq n$, thus (x_n) is a bounded sequence in $[a,b]$, hence there exists a convergent subsequence (x_{n_k}) of (x_n). Suppose, (x_{n_k}) converges to $c \in [a,b]$, then $\exists\ \epsilon > 0$ such that f is bounded in $N(c;\epsilon) \subseteq [a,b]$. Suppose that for some $M > 0$, $|f(x)| \leq M$ for all $x \in N(c;\epsilon)$. Since (n_k) is increasing $\exists\ n_p > M$. And $N(c;\epsilon)$ contains a tail $x_{n_q}, x_{n_{q+1}}, x_{n_{q+2}}, ...$ of $(x_{n_k}) \Rightarrow x_{n_k} \in (c-\epsilon, c+\epsilon)\ \forall\ k \geq r = \max\{p,q\}$ i.e. $x_{n_r} \in N(c;\epsilon) \Rightarrow f(x_{n_r}) > n_r \geq n_p > M$, a contradiction. $\square$

4.6.10 Problem. Suppose that a function f is locally bounded at each point of a closed and bounded set E. Then f is bounded on the whole of the set E.

4.6.10.1 Solution. 1. If f is not bounded on E there must be a sequence of points (x_n) chosen from E so that $|f(x_n)| > n\ \forall\ n$. If such a sequence could not be chosen, then at some stage, N say, there are no more points with $|f(x_N)| > N$ and N is an upper bound. By compactness there is a convergent subsequence (x_{n_k}) converging to a point $z \in E$. By the local boundedness assumption there is an open interval $(z-\delta, z+\delta)$ and a number M_z so that $|f(t)| \leq M_z$ whenever $t \in E$ and inside that interval. But for all sufficiently large values of k, the point x_{n_k} must belong to the interval $(z-\delta, z+\delta)$. The two statements

$$f(x_{n_k}) > n_k \text{ and } f(x_{n_k}) \leq M_z$$

cannot both be true for all large k and so we have reached a contradiction, proving the statement.

4.6.10.2 Solution. 2. Suppose that f is not bounded on E. Since E is bounded we may assume that E is contained in some interval $[a,b]$. Divide that interval in half, forming two subintervals of the same length, namely $(b-a)/2$. At least one of these intervals contains points of E and f is unbounded on that interval. Call it $[a_1,b_1]$. Now do the same to the interval $[a_1,b_1]$. Divide that interval in half, forming two subintervals of the same length, namely $(b-a)/4$. At least one of these intervals contains points of E and f is unbounded on that interval. Call it $[a_2,b_2]$. Continue this process inductively, producing a descending sequence of intervals $([a_n,b_n])$ so that the n-th interval $[a_n,b_n]$ has length $(b-a)/2^n$, contains points of E, and f is unbounded on $E \cap [a_n,b_n]$. By the Cantor intersection property there is a single point $z \in E$ contained in all of these intervals. But by our local boundedness assumption there is an interval $(z-c, z+c)$ so that f is bounded on the points of E in that interval. For any large enough value of n, though, the interval $[a_n,b_n]$ would be contained inside the interval $(z-c, z+c)$. This would be impossible and so we have reached a contradiction, proving the statement. $\square$

4.6.11 Problem. Prove that a function f is locally bounded on a compact interval $[a,b]$.

4.6.11.1 Solution. Let $S = \{a < x \leq b; f \text{ is bounded on } [a,x]\}$. Using these following steps, construct a proof of the result on local boundedness:

1. Show that $S \neq \emptyset$.

2. Show that if $z = \sup S$, then $a < z \leq b$.

3. Show that $z \in S$.

4. Show that $z = b$ by showing that $z < b$ is impossible.

4.6.12 Problem. Prove that, no periodic function (unless it is a constant) can be a rational function.

4.6.12.1 Solution. Let $f(x) = \frac{p(x)}{q(x)}, (q(0) \neq 0)$, where p and q are polynomials, and that $f(x) = f(x + w)$ for all values of x. Let $f(0) = a$, then the equation of the n-th degree $p(x) - aq(x) = 0$ is satisfied by the $n + 1$ values of x, viz $0, w, 2w, .., nw$. Hence by the fundamental theorem of algebra, $f(x)$ must be identically equal to a for all values of x. $\square$

4.6.13 Problem. Consider the function $f(x) = (ax + b)/(cx + d)$; where $ad - bc \neq 0$: Show directly that f is strictly increasing (resp., strictly decreasing) if $ad - bc > 0$ (resp., $ad - bc < 0$) and find $\sup(f)$ and $\inf(f)$ in each case.

4.6.13.1 Solution. Hint: First look at the case where $c = 0$: Next, assume that $c \neq 0$ and reduce to the case where $f(x) = \alpha + \beta/(cx + d)$ for some α and β. $\square$

4.6.14 Problem. Give an example of a one-to-one function $f : (0, \infty) \to \mathbb{R}$ that is not monotone.

4.6.14.1 Solution. Consider the function $f : (0, \infty) \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x, & \text{if } 0 \leq x \leq 1 \\ 3 - x, & \text{if } 1 < x < 2 \\ x, & \text{if } x \geq 2. \end{cases}$$

$\square$

4.6.15 Problem. For any real-valued function f, the following relation holds:

$$\{x; f(x) \geq k\} = \bigcap_{n=1}^{\infty} \left\{x; f(x) > k - \frac{1}{n}\right\}.$$

4.6.15.1 Solution. Suppose x is such that $f(x) \geq k$. Then, for every positive n, $f(x) > k - \frac{1}{n}$. On the other hand, if $f(x) > k - \frac{1}{n}$ for every n, then $f(x) \geq k$. $\square$

4.6.16 Problem. Let I be a nondegenerate interval and let $f : I \to \mathbb{R}$ be an injective function with intermediate value property. Then, f is strictly monotone.

4.6.16.1 Solution. Suppose that f is not strictly monotone. Then, there exist $a, b, c \in I$ such that $a < b < c$ and $f(b)$ is not between $f(a)$ and $f(c)$. In other words, one of the following cases may occur:

1. $f(b) < f(a) < f(c)$

2. $f(a) < f(c) < f(b)$

3. $f(b) < f(c) < f(a)$

4. $f(c) < f(a) < f(b)$.

Suppose the case (1) and let $\lambda = f(a)$. Since f has the intermediate value property, there exists $\alpha \in (b, c)$ such that $\lambda = f(\alpha)$. Since $\alpha \neq a$, this fact contradicts the injectivity of f. The other cases treat similarly. □

4.6.17 Problem.

1. Give an example of two functions $f, g : \mathbb{R} \to \mathbb{R}$ such that $f \neq g$, but such that $f \circ g = g \circ f$.
2. Give an example of three functions $f, g, h : \mathbb{R} \to \mathbb{R}$ such that $f \circ (g + h) \neq f \circ g + f \circ h$.

4.6.17.1 Solution.

1. Let $f(x) = 2x, g(x) = 3x$.
2. Let $f(x) = x^2, g(x) = x, h(x) = 1$. □

4.6.18 Problem. Give an example of a function $f : [0, 1] \to [0, 1]$ such that f is bijective but not monotone. Can you find a monotone and injective function that is not surjective? Or monotone and surjective but not injective?

4.6.18.1 Solution.

- Consider $f(x) = \begin{cases} 1, & \text{if } x = 0 \\ x, & \text{if } 0 < x < 1 \\ 0, & \text{if } x = 1 \end{cases}$.
- Consider $f(x) = \frac{1}{6}(x + 3)$, for $0 \leq x \leq 1$.
- Consider $f(x) = \begin{cases} x, & \text{if } 0 \leq x \leq \frac{1}{2} \\ \frac{1}{2}, & \text{if } \frac{1}{2} < x < \frac{1}{3} \\ \frac{3x+1}{4}, & \text{if } \frac{1}{3} \leq x \leq 1 \end{cases}$. □

4.6.19 Problem. Check whether the following functions are periodic; if yes, find their basic periods T, if any.

1. $f(x) = \sin^2 x \ \forall\, x \in \mathbb{R}$
2. $f(x) = \sin x^2 \ \forall\, x \in \mathbb{R}$
3. $f(x) = \sin |x| \ \forall\, x \in \mathbb{R}$
4. $f(x) = |\sin x| \ \forall\, x \in \mathbb{R}$
5. $f(x) = x - [x] \ \forall\, x \in \mathbb{R}$
6. $\mathfrak{D}(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \in \mathbb{Q}^C \end{cases}$

4.6.19.1 Solution.

1. $\sin^2 x = \frac{1}{2}(1 - \cos 2x), \ \forall\, x \in \mathbb{R}$. So, the function $g(x) = \cos 2x$, is periodic with the basic period $2\pi/2 = \pi$. Hence, the function f is also periodic with the basic period π.

2. The zeros of the function f are of the form $\pm\sqrt{k\pi}$ $\forall$ $k \in \mathbb{N}$. Let us show that the distance between the zeros of f tends to zero as $k \to \infty$,
$$\lim_{k\to\infty} \sqrt{(k+1)\pi} - \sqrt{k\pi} = \lim_{k\to\infty} \frac{\pi}{\sqrt{(k+1)\pi} + \sqrt{k\pi}} = 0.$$
This implies that the function f is not periodic.
3. not periodic.
4. periodic with basic priod π.
5. periodic with basic period 1.
6. Let us prove first that every rational number r is a period of the function D. Namely, since the sum of two rational numbers is again rational, it follows that $\mathfrak{D}(x+r) = 1 = \mathfrak{D}(x)$ for every rational number x, while the sum of a rational and an irrational number is an irrational number. Hence $\mathfrak{D}(x+r) = 0 = \mathfrak{D}(x)$ for every irrational number x. Thus we proved that every rational number r is the period of $\mathfrak{D}$. However, since there does not exist a smallest positive rational number, it follows that the Dirichlet function has no basic period. (Let us add that no irrational number is the period of $\mathfrak{D}$—check that!) □

4.6.20 Problem. Consider the function $f : \mathbb{R} \to \mathbb{R}$, defined by
$$f(x) = \begin{cases} 0, & \text{if } x \notin \mathbb{Q} \\ 1, & \text{if } x = 0 \\ \frac{1}{q}, & \text{if } x = p/q \in \mathbb{Q} \text{ with } (p,q) = 1. \end{cases}$$

1. Find $f(n)$ each integer n.
2. Find the three solutions to the equation $f(x) = 1/3$.
3. Find the solutions to the equation $f(x) = 1/7$ that lie in $(3, 4)$.
4. Prove that the set of all solutions to the equation $f(x) = 1/5$ is countably infinite.
5. Let $(a, b) \subseteq \mathbb{R}$, and $\epsilon > 0$. Prove that $E = \{x \in (a, b) : f(x) \geq \epsilon\}$ is a finite set.

4.6.20.1 Solution.

1. $f(n) = 1$ $\forall$ $n \in \mathbb{Z}$.
2. $\frac{1}{3}, \frac{2}{3}, \frac{5}{3}$.
3. $3 + \frac{1}{7}, 3 + \frac{2}{7}, 3 + \frac{3}{7}, 3 + \frac{4}{7}, 3 + \frac{5}{7}, 3 + \frac{6}{7}$.
4. $\{1/5, 2/5, 3/5, 4/5\} + \mathbb{Z}$ which is countably infinite.
5. Let $\epsilon > 0$. Then $\exists$ $N \in \mathbb{N}$ such that $1/N < \epsilon$. Let $M = \min\{n \in \mathbb{N}; n\epsilon > 1\}$, then $1/N \leq 1/M < \epsilon$, so the set
$$\left\{\frac{1}{M-1}, \frac{1}{M-2}, \dots\right\} \bigcap (0, \epsilon) = \emptyset.$$
Thus the set of rationals whose denominator is less than $1/(M-1), 1/(M-2), 1/(M-3), \dots$ are all greater than ϵ, and they are finite in number. □

4.6.2 Note. For the above function $ran(f) = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, .., \frac{1}{n}, ...\right\}$, and for a fixed prime $p \in \mathbb{N}$, we have

$$f^{-1}\left(\frac{1}{p}\right) = \mathbb{Z} + \left\{\frac{1}{p}, \frac{2}{p}, .., \frac{p-1}{p}\right\}.$$

4.6.21 Problem. Decide whether the following statements are true or false (justify). Suppose that $f, g : \mathbb{R} \to \mathbb{R}$.

1. If f is odd, then $|f|$ is odd.
2. For any function f, the function $x \mapsto \frac{1}{2}\{f(x) + f(-x)\}$ is even.
3. If f is monotone, it is not even.
4. If f is periodic, it is not monotone.
5. If f is one-to-one, it is not even.
6. If f is one-to-one, it is not periodic.
7. If g is periodic, then for any function f, the function $f \circ g$ is periodic.
8. If f and $f \circ g$ are periodic, then g is either periodic or is a polynomial of degree 1.
9. If f is periodic and g is a polynomial of degree 1, then $f \circ g$ is periodic.

4.6.21.1 Solution.

1. False. Consider $f(x) = x$, then $|f|(x) = |x|$ is not odd.
2. Suppose $g(-x) = \frac{1}{2}\{f(-x) + f(x)\} = g(x)$ which is even.
3. False. Now f is even implies $f(x) = f(-x)\ \forall\ x \in \mathbb{R} \Rightarrow\ f$ is not monotone.
4. False. f is periodic implies $\exists\ p \in \mathbb{R}$ such that $f(x+p) = f(x)\ \forall\ x \in \mathbb{R} \Rightarrow\ f$ is not monotone.
5. False. Let f be one-one, so $f(x) = f(-x)\ \forall\ x \in \mathbb{R} \Rightarrow x = -x \Rightarrow x = 0$, which is impossible.
6. False. Let f be periodic, so $f(x+p) = f(x)\ \forall\ x \in \mathbb{R} \Rightarrow f$ is not one-one.
7. Now, let g be periodic then $\exists\ p \in \mathbb{R}$ such that $g(x+p) = g(x)\ \forall\ x \in \mathbb{R}$, and for any function f, $(f \circ g)(x+p) = f(g(x+p)) = f(g(x)) = (f \circ g)(x)$ implies $f \circ g$ is periodic.
8. Suppose that f has a period p and $f \circ g$ has a period q.

$$\begin{aligned}
&(f \circ g)(x+q) = (f \circ g)(x) \\
\Rightarrow &f(g(x+q)) = f(g(x)) \\
\Rightarrow &g(x+q) - g(x) = kp,\ k \in \mathbb{Z}.
\end{aligned}$$

   Now, $k = 0$ implies $g(x+q) = g(x)$ i.e. g is periodic or $g(x+q) - g(x)$ is a non-zero constant which shows that g is linear.

9. Suppose that f has a period p and $g(x) = ax + b$. Then

$$\begin{aligned}(f \circ g)(x+p) &= f(a(x+p)+b) \\ &= f(ap + (ax+b)) \\ &= f(ap + g(x)) = f(g(x)) \text{ if } a \in \mathbb{Z} \\ &= (f \circ g)(x). \quad \square\end{aligned}$$

4.6.22 Problem. If $f, g : \mathbb{R} \to \mathbb{R}$ are odd and even respectively, then decide whether the following functions are odd, even or neither (justify).
$f + g,\ fg,\ f \circ g,\ f/g,\ f \circ g \circ f.$

4.6.22.1 Solution.

1. Neither; $(f+g)(-x) = f(-x) + g(-x) = -f(x) + g(x) = (-f+g)(x)$
2. Odd; $(fg)(-x) = f(-x)g(-x) = -(fg)(x)$.
3. Even; $(f \circ g)(-x) = f(g(-x)) = f(g(x)) = (f \circ g)(x)$
4. Odd; $(f/g)(-x) = f(-x)/g(-x) = -f(x)/g(x) = -(f/g)(x)$.
5. Odd; $(f \circ g \circ f)(-x) = f \circ (g(f(-x))) = f(g(-f(x))) = f(-g(f(x))) = f(-g(f(x))) = -(f \circ g \circ f)(x)$. $\square$

4.6.23 Problem. If f is odd and g is a function with $dom(g) = ran(f)$ such that $g \circ f$ is odd. Prove that g is odd.

4.6.23.1 Solution. Let $x \in dom(g)$. then $\exists\, y \in dom(f)$ such that $f(y) = x$. Hence $-y \in dom(f)$ and $f(-y) = -f(y)$, so that $-x = -f(y) = f(-y) \in ran(f) = dom(g)$. Further,

$$g(-x) = g(f(-y)) = (g \circ f)(-y) = -(g \circ f)(y)) = -g(f(y)) = -g(x). \quad \square$$

4.6.24 Problem. Give an example of 1-1 function and odd which is not monotone.

4.6.24.1 Solution. Consider the function $f : [-a, a] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} a, & \text{if } x = -a \\ x, & \text{if } x \in (-a, a) \\ -a, & \text{if } x = a \end{cases},$$

shows that $a > -a \Rightarrow f(a) < f(-a)$. $\square$

4.6.25 Problem. Let $f : \mathbb{R} \to \mathbb{R}$ and $\alpha \in \mathbb{R}^+$ Prove that if f and g, where $g(x) = f(x+\alpha)$ are both even, then 2α is a period of f.

4.6.25.1 Solution. Now, $g(-x) = g(x) \Rightarrow f(-x+\alpha) = f(x+\alpha)$, now replacing x by $x+\alpha$ we get $f(-x) = f(x+2\alpha) \Rightarrow f(x) = f(x+2\alpha)$. $\square$

4.6.26 Problem. Prove that if f is periodic with period λ, then $|f|$ is periodic with a period λ.

4.6.26.1 Solution. Here $|f|(x+\lambda) = |f(x+\lambda)| = |f(x)| = |f|(x)$. $\square$

4.6.27 Problem. (IMO,1968) Let $a > 0$ be a real number and $f(x)$ a real valued function satisfying

$$f(x+a) = \frac{1}{2} + \sqrt{f(x) - f^2(x)}.$$

1. Prove that the function f is periodic.
2. Give an example of such a non-constant function for $a = 1$.

4.6.27.1 Solution.

1. From the relation,

$$\left(f(x+a) - \frac{1}{2}\right)^2 = f(x) - f^2(x),$$

we obtain

$$(f(x) - f^2(x)) + (f(x+a) - f^2(x+a)) = \frac{1}{4}.$$

Subtracting the above relation for $x + a$ in place of x we get

$$f(x) - f^2(x) = f(x+2a) - f^2(x+2a),$$

which implies

$$\left(f(x+a) - \frac{1}{2}\right)^2 = f(x) - f^2(x).$$

Since $f(x) \geq 1/2$ holds for all x by the condition of the problem, we conclude that $f(x+2a) = f(x)$. We see that the period of f is $2a$. □

4.6.27.2 Solution.

1. We expect the period to be related to a. Iterating the relation from the statement gives

$$\begin{aligned} f(x+2a) &= \frac{1}{2} + \sqrt{f(x+a) - f^2(x+a)} \\ &= \frac{1}{2} + \sqrt{\frac{1}{2} + \sqrt{f(x) - f^2(x)} - \frac{1}{4} - \sqrt{f(x) - f^2(x)} - (f(x) - f^2(x))} \\ &= \frac{1}{2} + \sqrt{\left(\frac{1}{2} - f(x)\right)^2} = \frac{1}{2} + \left|f(x) - \frac{1}{2}\right| \\ &= f(x) \text{ or } 1 - f(x). \end{aligned}$$

The defining relation shows that $f(x) \geq \frac{1}{2}$ for all x. Hence the above computation implies $f(x+2a) = f(x)$ for all x, which proves that f is periodic.
or
An alternative solution due to R. Stong that avoids the use of square roots. Rewrite and square to obtain $f(x) - f^2(x) + f(x+a) - f^2(x+a) = 1/4$. Replacing x by $x + a$ in this formula gives $f(x+a) - f^2(x+a) + f(x+2a) - f^2(x+2a) = 1/4$. Subtracting gives

$$\begin{aligned} 0 &= f(x) - f^2(x) - f(x+2a) + f^2(x+2a) \\ &= [f(x+2a) - f(x)][1 - f(x) - f(x+2a)]. \end{aligned}$$

Since the original defining equation gives $f(x) \geq 1/2$, the second factor is non-zero unless $f(x+2a) = f(x) = 1/2$. Either because both are $1/2$ or by cancelling, we get $f(x+2a) = f(x)$.

2. An example of such a function is

$$f(x) = \begin{cases} \frac{1}{2} & \text{if } 2n \leq x < 2n+1 \\ 1 & \text{if } 2n+1 \leq x \leq 2n+2 \end{cases}$$

where $n \in \mathbb{Z}$. Another example is the constant function $f(x) = \frac{1}{2} + \frac{1}{2\sqrt{2}}$. □

4.6.28 Problem. Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded function wsuch that

$$f\left(x + \frac{1}{6}\right) + f\left(x + \frac{1}{7}\right) = f(x) + f\left(x + \frac{13}{42}\right).$$

Show that f is periodic.

4.6.28.1 Solution. Define $g : \mathbb{R} \to \mathbb{R}$, by $g(x) = f(x + 1/6) - f(x)$. The condition gives that $g(x + 1/7) = g(x)$. Therefore $g(x + 1) = g(x)$. Now let $h : \mathbb{R} \to \mathbb{R}, h(x) = f(x + 1) - f(x) = g(x) + g(x + 1/6) + ... + g(x + 5/6)$, then we also have that $h(x + 1) = h(x)$. Thus we obtain $h(x + k) = h(x)\ \forall\ k \in \mathbb{N}$. Since $f(x + k) - f(x) = h(x) + h(x + 1) + ... + h(x + k - 1)$, we obtain $f(x+k)-f(x) = kh(x)$. And because f is bounded, $kh(x)$ is bounded as well for all positive integers k, which is possible only if h is identically equal to zero. It follows that $f(x+1) = f(x)$ for all x, so f is periodic with period 1. □

4.6.29 Problem. Let $f : \mathbb{R} \to \mathbb{R}$ be a function that satisfies

1. $f(x + y) + f(x - y) = 2f(x)f(y),\ \forall\ x, y \in \mathbb{R}$;
2. there exists x_0 with $f(x_0) = -1$.

Prove that f is periodic.

4.6.29.1 Solution. (M. Martin) Substituting $x = y = 0$, we obtain $f^2(0) = f(0)$, so $f(0)$ is equal to 0 or 1. If $f(0) = 0$, then let $x = x_0$ and $y = 0$, then we obtain $2f(x_0) = 2f(x_0)f(0)$ and $f(0) = 0$ implies $-1 = f(x_0)f(0) = f(x_0).0 = 0$, which cannot happen. Thus $f(0) = 1$.
For $x = y = x_0$, we obtain $f(2x_0) = 1$. This suggests that $2x_0$ might be a period for f. Let us show that this is indeed the case.
Replace x by $x + 2x_0$ and y by $x - 2x_0$ to obtain

$$f(2x) + f(4x_0) = 2f(x + 2x_0)f(x - 2x_0).$$

Since $f(2x) = 2f^2(x) - 1$ and $f(4x_0) = 2f^2(2x_0) - 1 = 1$, the above relation becomes $f(x + 2x_0)f(x - 2x_0) = f^2(x)$. Similarly, for x arbitrary and $y = 2x_0$, we obtain $f(x+2x_0)+f(x-2x_0) = 2f(x).f(x - 2x_0) = f(x + 2x_0) = f(x)$, so f has period $2x_0$. □

4.6.30 Problem. (French Mathematical Olympiad, 1996) Let a, b be positive integers with a odd. Define the sequence (u_n) as follows:

$$u_0 = b,\ u_{n+1} = \begin{cases} \frac{1}{2}u_n & \text{if } u_n \text{ is even} \\ u_n + a & \text{otherwise.} \end{cases}$$

1. Show that $u_n \leq a$ for some $n \in \mathbb{N}$.

2. Show that the sequence (u_n) is periodic from some point onwards.

4.6.30.1 Solution.

1. Suppose $u_n > a$. If u_n is even, then $u_{n+1} = u_n/2 < u_n$. If u_n is odd, $u_{n+2} = (u_n + a)/2 < u_n$. Hence for each term greater than a, there is a smaller subsequent term. These form a decreasing subsequence, which must eventually terminate, and this happens only if $u_n \leq a$.

2. We will show that infinitely many terms of the sequence are less than $2a$. Suppose this is not true, and let u_m be the largest with this property. If u_m is even, then $u_{m+1} = u_m/2 < 2a$. If u_m is odd, then $u_{m+1} = u_m + a$ is even; hence $u_{m+2} = (u_m + a)/2 < 3a/2 < 2a$, which is again impossible. This shows that there are infinitely many terms less than $2a$. An application of the pigeonhole principle with infinitely many pigeons shows that some term (u_n) repeats, leading to a periodic sequence. □

4.6.31 Problem. Let $f : I \to \mathbb{R}$ be a function and $c \in I$. If f is continuous at c, then f is locally bounded at c.

4.6.31.1 Solution. Suppose f is continuous at c, then $\forall\ \epsilon > 0\ \exists\ \delta > 0$ such that $x \in B(c; \delta) \Rightarrow f(x) \in B(f(c); \epsilon)$ which shows that f is locally bounded at c. □

4.6.32 Problem. Prove that, f is bounded implies f is locally bounded. The converse is false. Give an example.

4.6.32.1 Solution. $f(x) = x$ is locally bounded on $\mathbb{R}$ but not bounded. □

4.6.33 Problem. Let X, Y be non-empty sets and $h : X \times Y \to \mathbb{R}$ have bounded range in $\mathbb{R}$. Let $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$ be defined by $f(x) = \sup\{h(x, y); y \in Y\}$ and $g(y) = \inf\{h(x, y); x \in X\}$. Prove that

$$\sup\{g(y); y \in Y\} \leq \inf\{f(x); x \in X\}.$$

4.6.33.1 Solution. Let $\sup_y \inf_x h(x, y) = \alpha$ and $\inf_x \sup_y h(x, y) = \beta$. We prove that $\beta \geq \alpha$. If possible, let $\alpha > \beta$ and $\alpha - \beta = \epsilon > 0$. Hence $\exists\ x_0 \in X$ such that $\sup_y h(x_0, y) < \beta + \frac{\epsilon}{2}$ and $\exists\ y_0 \in Y$ such that $\inf_x h(x, y_0) > \alpha - \frac{\epsilon}{2}$. In particular,

$$h(x_0, y_0) < \sup_y h(x_0, y) < \beta + \frac{\epsilon}{2}$$

and

$$h(x_0, y_0) > \inf_x h(x, y_0) > \alpha - \frac{\epsilon}{2}$$

i.e.,

$$\alpha - \frac{\epsilon}{2} < h(x_0, y_0) < \beta + \frac{\epsilon}{2} \Longrightarrow \alpha - \beta < \epsilon \text{ (a contradiction)}.$$

Hence $\alpha \leq \beta$. □

4.6.34 Problem. Let X, Y be non-empty sets and let $h : X \times Y \to \mathbb{R}$ have a bounded range in $\mathbb{R}$. Let $f : X \to \mathbb{R}, g : Y \to \mathbb{R}$ be defined by

$$f(x) = \sup\{h(x, y); y \in Y\};\ \ g(y) = \inf\{h(x, y); x \in X\}.$$

Prove that

$$\sup\{g(y); y \in Y\} \leq \inf\{f(x); x \in X\}$$

i.e.

$$\sup_y \inf_x h(x,y) \le \inf_x \sup_y h(x,y);$$

and

$$\sup_y \sup_x h(x,y) = \sup_x \sup_y h(x,y) = \sup_{x,y} h(x,y).$$

4.6.34.1 Solution. Let $\sup_y \inf_x h(x,y) = \alpha$ and $\inf_x \sup_y h(x,y) = \beta$ We prove that $\alpha \le \beta$. If possible, let $\alpha > \beta$. Suppose $\epsilon = \alpha - \beta > 0$.
Hence $\exists\ x_0 \in X$ and $y_0 \in Y$ such that

$$\sup_y h(x_0,y) < \beta + \frac{\epsilon}{2},\ \text{and}\ \inf_x h(x,y_0) > \alpha - \frac{\epsilon}{2}$$

in particular,

$$h(x_0,y_0) < \sup_y h(x_0,y) < \beta + \frac{\epsilon}{2}, \text{and } h(x_0,y_0) > \inf_x h(x,y_0) > \alpha - \frac{\epsilon}{2},$$

i.e $\alpha - \frac{\epsilon}{2} < h(x_0,y_0) < \beta + \frac{\epsilon}{2} \Rightarrow \alpha - \beta < \epsilon$ (a contradiction).
Hence $\alpha \le \beta$.
Another part:
Let $\alpha = \sup_{x,y} h(x,y)$, $\beta = \sup_x \sup_y h(x,y); \gamma = \sup_y \sup_x h(x,y)$. If possible, let $\alpha > \beta$. Suppose $\epsilon = \alpha - \beta > 0$. Now $\alpha \ge h(x,y); \forall\ x \in X, y \in Y \Rightarrow \alpha \ge h(x,y); \forall\ y \in Y \Rightarrow \alpha\ \ge \sup_y h(x,y)$. So $\exists\ y_0 \in Y,$ such that$\beta - \epsilon < \sup_y h(x,y_0) \le \alpha \Rightarrow \beta - \alpha < \epsilon$, a contradiction.In a similar way $\alpha > \beta$ give rise to a contradiction. Hence $\alpha = \beta$ and also $\alpha = \gamma$ implies $\alpha = \beta = \gamma$. □

4.6.35 Problem. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ such that the set $\{x \in \mathbb{R}; f(x) = 0\}$ is neither open nor closed.

4.6.35.1 Solution. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational.} \end{cases} \qquad \square$$

4.6.36 Problem. Let $f : [a,b] \to \mathbb{R}$, be a function and suppose that $\forall\ \epsilon > 0$, the function g defined by $g(x) = f(x) + \epsilon x$ is increasing on $[a,b]$. Prove that f is increasing on $[a,b]$.

4.6.36.1 Solution. If possible, let f be decreasing, then $\exists\ x_1, x_2 \in [a,b]$ such that $x_1 > x_2 \Rightarrow f(x_1) \le f(x_2)$. Let $0 < \epsilon < \frac{f(x_2)-f(x_1)}{x_1-x_2}$, then

$$\begin{aligned} &\epsilon(x_1 - x_2) < f(x_1) - f(x_2) \\ &\Rightarrow \epsilon x_1 - \epsilon x_2 < f(x_1) - f(x_2) \\ &\Rightarrow \epsilon x_1 + f(x_1) < f(x_2) + \epsilon x_2 \\ &\Rightarrow g(x_1) < g(x_2), \end{aligned}$$

shows that g is decreasing, a contradiction. Hence f is increasing on $[a,b]$. □

4.6.37 Problem. Let $f(x) = \lim_{n\to\infty} \frac{x^n}{1+x^n}$, when this limit exists.

1. For what values of x this limit exists?
2. For what values of x the function is continuous?

3. Compute $f(x)$ for all x in the domain of f.

4.6.37.1 Solution.

1. $f(x) = \lim_{n\to\infty} \frac{x^n}{1+x^n}$ exists, if $x \geq 0$. Domain of f is $[0,\infty)$. f is given by
2. f is continuous on $[0,1) \cup (1,\infty)$.
3. $$f(x) = \begin{cases} 0, & \text{if } 0 \leq x < 1 \\ 1/2, & \text{if } x = 1 \\ 1, & \text{if } x > 1. \end{cases} \qquad \Box$$

4.6.38 Problem. Prove that the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \inf\left\{\left|x - \frac{1}{n}\right|, \ \forall\ n \in \mathbb{N}\right\}$$

is continuous on $\mathbb{R}$.

4.6.38.1 Solution. Let $A = \left\{\frac{1}{n}; n \in \mathbb{N}\right\}$ in the previous problem, then proceed.

4.6.39 Problem. Find an injective function $f : (0,1) \to 2^{\mathbb{N}}$ by using binary expansions.Then define a surjective function $g : 2^{\mathbb{N}} \to (0,1)$ using similar idea.Though g is not one-one, it is "two-one" in certain sense; why?

4.6.39.1 Solution. Every $x \in (0,1)$ has a binary expansion that $\exists$ a sequence (a_n) such that

$$x = \sum_{n=0}^{\infty} a_n 2^{-n}$$

where $a_n \in \{0,1\}\ \forall\ n \in \mathbb{N}$. (The expansion is typically written $0.a_1a_2....$) It may be assumed that $a_n's$ do not have an "infinite trail" of $1's$, i.e. (a_n) does not converge to 1; this ensures that the binary expansions are unique. Now define $f : (0,1) \to 2^{\mathbb{N}}$ by

$$f(x) = \{n \in \mathbb{N}; a_n = 1\}$$

where (a_n) is the binary expansion of x (unique in the above sense). It easy to see that f is injective. Now define $g : 2^{\mathbb{N}} \longrightarrow (0,1)$ as follows.
If $S \subseteq \mathbb{N}$, define $g(S) = x$, if $S \neq \emptyset$ and $S \neq \mathbb{N}$,define

$$g(S) = x$$

where x has the binary expansion (a_n) given by

$$a_n = \begin{cases} 1, & \text{if } n \in S \\ 0, & \text{if } n \notin S. \end{cases}$$

Also define $g(\emptyset) = \frac{1}{3}$ and $g(\mathbb{N}) = \frac{2}{3}$. It is easy to thst g is surjective. Furthermore g is two-to-one in the sense that

$$|\{S; g(S) = x\}| \leq 2.$$

The latter follows from the fact: every real number has at most two distinct binary expansions, and if a real number has two binary expansions then it has a terminated binary expansion. (This is the reason for taking 1/3 and 2/3 above; they have no terminating binary expansion.) □

4.6.40 Problem. Assume that $f : [a, b] \to \mathbb{R}$ is increasing, and define $g : (a, b) \to \mathbb{R}$ by $g(x) = \sup\{f(y); y < x\}$. Prove that $g(c) \leq f(c)$. Also prove that if $g(c) \neq f(c)$, then f does not have a limit at c.

4.6.40.1 Solution. Let $c \in (a, b)$, then $\forall y < c \Rightarrow f(y) < f(c)$. So the set $S = \{f(y); y < c\}$ is bounded above by $f(c)$, hence $\sup S \leq f(c)$. i.e. $g(c) \leq f(c)$. Again if $g(c) \neq f(c)$ then $g(c) < f(c)$. We show that f does not have a limit at c. Now for each $n \in \mathbb{N}$, we get sequences u_n and v_n such that $c - 1/n < u_n < c < v_n < c + 1/n$, then $f(u_n) \leq g(c) < f(c) \leq f(v_n)$ and so $f(v_n) - f(u_n) \geq f(c) - g(c) > 0$. From the above we see that u_n is increasing and v_n is decreasing, and both converging to c, but $f(v_n) - f(u_n)$ does not converge to 0. Hence f does not have a limit at c. □

4.6.41 Problem. Determine the largest interval I where the map f defined by

$$f(x) = \sqrt{|x-2| - |x| + 2}$$

is invertible. Write the expression of the inverse function of f restricted to I.

4.6.41.1 Solution. Since

$$f(x) = \begin{cases} 2 & \text{if } x \leq 0 \\ \sqrt{4-2x} & \text{if } 0 < x \leq 2 \\ 0 & \text{if } x > 2, \end{cases}$$

the required interval I is [0,2]. In addition $f([0, 2]) = [0, 2]$, so $f^{-1} : [0, 2] \to [0, 2]$. By putting $y = \sqrt{4-2x}$ we obtain $x = \frac{4-y^2}{2}$, which implies $f^{-1}(x) = 2 - \frac{1}{2}x^2$. □

4.6.42 Problem. Suppose that $f : I \to \mathbb{R}$ satisfies the condition

$$f\left(\frac{s+t}{2}\right) \leq \frac{1}{2}(f(s) + f(t)) \;\forall\, s, t \in I \tag{4.3}$$

1. Show that $f(\lambda s + (1-\lambda)t \leq \lambda f(s) + (1-\lambda)f(t)$ holds $\forall\, s, t \in I$ and all $\lambda \in [0, 1]$ of the form $\lambda = m/2^n$ with integers $m \geq 0$ and $n \geq 1$.

2. Show that if f satisfies (4.3) and is continuous, then f is convex.

4.6.42.1 Solution.

1. Hint: Use induction and the identity

$$\frac{m}{2^n}s + \left(1 - \frac{m}{2^n}\right)t = \frac{1}{2}\left[\frac{m}{2^{n-1}}s + \left(1 - \frac{m}{2^{n-1}}\right)t + t\right]$$

2. Hint: Show that $\{m/2^n; m/2^n \leq 1, m \in \mathbb{N}_0, n \in \mathbb{N}\}$ is dense in $[0, 1]$ and use part (1). □

4.6.43 Problem. Let $f : [0,1] \to [0,1]$ be a function defined as follows $f(x) = .b_1b_2...b_n...$ when $x = .a_1a_2...a_n...$, and $b_i = 9 - a_i$ $(i = 1, 2, .., n, ...)$. Show that $f(x) + f(1-x) = 1$.

4.6.43.1 Solution. Here $f(x) + x = .b_1b_2..b_n.... + .a_1a_2...a_n... = .9999... = 1$. Hence $f(x) = 1 - x$. So $f(x) + f(1-x) = 1 - x + x = 1$. □

4.6.44 Problem. For any non-constant function defined on $\mathbb{R}$ there exists an interval where the function admits of inverse.

4.6.44.1 Solution. False. Dirichlet's function $\mathfrak{D} : \mathbb{R} \to \mathbb{R}$ defined by

$$\mathfrak{D}(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \in \mathbb{Q}^C, \end{cases}$$

but it has repeated values in any interval. □

4.6.45 Problem. Is it true that if $f : [0,1] \to [0,1]$ is

1. monotone increasing
2. monotone decreasing

then there exists an $x \in [0,1]$ for which $f(x) = x$?

4.6.45.1 Solution.

1. Yes. Let $A = \{x \in [0,1]; f(x) > x\}$. If $f(0) = 0$ we are done, if not then A is non-empty (0 is in A) bounded, so it has supremum, say a. Let $b = f(a)$.
   - $a < b$: Then, using that f is monotone and a was the sup, we get $b = f(a) \le f((a+b)/2) \le (a+b)/2$, which contradicts $a < b$.
   - $a > b$: Then we get $b = f(a) \ge f((a+b) = 2) > (a+b)/2$ contradiction. Therefore we must have $a = b$.
2. No. Let, for example,

$$f(x) = \begin{cases} 1 - \frac{x}{2} & \text{if } x \le \frac{1}{2} \\ \frac{1}{2} - \frac{x}{2} & \text{if } x > \frac{1}{2}. \end{cases} \quad \square$$

4.6.46 Problem. Let $y_1, y_2, ..., y_7 \in \mathbb{R}$. Then show that $\exists\, y_i, y_j$ among seven numbers such that

$$0 \le \frac{y_i - y_j}{1 + y_iy_j} \le \frac{1}{\sqrt{3}}.$$

4.6.46.1 Solution. Consider the function $f : I = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$ defined by $f(x) = \tan x$, since f is surjective, so there exists $x_i \in I$ such that $y_i = f(x_i) = \tan x_i; 1 \le i \le 7$. Now divide I into six equal intervals. So, by Pigeon Hole Principle, we get at least two x_i, x_j (say) lie in the interval of length $\frac{\pi}{6}$, i.e. $0 \le x_i - x_j \le \frac{\pi}{6}$. Hence, as f is increasing, we get

$$\begin{aligned} &\tan 0 \le \tan(x_i - x_j) \le \tan\frac{\pi}{6} \\ \Rightarrow &0 \le \frac{\tan x_i - \tan x_j}{1 + \tan x_i \tan x_j} \le \frac{1}{\sqrt{3}} \\ \Rightarrow &0 \le \frac{y_i - y_j}{1 + y_iy_j} \le \frac{1}{\sqrt{3}}. \quad \square \end{aligned}$$

4.7 Additional Exercises on Chapter 4.

4.7.1 Exercise. Decide whether the following statements are true or false (justify). Suppose that $f, g : \mathbb{R} \to \mathbb{R}$.

1. Show that the product of two increasing functions may not be increasing. Find a condition that guarantees that the product of two increasing functions is an increasing function.
2. If the functions f, g are strictly increasing, so are $f + g, fg, f \circ g$.
3. Give an example to show that the difference of two monotone functions may not be a monotone function.
4. If f, g are monotone, then $f \circ g$ is monotone.
5. Let I be an interval and f be a positive increasing function and g be a positive decreasing function defined on I, Give an example to show that fg may be strictly increasing on I and show thatfg may be strictly decreasing on I.
6. Give an example of a function $f : [a, b] \to \mathbb{R}$ such that f is increasing on (a, b) but not an increasing on $[a, b]$.

4.7.2 Exercise. For every real number x and every integer n,

$$[x + n] = [x] + n.$$

4.7.3 Exercise. Suppose that $f : [a, b] \to \mathbb{R}$ is monotone on $[a, b]$. Prove that f is bounded on $[a, b]$.

4.7.4 Exercise. Let $I = (a, b)$ and $p(x)$ be the polynomial of degree 2.

1. Find a specific polynomial $q(x)$ so that $q(I)$ is an open interval.
2. Find a specific polynomial $q(x)$ so that $q(I)$ is a half-open interval.
3. Is there a polynomial $q(x)$ so that $q(I)$ is a closed interval.

4.7.5 Exercise. Show that $f : \mathbb{R} \to \mathbb{R}$ is increasing iff f is increasing at every $p \in \mathbb{R}$.

4.7.6 Exercise. Suppose that $f : (a, b) \to \mathbb{R}$ is monotone on (a, b). Give an example that f may not be bounded on (a, b).

4.7.7 Exercise. In general, the monotone functions on a subset X of $\mathbb{R}$ do not form a vector space. (The sum of two monotone functions need not be monotone.) Let $\mathscr{M}$ denote the linear space of functions generated by the monotone functions. Prove that $f \in \mathscr{M}$ if and only iff $\mathscr{M}$ is the difference of two increasing functions.

4.7.8 Exercise. Let $\mathscr{F}$ be a set of increasing functions on $[0, 1]$. If $g(x) = \sup\{f(x); f \in \mathscr{M}\}$ is finite for all $x \in [0, 1]$, then the function g is increasing. If $h(x) = \inf\{f(x); f \in \mathscr{M}\}$ is finite for all $x \in [0, 1]$, then the function h is increasing. (a similar result for decreasing functions.)

4.7.9 Exercise. Give an example of a function $f : [0, 1] \to \mathbb{R}$ for which $f([0, 1])$ is a countably infinite union of disjoint intervals.

4.7.10 Exercise. If f is 1-1 and odd. Prove that f^{-1} is odd.

4.7.11 Exercise. The period of the function " sin " is 2π, while the period of $|\sin|$ is π. Is it true that for every periodic function with period μ is greater than or equal to $\frac{1}{2}\mu$?

4.7.12 Exercise. If a function $f : \mathbb{R} \to \mathbb{R}$ is not injective and there exists a function $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that $f(x + y) = g(f(x), y)\ \forall\ x, y \in \mathbb{R}$, show that f is periodic.

4.7.13 Exercise. Show that $f : \mathbb{R} \to \mathbb{R}$ is increasing iff f is locally increasing at every $p \in \mathbb{R}$.

4.7.14 Exercise. Prove that the sum and the product of two bounded functions is a bounded function.

4.7.15 Exercise. Find general conditions on a bounded function f that guarantee the function $\frac{1}{f}$ is also a bounded function.

4.7.16 Exercise. Prove directly that $(2) \Rightarrow (3), (2) \Rightarrow (4)$ and $(3) \Leftrightarrow (4)$ the following four conditions on a set $A \subseteq \mathbb{R}$ are equivalent:

1. A is closed and bounded.
2. Every infinite subset of A has a limit point in A.
3. Every sequence of points from A has a subsequence converging to a point in A.
4. Every open cover of A has a finite subcover.

4.7.17 Exercise. Show that a function that is locally increasing at every point in $\mathbb{R}$ must be increasing; that is, that $f(x) < f(y)$ for all $x < y$.

4.7.18 Exercise. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ that is not locally bounded at any point.

4.7.19 Exercise. Let (a_n) be a sequence of real numbers such that $\lim_{n\to\infty} a_n = \infty$. For any real number x define integer-valued function $f(x)$ as the smallest positive integer n for which $a_n \geq x$. Then show that, for any integer $n \geq 1$ and any real number x, $f(a_n) \geq n$ and $a_{f(x)} \geq x$.

4.7.20 Exercise. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 1 - |1 - 2x|$. The n-th iterate of f is $f^{(n)} = f \circ f \circ .. \circ f$ (n-times). The **orbit** of a point x is

$$\left\{x, f(x), f(f(x)), ..., f^{(n)}(x), ...\right\}.$$

1. If x is rational prove that the orbit of x is a finite set.
2. If x is irrational what is the orbit?

4.7.21 Exercise. Let $f : [0, 1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 0, & \text{if } x = 0 \\ (-1)^n, & \text{if } \frac{1}{n+1} < x < \frac{1}{n}. \end{cases}$$

At which points $a \in [0, 1]$ does $\lim_{x\to a} f(x)$ exists? What about the function $g : [0, 1] \to \mathbb{R}$ be defined by

$$g(x) = \begin{cases} 0, & \text{if } x = 0 \\ (-1)^n/n, & \text{if } \frac{1}{n+1} < x < \frac{1}{n}. \end{cases}$$

4.7.22 Exercise. Prove that the sum and the product of two bounded functions is a bounded function.

4.7.23 Exercise. Find general conditions on a bounded function f that guarantee the function $\frac{1}{f}$ is also a bounded function.

4.7.24 Exercise. Suppose that $f : [a, b] \to \mathbb{R}$ is monotone on $[a, b]$. Prove that f is bounded on $[a, b]$.

4.7.25 Exercise. Suppose that $f : (a, b) \to \mathbb{R}$ is monotone on (a, b). Give an example that f may not be bounded on (a, b).

4.7.26 Exercise. Give an example of a function $f : (0, 2) \to \mathbb{R}$ such that f has relative maximum value at 1, but not bounded above on $(0, 2)$.

4.7.27 Exercise. Let I be an interval and $f : I \to \mathbb{R}$, suppose $a \in I$. Consider the following statement: the function f has a relative maximum value at a if there exists an interval $J \subseteq I$ such that $a \in J$ and $f : J \to \mathbb{R}$ has maximum value at a. Show that this statement is false. Then determine how to modify the staement so that it is true.

4.7.28 Exercise. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ which is unbounded in every open interval.

4.7.29 Exercise. Give an example of a function $f : [0, 1] \to \mathbb{R}$ such that $S = \{a \in [0, 1]$ and $\lim_{x\to a} f(x)$ does not exist$\}$ is countably infinite.

4.7.30 Exercise. Let $f : [a, b] \to \mathbb{R}$ be monotone, then show that f can be uniformly approximated on $[a, b]$ by step-functions.

4.7.31 Exercise. Let $f : [a, b] \to \mathbb{R}$ be continuous, show that f can be uniformly approximated on $[a, b]$ by polygonal functions.

4.7.32 Exercise. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous.

1. If $f(0) > 0$, show that $\exists a \in \mathbb{R}$ such that $f(x) > 0, \forall x \in (-a, a)$.
2. If $f(x) \geq 0$ for all rational x, show that $f(x) \geq 0, \forall x \in \mathbb{R}$. Is it true when "$\geq 0$ " replaced by "> 0 ".

4.7.33 Exercise. Let $f : [a, b] \to \mathbb{R}$ be continuous at $c \in [a, b]$ and suppose that $f(c) > 0$. Prove that $\exists\ m > 0$ and an interval $[u, v] \subseteq [a, b]$ such that $c \in [u, v]$ and $f(x) \geq m, \forall x \in [u, v]$.

4.7.34 Exercise. Let (r_n) be the enumeration of $\mathbb{Q}$ and (v_n) be a sequence of non-zero real numbers that converges to 0. Define a function

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is irratioal,} \\ v_n, & \forall n \in \mathbb{N} \end{cases}$$

Show that f is continuous everywhere except for the set $\mathbb{Q}$.

4.7.35 Exercise. Let $I = (a, b)$ and $p(x)$ be the polynomial of degree 2.

1. Find a specific polynomial $q(x)$ so that $q(I)$ is an open interval.

2. Find a specific polynomial $q(x)$ so that $q(I)$ is a half-open interval.

3. Is there a polynomial $q(x)$ so that $q(I)$ is a closed interval.

4.7.36 Exercise. Let I be an interval and $f : I \to \mathbb{R}$ be a function. Suppose $c \in I$, then f is said to be **locally bounded** at c if there exists $M > 0$ and $\delta > 0$ such that $|f(x)| \leq M, \forall x \in B(c;\delta)$. Let $f : I \to \mathbb{R}$ be a function and $c \in I$. If f is continuous at c, then f is locally bounded at c.

4.7.37 Exercise. For each positive integer k, let

$$f_k(x) = \begin{cases} 1/(1-kx), & \text{if } 0 \leq x < 1/k \\ 0, & \text{if } x \geq 1/k; \end{cases}$$

and let f be a function defined by $f(x) = \sum_{k=1}^{\infty} f_k(x), \forall\ x \in [0,1]$. Note that, for each value of x there are only a finite number of terms in this sum that are non-zero. Find the set of points at which f is not locally bounded.

4.7.38 Exercise. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is irrational,} \\ (-1)^m n, & \forall\ x = m/n; m, n \in \mathbb{Z}, n > 0, (m,n) = 1 \end{cases}$$

Prove that f is not locally bounded at any point of $(0,1)$.

4.7.39 Exercise. Let I be an interval and $f : I \to \mathbb{R}$ be a function. Suppose $c \in I$, if f has one-sided limits at c, then show that f is locally bounded at c.

4.7.40 Exercise. Let $f : [a,b] \to \mathbb{R}$ be a function. Suppose that f is locally bounded at each point of $[a,b]$. Prove that f is bounded in of the following ways:

1. Use B-W theorem

2. Let $S = \{x \in [a,b]; f \text{ is bounded on } [a,x]\}$. Prove that S is non-empty $\sup S \in S$, and $\sup S = b$.

3. Suppose that f is not bounded on $[a,b]$. Using bisection method, show that there exists a nested sequence $([a_n,b_n])$ of intervals such that f is not bounded on $[a_n,b_n]$ for each $n \in \mathbb{N}$. Then use Nested Intervals theorem to obtain a contradiction.

4.7.41 Exercise. Evaluate the following limit, $\lim_{x\to x_0} \chi_{\mathbb{Q}}(x)\ \forall\ x_0 \in \mathbb{R}$

4.7.42 Exercise. Using $\lim_{x\to 0} \frac{\sin x}{x} = 1$; show that $\lim_{x\to 0} \frac{\sin^{(n)} x}{x} = 0$; where we have used the "n-th iterate", $\sin^{(n)}$; defined by $\sin^{(n)} = \sin \circ \sin \circ .. \circ \sin$, with n iterations.

4.7.43 Exercise. Show that, if $\lim_{x\to 0} \frac{f(x)}{x} = l \in \mathbb{R}$ and $a \neq 0$; then $\lim_{x\to 0} \frac{f(ax)}{x} = al$. What if $a = 0$?

4.7.44 Exercise. Show that, if $\lim_{x\to a-} f(x) < \lim_{x\to a+} f(x)$; then there is a $\delta > 0$ such that $|x-a| < \delta, |y-a| < \delta$; and $x < a < y$; imply $f(x) < f(y)$. Is the converse also true?

4.7.45 Exercise. Let $f : \mathbb{R} \to \mathbb{R}$ be one-to-one. Show that there exists a $c \in \mathbb{R}$ such that $f(c) - (f(c))^2 < 1/4$.

4.7.46 Exercise. Let E be a closed nowhere dense set and define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1, & \text{if } x \in E \\ 0, & \text{if } x \notin E. \end{cases}$$

Prove that the set of points at which f is not continuous is E.

4.7.47 Exercise. Let E be a set that can be expressed as $\bigcup_{n=1}^{\infty} E_n$, where each E_n is closed and nowhere dense.(Note that E may not be nowhere dense.) Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0, & \text{if } x \notin E \\ \frac{1}{p}, & \text{if } x \in E, \text{where } p = \min\{n; x \in E_n\}. \end{cases}$$

1. Prove that the set of points at which f is continuous is $\mathbb{R} \setminus E$.
2. Prove that the set $\{x \in \mathbb{R}; f(x) \geq r\}$ is closed for every real number r.

4.7.48 Exercise. For each positive integer k, let

$$f_k(x) = \begin{cases} 1/(1-kx), & \text{if } 0 \leq x < 1/k \\ 0, & \text{if } x \geq 1/k; \end{cases}$$

and let f be a function defined by $f(x) = \sum_{k=1}^{\infty} f_k(x), \forall\ x \in [0,1]$. Note that, for each value of x there are only a finite number of terms in this sum that are non-zero. Find the set of points at which f is not locally bounded.

4.7.49 Exercise. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is irratioal,} \\ (-1)^m n, & \forall\ x = m/n; m, n \in \mathbb{Z}, n > 0, (m,n) = 1 \end{cases}$$

Prove that f is not locally bounded at any point of $(0,1)$.

4.7.50 Exercise. Let I be an interval and $f : I \to \mathbb{R}$ be a function. Suppose $c \in I$, if f has one-sided limits at c, then show that f is locally bounded at c.

4.7.51 Exercise. Let $f : [a,b] \to \mathbb{R}$ be a function. Suppose that f is locally bounded at each point of $[a,b]$. Prove that f is bounded in of the following ways:

1. Use B-W theorem.
2. Let $S = \{x \in [a,b]; f \text{ is bounded on } [a,x]\}$. Prove that S is non-empty $\sup S \in S$, and $\sup S = b$.
3. Suppose that f is not bounded on $[a,b]$. Using bisection method, show that there exists a nested sequence $([a_n,b_n])$ of intervals such that f is not bounded on $[a_n,b_n]$ for each $n \in \mathbb{N}$. Then use Nested Intervals theorem to obtain a contradiction.

4.7.52 Exercise. Give an example of a one-to-one function $f : (0,\infty) \to \mathbb{R}$ that is not monotone.

4.7.53 Exercise. Show that the following four conditions on a set $A \subseteq \mathbb{R}$ are equivalent:

1. A is closed and bounded.
2. Every infinite subset of A has a limit point in A.
3. Every sequence of points from A has a subsequence converging to a point in A.
4. Every open cover of A has a finite subcover.

4.7.54 Exercise. A function $f : \mathbb{R} \to \mathbb{R}$ is said to have an **isolated zero** at a point x_0 if $f(x_0) = 0$ but x_0 is an isolated point in the set $\{x : f(x) = 0\}$. Show that a function can have at most countably many isolated zeros.

4.7.55 Exercise. Let f be defined as follows:

$$f(x) = \begin{cases} x & \text{if } 0 \leq x < 1 \text{ and } 2 \leq x < 3 \\ -x & \text{if } 1 \leq x < 2 \text{ and } 3 \leq x < 4 \end{cases}$$

and in general, if $n \in \mathbb{N}$,

$$f(x) = (-1)^{n+1}x, \text{ if } \; n - 1 \leq x < n.$$

Answer the following:

1. What is the $dom(f)$? Call it H.
2. What is the $ran(f)$? Call it K.
3. Is f strictly monotone?
4. Is f one-one?
5. Does f^{-1} exist? If not, why not? If so,
   (a) State the domain and range of f^{-1}.
   (b) Try to formulate the definition of f^{-1} and verify that $f^{-1} \circ f$ is the identity mapping on H and $f \circ f^{-1}$ is the identity mapping on K.

Chapter 5

Sequence of Real Numbers

The heart of Mathematics is its problems.
- Paul Halmos.

5.1 Sequences, Subsequences

5.1.1 Definition. A **sequence** in a set S is any function $f : \mathbb{N} \to S$ so, a sequence in $\mathbb{R}$ is any function $X : \mathbb{N} \to \mathbb{R}$. Thus a sequence is often understood to refer to a countably indexed set.

The functional value $X(n)$ is denoted by $x_n \ \forall \ n \in \mathbb{N}$, in the sense that X is the selection procedure of indexing some members of $\mathbb{R}$ by $\mathbb{N}$.

5.1.2 Remark. A sequence in $\mathbb{R}$ can also be determined by any function f on $\mathbb{R}$ and may be thought of as a composition function of an inclusion function $\iota : \mathbb{N} \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ i.e. by restricting its domain on $\mathbb{N}$ and thus $(f \circ i)(n) = f|_{\mathbb{N}}(n) = x_n$, as for example, if $f(x) = (-1)^{[x]}; x \geq 1$, then $f(n) = (-1)^n$. We denote the set of all sequences in $\mathbb{R}$ by $\mathbb{R}^{\mathbb{N}}$. As $\mathbb{R}^{\mathbb{N}}$ is the cartesian product of $\mathbb{N}$ copies of $\mathbb{R}$, and $X \in \mathbb{R}^{\mathbb{N}} \Rightarrow X = (x_1, x_2, ..x_n, ..)$ so we shall denote a sequence X by $(x_1, x_2, ..x_n, ..)$ or by $(x_n)_{n\in\mathbb{N}}$ or $(x_k)_{k\in\mathbb{N}}$ or simply by (x_n) or by (x_k). Another convention in referring to the terms of a sequence $X \in \mathbb{R}^{\mathbb{N}}$, as a range of sequence in $\mathbb{R}$, thus the notation $\{x_n\}$ stands for the set $\{x_n; n \in \mathbb{N}\}$.

5.1.3 Definition. A sequence $Y = (y_k)$ is said to be a **subsequence** of a sequence $X = (x_n)$ iff there exists an increasing function $n : \mathbb{N} \to \mathbb{N}$, such that $Y = X \circ n \Rightarrow Y(k) = X(n(k)) \Rightarrow y_k = x_{n_k} \forall \ k \in \mathbb{N}$. It is traditional to write the value of Y at k as x_{n_k}. We simply say that (x_{n_k}) is a subsequence of (x_n).

Again, $Z = (z_p)$ is a subsequence of $Y = (y_k)$, if there exists an increasing function $p : \mathbb{N} \to \mathbb{N}$ such that

$$Z = Y \circ p = (X \circ n) \circ p \Rightarrow z_k = Z(k) = (Y \circ p)(k) = ((X \circ n) \circ p)(k) = x_{n_{p_k}},$$

which shows that $Z : \mathbb{N} \to \mathbb{R}$, i.e. $Z = (z_k)$ is a subequence of $X = (x_n)$. Since the identity function $\iota : \mathbb{N} \to \mathbb{N}$ is an increasing function, so $X = (x_n)$ is a subsequence of itself, and we call it a **trivial** subsequence.

5.2 Bounded sequences, Monotone sequences

5.2.1 Definition. A sequence (x_n) in $\mathbb{R}$ is said to be **bounded** if $\exists$ a positive number M such that $|x_n| < M$ for all $n \geq N$. Equivalently, a sequence (x_n) is said to be **bounded** if $\exists$ numbers $M, m \in \mathbb{R}$ such that $m \leq x_n \leq M$ for all $n \geq N$.

5.2.2 Definition. A sequence (x_n) in $\mathbb{R}$ is said to be **eventually or ultimately bounded** if $\exists$ a positive number M and a positive integer N such that $|x_n| < M$ for all $n \geq N$. Prove that a sequence is bounded iff it is eventually bounded.

5.2.3 Definition. A sequence $X \in \mathbb{R}^{\mathbb{N}}$ is called

1. **increasing** if $x_{n+1} \geq x_n, \ \forall \ n \in \mathbb{N}$.
2. **strictly increasing** if $x_{n+1} > x_n, \ \forall \ n \in \mathbb{N}$.
3. **decreasing** if $x_{n+1} \leq x_n, \ \forall \ n \in \mathbb{N}$.
4. **strictly decreasing** if $x_{n+1} < x_n, \ \forall \ n \in \mathbb{N}$.

$X \in \mathbb{R}^{\mathbb{N}}$ is called **monotone** if X is either increasing or decreasing and X is called **strictly monotone** if X is either strictly increasing or strictly decreasing.

5.2.4 Definition. A function $f : \mathbb{R} \to \mathbb{R}$ is called **increasing** at $p \in \mathbb{R}$ if there exists an $\epsilon > 0$ such that$f(x) \leq f(p) \leq f(y)$, for all $x \in (p - \epsilon, p)$ and $y \in (p, p + \epsilon)$.

5.3 Limsup (Upper limit), Liminf (Lower limit)

[**Limit superior or limit inferior** of a bounded sequence of real numbers.]

5.3.1 Definition.

1. Let (x_n) be a bounded sequence in $\mathbb{R}$. A real number U is said to be the **limit superior** if U satisfies the following conditions:
   (a) Given $\epsilon > 0 \ \exists \ m \in \mathbb{N}$ such that $n \geq m \Rightarrow x_n < U + \epsilon$.
   (b) Given $\epsilon > 0$ and $\forall \ p \in \mathbb{N} \ \exists \ n > p$ such that $x_n > U - \epsilon$.
2. Let (x_n) be a bounded sequence in $\mathbb{R}$. A real number V is said to be the **limit superior** if
$$V = \inf_{k \geq 1} \sup_{n \geq k} x_n.$$
3. If S be the set of all subsequential limits of (x_n), then **limit superior** of (x_n) is defined by $W = \sup S$.

We show that the three definitions are equivalent in the worked out problems.

5.3.2 Definition.

1. Let (x_n) be a bounded sequence in $\mathbb{R}$. A real number L is said to be the **limit inferior** if L satisfies the following conditions:

(a) Given $\epsilon > 0$ $\exists$ $m \in \mathbb{N}$ such that $n \geq m \Rightarrow x_n > L - \epsilon$.

(b) Given $\epsilon > 0$ and $\forall$ $p \in \mathbb{N}$ $\exists$ $n > p$ such that $x_n < L + \epsilon$.

2. Let (x_n) be a bounded sequence in $\mathbb{R}$. A real number V is said to be the **limit inferior** if

$$V = \sup_{k \geq 1} \inf_{n \geq k} x_n.$$

3. If S be the set of all subsequential limits of (x_n), then **limit inferior** of (x_n) is defined by $W = \inf S$.

5.4 Limit of a function: Notion of nearness

[1] A real number $x \in \mathbb{R}$ is said to be ϵ**-near to** $a \in D \subseteq \mathbb{R}$ iff $|x - a| < \epsilon$ for $\epsilon > 0$ and $x \in \mathbb{R}$ is said to be **arbitrary-near** to a iff $|x - a| < \epsilon$ for all $\epsilon > 0$, Now one can think it is possible only when a is a limit point of some set. If a is not a limit point, then a is only point near to a.
Let $D \subseteq \mathbb{R}$ and $a \in D'$ (the derived set of D), and a function $f\colon D \to \mathbb{R}$. Suppose $l \in \mathbb{R}$. If it is possible that $f(x)$ can be made arbitrary near l while x is sufficiently near a, then we say that f has a limit l when x approaches to a. Note that the point a may or may not belong to D. Now, we write it as a definition below.

5.4.1 Definition. Let $D \subseteq \mathbb{R}$ and $a \in D'$, then a function $f\colon D \to \mathbb{R}$ is said to have a **limit** l at a iff for every $\epsilon > 0$, $\exists$ $\delta > 0$ such that $f(\hat{B}(a;\delta) \cap D) \subseteq B(l;\epsilon)$. In other words, $f(x)$ tends to a limit l as x tends to a, or $f(x)$ converges to l as x tends to a iff for each nbhd. V of l we can find a deleted nbhd. $\hat{U} = U \setminus \{a\}$ of a such that $f(\hat{U} \cap D) \subseteq V$.

The above is equivalent to the statement

$$\forall\, \epsilon > 0\ \exists\, \delta > 0 \text{ such that } x \in \hat{U} \cap D \Rightarrow f(x) \in B(l;\epsilon). \tag{5.1}$$

We write the above, symbolically, as $\lim_{x\to a} f(x) = l$, read as $f(x)$ tends to a limit l as x tends to a, Which is equivalent to the statement

$$\forall\, \epsilon > 0\ \exists\, \delta > 0 \text{ such that } x \in D \text{ and } 0 < |x - a| < \delta \Rightarrow |f(x) - l| < \epsilon. \tag{5.2}$$

5.4.2 Remark. This definition of limit is due to Karl Theodor Wilhelm Weierstrass (31 October 1815 Ostenfelde to 19 February 1897 Berlin).

The value of $f(a)$ is irrelevant to the existence of $\lim_{x\to a} f(x)$ and the limit, if it exists, may or may not equal to $f(a)$.

5.4.3 Example. Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} x, & x \neq 0 \\ 1, & x = 0. \end{cases}$$

Then $\lim_{x\to a} f(x) = 0$. For given $\epsilon > 0$, put $\delta = \epsilon$. If $0 < |x - 0| < \delta$, then $|f(x) - 0| = |x| < \epsilon$, as required.

[1] see special topics

5.4.4 Example. Suppose we want to find a $\delta > 0$ such that

$$|x| < \delta \Rightarrow |x^2 + 3x| < 1.$$

Before starting, we remember that we are asked to find a $\delta > 0$ with the required property, not the greatest possible one. We may therefore impose any convenient additional assumptions.
Suppose that $|x| < \delta$. Then

$$\begin{aligned} |x^2 + 3x| = |x||x + 3| &< \delta|x + 3| \\ &\leq \delta(|x| + |3|) \\ &< \delta(\delta + 3) \\ &\leq \delta(1 + 3) = 4\delta && \text{if } \delta \leq 1 \\ &\leq 1 && \text{if } \delta \leq 1/4. \end{aligned}$$

Therefore, if we choose in the first $\delta \leq 1$ and in the second $\delta \leq 1/4$, we would obtain

$$|x| < \delta \Rightarrow |x^2 + 3x| < 1.$$

The arbitrary assumption $\delta \leq 1$ is done purely for convenience, since it avoids a more complicated calculations in finding a δ such that $\delta^2 + \delta \leq 1$.
If we suppose $\delta \leq 2$ instead of $\delta \leq 1$, we would have obtained a different, but equally correct, a set of constrants on δ. The **key step** is that in which we obtain $|x^2 + 3x| \leq \delta|x + 3|$. From this point onwards it is enough to find a constant k, independent of x and δ, such that

$$|x| < \delta \Rightarrow |x + 3| < k,$$

from where we can write

$$|x^2 + 3x| < k\delta$$

and if we impose the additional constraint of $\delta \leq 1/k$ we shall have solved the problem.
If we do the same problem again, differently, we might obtain

$$\begin{aligned} |x| < \delta \Rightarrow |x^2 + 3x| &< \delta|x + 3| \leq \delta(|x| + 3) < \delta(\delta + 3) \\ \Rightarrow |x^2 + 3x| &< (7/2)\delta, \text{ if } \delta \leq 1/2 \\ &\leq 1, \text{ if } \delta \leq 2/7. \end{aligned}$$

Thus, if $\delta = 2/7$ we get

$$|x| < 2/7 \Rightarrow |x^2 + 3x| < 1.$$

The reader is invited to show that the largest value of δ for which

$$|x| < \delta \Rightarrow |x^2 + 3x| < 1 \text{ is } \frac{\sqrt{13} - 3}{2}.$$

5.4.5 Example. Let $\epsilon > 0$. Find $\delta > 0$ such that $|x - 1| < \delta$ and $|y - 1| < \delta$ together imply $|xy - 1| < \epsilon$.
Let $|x - 1| < \delta_1$ and $|y - 1| < \delta_1$, then

$$\begin{aligned} |xy - 1| = |x(y - 1) + (x - 1)| &\leq |x||y - 1| + |x - 1| < \delta_1(|x| + 1) \\ &\leq \delta_1(|x - 1| + 2) < \delta_1(\delta_1 + 2) \\ &< 3\delta_1, \text{ if } \delta_1 < 1 \\ &< \epsilon, \text{ if } \delta_1 < \epsilon/3. \end{aligned}$$

Therefore, if $\delta < \min\{1, \epsilon/3\}$, then $|x - 1| < \delta$ and $|y - 1| < \delta$ together imply $|xy - 1| < \epsilon$.

Now, we generalise the above definition

5.4.6 Definition (Right-hand Limit). Let $D \subseteq \mathbb{R}$ and $(a, a+\delta) \cap D \neq \emptyset\ \forall\ \delta > 0$. Then a function $f\colon D \to \mathbb{R}$ is said to have a right-hand limit l at a iff for every $\epsilon > 0$, $\exists\ \delta > 0$ such that $|f(x) - l| < \epsilon\ \forall\ x \in (a, a+\delta)$ In other words, $f(x)$ tends to a limit l as x tends to $a+$. Symbolically, $\lim_{x\to a+} f(x) = l$.

5.4.7 Definition (Left-hand Limit). Let $D \subseteq \mathbb{R}$ and $(a-\delta, a) \cap D \neq \emptyset\ \forall\ \delta > 0$. Then a function $f\colon D \to \mathbb{R}$ is said to have a left-hand limit l at a iff for every $\epsilon > 0$, $\exists\ \delta > 0$ such that $|f(x) - l| < \epsilon\ \forall\ x \in (a-\delta, a)$ In other words, $f(x)$ tends to a limit l as x tends to $a-$. Symbolically, $\lim_{x\to a-} f(x) = l$.

Thus we can say

5.4.8 Theorem. $\lim_{x\to a} f(x) = l$ iff both Right-hand limit and Left-hand limit exists.

5.4.9 Lemma. The following are equivalent:

1. $\lim_{x\to a} f(x) = l$.
2. If (x_n) be any sequence such that (x_n) converges to a, and $x_n \neq a$ for all but finite number of values of $n \in \mathbb{N}$, then $(f(x_n))$ converges to l.

5.4.10 Proposition. If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exists, then

1. $\lim_{x\to a}(f(x) + g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
   But, $\lim_{x\to a}(f(x) + g(x))$ exists does not imply that both $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. e.g. Consider $f(x) = 1 + \frac{1}{x}$ and $g(x) = 1 - \frac{1}{x}$
2. $\lim_{x\to a}(f(x).g(x)) = \lim_{x\to a} f(x). \lim_{x\to a} g(x)$
   But, $\lim_{x\to a}(f(x).g(x))$ exists does not imply that both $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. e.g. Consider $f(x) = \frac{1}{x}$ and $g(x) = x$
3. $\lim_{x\to a}(f(x)/g(x)) = \lim_{x\to a} f(x)/\lim_{x\to a} g(x)$, provided $\lim_{x\to a} g(x) \neq 0$.
4. $\lim_{x\to a} f(x)^{g(x)} = \lim_{x\to a} f(x)^{\lim_{x\to a} g(x)}$, provided both limits are not 0.
5. $\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x))$, is not true, in general.

5.4.11 Remark. This definition appears in Cauchy's *Cours d'analyse*, although it is stated with a minimal use of symbols. The symbol ϵ appears elsewhere in the book and is considered to be the first letter of the French word "erreur" (error). He also used the letter δ to denote a small quantity. It is believed that it comes from the word "différence" (French for difference).

5.4.12 Note. The first use of the abbreviation lim. (with a period at the end) was by a Swiss mathematician, Simon L' Huilier (1750–1840), in 1786. German mathematician Karl Weierstrass (1815–1897), who is often cited as the "father of modern analysis," used it (without a period) as early as 1841, but it did not appear in print until 1894. In the 1850s, he began to write $\lim_{x=c}$, and it appears that we owe the arrow (instead of the equality) to two English mathematicians. John Gaston Leathem (1871–1923) pioneered its use in 1905, and Godfrey Harold Hardy (1877–1947) made it popular through his 1908 textbook: A Course of Pure Mathematics. Weierstrass was supposed to study law and finance, but instead spent time studying mathematics. That is why he did not

get a degree, and he started his career as a high school teacher. He spent about 15 years there, until his mathematical work brought him fame: an honorary doctoral degree from the University of Königsberg and a position of professor at the University of Berlin, which was considered the leading university in the world to study mathematics.

5.4.13 Note. Definition of limit appeared for the first time in 1821, in a textbook Cours d'analyse (A Course of Analysis), by a French mathematician Augustin-Louis Cauchy (1789-1857). It completely changed mathematics by introducing the rigor that was lacking in the work of his contemporaries. We celebrate Newton and Leibniz for inventing calculus, but Cauchy took a giant step toward making it a rigorous discipline.
Isaac Newton (1642–1727) was an English physicist, mathematician, and astronomer. He has been considered by many to be the greatest scientist who ever lived. Gottfried Leibniz (1646–1716) was a German philosopher and mathematician. He belongs to the Pantheon of Mathematics for inventing calculus (independently of Newton). Cauchy was one of the most prolific writers with approximately eight hundred research articles and five complete textbooks. His writings cover the entire range of mathematics and mathematical physics. It is believed that more concepts and theorems have been named for Cauchy than for any other mathematician.

5.5 Infinite limits:

Infinite limits can defined as usual in the following ways.

1. $\lim_{x\to\infty} f(x) = l$ iff for every $\epsilon > 0$, ϵ-nbhd. $B(l;\epsilon)$ of l $\exists$ a nbhd. (M,∞) of ∞ such that $f(M,\infty) \subseteq B(l;\epsilon)$ i.e. in other words, $\exists\ M > 0$ such that $x > M \Rightarrow |f(x) - l| < \epsilon$. The following explains the situation:
To prove $\lim_{x\to\infty} \frac{1}{x} = 0$, let $M \in \mathbb{R}$ and $x > M$ then $1/x < 1/M \Rightarrow |1/x - 0| < 1/M$, now choose $\epsilon = 1/M$

2. $\lim_{x\to-\infty} f(x) = l$ iff $\forall\ \epsilon > 0\ \exists\ M \in \mathbb{R}$ such that $f(-\infty, M) \subseteq B(l;\epsilon)$ i.e. $x < M \Rightarrow |f(x)-l| < \epsilon$. Example: $\lim_{x\to\infty} \frac{1}{x^2} = 0$

3. $\lim_{x\to\infty} f(x) = \infty$ iff $\forall\ N \in \mathbb{R}$, $\exists\ M \in \mathbb{R}$ such that $f(M,\infty) \subseteq (N,\infty)$ i.e. $x > M \Rightarrow f(x) > N$. Example: $\lim_{x\to\infty} x^2 = \infty$.

4. $\lim_{x\to-\infty} f(x) = \infty$ iff $\forall\ N \in \mathbb{R}\ \exists\ M \in \mathbb{R}$ such that $f(-\infty, M) \subseteq (N,\infty)$ i.e. $x < M \Rightarrow f(x) > N$. Example: $\lim_{x\to-\infty} x^2 = \infty$

5. $\lim_{x\to a} f(x) = \infty$ iff $\forall\ M > 0\ \exists\ \delta > 0$ such that $f(B(a;\delta)) \subseteq (M,\infty)$ i.e. $x \in (a-\delta, a+\delta) \Rightarrow f(x) > M$. Example: $\lim_{x\to a} \frac{1}{|x-a|} = \infty$.

6. $\lim_{x\to-\infty} f(x) = -\infty$ iff $\forall\ N > 0\ \exists\ M > 0$ such that $f(-\infty, M) \subseteq (-\infty, N)$ i.e. $x < M \Rightarrow f(x) < N$. Example: $\lim_{x\to-\infty} x^3 = -\infty$.
And similarly for other limits.

5.6 Limit of a sequence

Any real number is made accessible through its rational approximations, for example, cutting off the decimals starting with $(n+1)$-th one. As n increases, these approximations come closer to the

given real number, a process that lies at the heart of the subject of convergence. The study of many algorithms (such as the Babylonian algorithm for extracting the square root) needs some theoretical considerations of convergence and limits, which can be found in this section.
The notion of a sequence of real numbers is motivated by the various algorithms that make available a certain object by its successive approximations in a class of well-behaved objects. From this point of view, the main problem in connection with a sequence is its behavior for large values of indices. Since a function $X : \mathbb{N} \to \mathbb{R}$ is a sequence in $\mathbb{R}$, we write

5.6.1 Definition. (**Topological version**) $X = (x_n) \in \mathbb{R}^{\mathbb{N}}$ $(X(n) = x_n)$ tends to a limit l as n tends to ∞, iff for each nbhd. U of l there exists a nbhd. N_∞ of ∞ such that $X(N_\infty \cap \mathbb{R}) \subseteq U$. Note that any set which contains a set of the form (a, ∞) is a nbhd. of ∞.

We see that, $N_\infty \cap \mathbb{R}$ is actually the set $(\lambda, \infty) \cap \mathbb{N} = \{m, m+1, ..\}$ for some $\lambda \in \mathbb{R}$, and $m \in \mathbb{N}$ with $m - 1 \leq \lambda < m$. If we take $U = (l - \epsilon, l + \epsilon)$ then $x_n \in (l - \epsilon, l + \epsilon)$ whenever $n \geq m$. i.e. If ϵ is chosen arbitrarily, then there exists $m \in \mathbb{N}$, such that $n \geq m \Rightarrow |x_n - l| < \epsilon$. Symbolically, we write $\lim_{n\to\infty} x_n = l$ or $x_n \to l$ as $n \to \infty$. or simply $\lim x_n = l$.

5.7 Convergent and Divergent sequences

5.7.1 Definition. A sequence (x_n) in $\mathbb{R}$ is said to **converge** to $x \in \mathbb{R}$, or x is said to be a limit of (x_n), if for every $\epsilon > 0$ there exists a natural number m such that for all $n > m$, the terms x_n satisfy $|x_n - x| < \epsilon$.
If a sequence has a limit, we say that the sequence is **convergent**; if it has no limit, we say that the sequence is **divergent**. We say (x_n) is **properly divergent** if either $\lim_{n\to\infty} x_n = \infty$ or $-\infty$.

5.7.2 Proposition. Let (x_n) be a sequence of real numbers. Then the following are equivalent:

1. The sequence (x_n) does not converge to $x \in \mathbb{R}$.
2. There exists an $\epsilon > 0$ such that for any $k \in \mathbb{N}$, there exists $n_k \in \mathbb{N}$ such that $n_k > k$ and $|x_{n_k} - x| \geq \epsilon$.
3. There exists an $\epsilon > 0$ and a subsequence (x_{n_k}) of (x_n) such that $|x_{n_k} - x| \geq \epsilon \; \forall \; k \in \mathbb{N}$.

5.7.3 Note. If a sequence $X = (x_n)$ has a limit l, then l may not be a limit point of $\{x_n\}$. e.g. Consider

$$x_n = \begin{cases} n, & \text{if } 1 \leq n \leq 4 \\ 5, & \text{if } n \geq 5 \end{cases}$$

Here X converges to 5, but $\{x_n; n \in \mathbb{N}\} = \{1, 2, 3, 4, 5\}$ and 5 is not a limit point $\{x_n; n \in \mathbb{N}\}$.

5.7.4 Definition. A real number l is an **limit point** or **accumulation point** of a sequence (x_n) if and only if either the sequence (x_n) has a stationary subsequence whose every element is equal to l, or every interval containing l has an infinite number of terms of the sequence (x_n) (or both), i.e. the limit point of the set $\{x_n; n \in \mathbb{N}\}$.

Here are two useful definitions:

5.7.5 Definition.

1. A sequence (x_n) is **eventually** or **ultimately** in a set $A \subseteq \mathbb{R}$ iff there exists an $m \in \mathbb{N}$ such that $x_n \in A$ for all $n \geq m$.
2. A sequence (x_n) is **frequently** in a set $A \subseteq \mathbb{R}$ iff for every $p \in \mathbb{N}$ there exists an $n \in \mathbb{N}$ such that $n \geq p \Rightarrow x_n \in A$.

"Ultimately" means "from some index onward"; "frequently" means "for infinitely many indices".

5.7.6 Definition. A sequence (x_n) in $\mathbb{R}$ is said to be **null sequence** if, for every positive real number ϵ, $|x_n| < \epsilon$, ultimately.

Now, it is convenient to state explicitly a result which is often used in in the proofs of convergence.

5.7.7 Lemma. Let (x_n) be a sequence in $\mathbb{R}$, and suppose that there is a positive real number k such that for a given $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $n \geq N \Rightarrow |x_n - l| < k\epsilon$. Then x_n converges to l.

Proof. Let $\epsilon > 0$, then $\frac{\epsilon}{k} > 0$, and if the stated condition holds, then $\exists\ N$ such that $n \geq N \Rightarrow |x_n - l| < k(\epsilon/k) = \epsilon$ as required. □

5.7.8 Note. There is no hard and fast rule for finding the limit of a sequence, but there exists a criterion for existence of a limit of a sequence.

The following was introduced by Cauchy and we honor him by calling this property that a sequence may have by his name.

5.7.9 Definition (Cauchy Sequence). A sequence (x_n) is said to be a **Cauchy sequence** iff for a given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $m, n \geq N \Rightarrow |x_m - x_n| < \epsilon$. Cauchy sequence is also termed as **fundamental sequence** or **regular sequence**.

5.7.10 Theorem. Cauchy Criterion: A sequence (x_n) in $\mathbb{R}$ converges iff it is a Cauchy sequence.

Proof. Suppose x_n converges to l. Then for any $\epsilon > 0, \exists\ N \in \mathbb{N}$ such that $n \geq N \Rightarrow |x_m - l| < \epsilon$. So for $m, n \geq N$

$$|x_m - x_n| = |x_m - l + l - x_n| \leq |x_m - l| + |l - x_n| < 2\epsilon$$

Hence (x_n) is a Cauchy sequence.
Conversely, suppose that (x_n) is a Cauchy sequence in $\mathbb{R}$. Let $\epsilon > 0$. Then $\exists\ N \in \mathbb{N}$ such that $m, n \geq N \Rightarrow |x_m - x_n| < \epsilon$. So for any $m \geq N$ we have $|x_m - x_N| < \epsilon$, and hence

$$|x_m| = |x_m - x_N + x_N| \leq |x_m - x_N| + |x_N| < \epsilon + |x_N|.$$

Thus $|x_n| \leq \max\{|x_1|, |x_1|, .., |x_{N-1}|, \epsilon + |x_N|\}$ for all $n \in \mathbb{N}$, so x_n is bounded.
Next, we consider the sets $S_m = \{x_n; n \geq m\}$ for each $m \in \mathbb{N}$. Since each S_m is bounded, by LUB axiom $\sup S_m$ exists, let $\sup S_m = t_m$, then we get

$$S_m \supseteq S_{m+1} \Rightarrow \sup S_m \geq \sup S_{m+1} \Rightarrow t_m \geq t_{m+1}\ \forall\ m \in \mathbb{N}.$$

Thus we get a monotonic decreasing sequence (t_n) and $t_m \geq x_m$, so (t_m) converges to its greatest lower bound l (say).
Finally, we prove that (x_n) converges to l. Let $\epsilon > 0$, then there exists $N_1 \in \mathbb{N}$ such that $m, n \geq N_1 \Rightarrow |x_m - x_n| < \epsilon$, and there exists $N_2 \in \mathbb{N}$ such that $m \geq N_2 \Rightarrow |t_m - l| < \epsilon$. Put $N = \max\{N_1, N_2\}$.

Since $t_N = \sup S_N$ then $\exists\, M \geq N$ such that $x_M > t_N - \epsilon$ also $x_M \leq t_N$. Since $x_M \in S_N$ and t_N is an upper bound of S_N, hence $|x_M - t_N| < \epsilon$. Now for any $n \geq N$, we get

$$|x_n - l| = |x_n - x_M + x_M - t_N + t_N - l| \leq |x_n - x_M| + |x_M - t_N| + |t_N - l| < 3\epsilon.$$

Thus (x_n) converges to l. $\square$

The above statement was proved by Cauchy in Cours d'analyse in 1821. Four years earlier, Bernard Bolzano (1781–1848), a Bohemian mathematician, philosopher, and a Catholic priest, explicitly stated the same result and gave an incomplete proof. It is very likely that Cauchy was aware of this work. Nevertheless, it should be emphasized that Cours d'analyse remains one of the most important and influential mathematics books ever written.
A further result about sequences we record here is known as Bolzano-Weierstrass theorem for a sequence.

5.7.11 Theorem. (**Bolzano-Weierstrass theorem**) Every bounded sequence of real numbers has at least one convergent subsequence.

5.7.12 Theorem (**Sandwich Theorem**). Suppose that $X = (x_n), Y = (y_n)$ and $Z = (z_n)$ are sequences of real numbers such that $x_n \leq z_n \leq y_n \;\forall\; n \in \mathbb{N}$, and that $\lim(x_n) = \lim(y_n)$. Then $Z = (z_n)$ is convergent and $\lim(x_n) = \lim(y_n) = \lim(z_n)$.

5.7.13 Theorem. Let (x_n) be a sequence of positive real numbers such that $L = \lim(x_{n+1}/x_n)$ exists. If $L < 1$, then (x_n) converges and $\lim(x_n) = 0$.

5.8 Oscillatory sequences

5.8.1 Definition. A sequence of real numbers which is not convergent to any value in the extended real number system is called an **oscillatory** sequence.

5.8.2 Example. A simple example of an oscilatory bounded sequence is

$$1, -1, 1, -1, ...$$

5.8.3 Example. An oscillatory sequence which is not bounded is

$$1, -2, 3, -4, 5, -6, ...$$

5.9 little o and Big O

5.9.1 Definition. We say that a function f is **asymptotically equivalent** to the function g as x approaches x_0 if there exists a function ϕ such that

$$f(x) = \phi(x).g(x); x \in (a, b), x \neq x_0, \text{and } \lim_{x \to x_0} \phi(x) = 1.$$

Then, we write $f(x) \sim g(x)$ as $x \to x_0$. A sufficient condition for the asymptotic equivalence $f(x) \sim g(x)$ as $x \to x_0$ is the equality $\lim_{x \to x_0} \frac{f(x)}{g(x)} = 1$.

5.9.2 Definition. We say that a function f is **small oh** to the function g as x approaches x_0 if there exists a function ϕ such that

$$f(x) = \phi(x).g(x); x \in (a,b), x \neq x_0, \text{and } \lim_{x \to x_0} \phi(x) = 0.$$

Then, we write $f = o(g)$ as $x \to x_0$. If $g(x) \neq 0 \ \forall\ x \neq x_0$, then a necessary and sufficient condition for the asymptotic relation $f(x) = o(g(x))$ as $x \to x_0$ is $\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0$.
In particular, if $g(x) \equiv 1$, then $f(x) = o(1)$ as $x \to x_0$. This means that f tends to zero as $x \to x_0$.

5.9.3 Definition. We say that a function f is **big oh** to the function g as x approaches x_0 if there exists a constant $K > 0$ such that

$$|f(x)| \leq K|g(x)|; x \in (a,b), x \neq x_0.$$

Then, we write $f(x) = O(g(x))$ as $x \to x_0$.
In particular, if $g(x) \equiv 1$, then $f(x) = O(1)$ as $x \to x_0$. This means, that for some $\delta > 0$, the function f is bounded on the set $(x_0 - \delta, x_0 + \delta) \setminus \{x_0\}$.
These three notions, namely the asymptotic equivalence, "big oh" and "small oh", can also be defined in the cases $x \to x_0-$, $x \to x_0+$, $x \to -\infty$, $x \to +\infty$.

5.9.4 Example. $\sin 3x \sim 3x, \arctan 2x \sim 2x, xe^x \sim x,$
$\ln(1 + 2x + \sin^2 x) \sim 2x + \sin^2 x \sim 2x$ when $x \to 0$.

5.9.5 Remark. $f(x) = o(g(x))$ as $x \to x_0$ implies $f(x) = O(g(x))$ as $x \to x_0$, but the converse is false in general.

5.9.6 Definition. (**Infinitesimal**). Let I be an interval, and let $x_0 \in I$. Suppose that f is a function defined on I (except possibly at x_0). We say that f is an **infinitesimal** (or infinitely small) at x_0 (or, as $x \to x_0$) if $f = o(1)(x \to x_0)$, i.e., if $\lim_{x \to x_0} f(x) = 0$.

5.9.7 Remark.

1. Using infinitesimals, we can rephrase many statements. Thus $f = o(g)(x \to x_0)$ can also be written as $f = g.o(1)(x \to x_0)$, which means (if g is nonzero near x_0) that f/g is an infinitesimal at $x = x_0$:

2. As $x \to 0$, we have an important sequence of infinitesimals, namely, the sequence of monomials $x, x^2, x^3, ..., x^n, ...$ It is obvious that the larger the exponent n; the faster x^n converges to 0: We shall see that many infinitesimals at 0 are equivalent to an infinitesimal of the form ax^n; where $n \in \mathbb{N}$ and a is a nonzero constant.

5.9.8 Definition. (**Bounded Away From Zero**). Let I be an interval, and let $x_0 \in I$. Suppose that f is a function defined on I (except possibly at x_0.) We say that f is **bounded away from zero** as $x \to x_0$ if there exists $\epsilon > 0$ such that $|f(x)| \geq \epsilon$.

5.9.9 Definition. We say that a sequence (x_n) is **asymptotically equivalent** to a sequence (y_n) as n tends to ∞, if $y_n \neq 0$ for every $n \in \mathbb{N}$ such that

$$\lim_{n \to \infty} \frac{x_n}{y_n} = 1.$$

Then we write $x_n \sim y_n$ as $n \to \infty$.

5.9.10 Definition. Let (b_n) be a given sequence of non-negative real numbers and let (a_n) be any sequence of real numbers. We say $a_n = O(b_n)$ as $n \to \infty$ (a_n is **big oh** of b_n) if there exist a fixed number K and $M \in \mathbb{N}$ such that $|a_n| \leq K b_n \ \forall\, n \geq M$. We also say $a_n = o(b_n)$ (a_n is **little oh** of b_n) as $n \to \infty$ if for each $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|a_n| \leq \epsilon b_n \ \forall\, n \geq N$.
In other words, if $b_n > 0 \ \forall\, n \in \mathbb{N}$, then "$a_n = O(b_n)$" means that the sequence (a_n/b_n) is bounded, and "$a_n = o(b_n)$" means that the sequence $(a_n/b_n) \to 0$ as $n \to \infty$. In paricular, "$a_n = O(1)$" that the sequence (a_n) is bounded, and "$a_n = o(1)$" means $(a_n) \to 0$ as $n \to \infty$.

In particular, if the sequences (a_n) and (b_n) diverge to ∞ and satisfy the condition

$$x_n = o(y_n) \text{ as } n \to \infty \ \Leftrightarrow \ \lim_{n\to\infty} \frac{x_n}{y_n} = 0.$$

then we say that the sequence (y_n) diverges faster to infinity than the sequence (x_n) and we write $x_n \prec y_n$ as $n \to \infty$.

The two following results are due to the German mathematician J. P. G. Lejeune Dirichlet.

5.9.11 Lemma. Let $\theta \in \mathbb{R}$ and $t \in \mathbb{N}$. Then there exist integers p and q so that $0 < q \leq t$ and

$$|q\theta - p| < \frac{1}{t}.$$

5.9.12 Theorem. Let $\theta \in \mathbb{R}$ and $t \in \mathbb{N}$. Then there exist integers p and q so that

$$\left|\theta - \frac{p}{q}\right| < \frac{1}{Qq} \leq \frac{1}{q^2} \text{ where } 0 < q \leq Q.$$

5.9.13 Theorem. Every sequence can have at most one limit.

5.9.14 Theorem. Every convergent sequence is bounded.

5.9.15 Theorem. Let $(a_n), (b_n)$ be convergent sequences with $\lim a_n = a, \lim b_n = b$, and let α be a real number. Then the sequences (αa_n) and $(a_n + b_n)$ are also convergent and:

1. $\lim_n(\alpha a_n) = \alpha \lim a_n$;
2. $\lim_n(a_n + b_n) = \lim a_n + \lim b_n$.

5.9.16 Theorem. Let $(a_n), (b_n)$ be convergent sequences with $\lim a_n = a, \lim b_n = b$. Then:

1. The sequence $(a_n b_n)$ is also convergent and $\lim(a_n b_n) = \lim a_n \lim b_n$.
2. If, in addition, $b_n \neq 0$ for all $n \in \mathbb{N}$ and if $b \neq 0$, then the sequence (a_n/b_n) is also convergent and $\lim(a_n/b_n) = \lim a_n / \lim b_n$.

5.9.17 Theorem. Let $(a_n), (b_n), (c_n)$ be sequences such that $\lim a_n = \lim c_n = L$ and suppose that, for all $n \in \mathbb{N}, a_n \leq b_n \leq c_n$. Then b_n is a convergent sequence and $\lim b_n = L$.

5.9.18 Proposition. Let a_n be a convergent sequence with $\lim a_n = a$, and suppose that $a_n \geq 0$ for all $n \in \mathbb{N}$. Then $a \geq 0$.

5.9.19 Corollary. Let $(a_n), (b_n)$ be two convergent sequences, let $\lim_n a_n = a, \lim_n b_n = b$, and suppose that $a_n \leq b_n$ for all $n \in \mathbb{N}$. Then $a = b$.

5.9.20 Theorem. (Monotone Convergence Theorem) If a sequence is increasing and bounded above, then it is convergent.

5.9.21 Theorem. (Bernoulli's Inequality) If $x > -1$ and $n \in \mathbb{N}$, then $(1+x)^n \geq 1 + nx$.

5.9.22 Note. This inequality carries the name of a Swiss mathematician Jacob Bernoulli (1654–1705), because it appeared in his work in 1689. Historians of mathematics point out at exactly the same result by Isaac Barrow (1630–1677), an English mathematician, except that the latter publication appeared almost 20 years earlier, in 1670. Jacob Bernoulli was one of the many prominent mathematicians in the Bernoulli family. Following his father's wish, he studied theology and, contrary to the desires of his parents, mathematics and astronomy. He became familiar with calculus through a correspondence with Leibniz, and he made significant contributions (separable differential equations), as well as in probability (Bernoulli trials). He founded a school for mathematics and the sciences at the University of Basel and worked there as a professor of mathematics for the rest of his life.

5.9.23 Theorem. Both the sequences $a_n = 1 + \frac{1}{1!} + + \frac{1}{2!} + ... + \frac{1}{n!}$ and $b_n = \left(1 + \frac{1}{n}\right)^n$ are convergent, and converges to the same limit e. (e is due to Euler.)

5.9.24 Note. The sequence a_n appeared for the first time in 1683, in the work of Jacob Bernoulli on compound interest. However, he only obtained that its limit lies between 2 and 3. Prior to that, e was present through the use of natural logarithms, but the number itself was never explicitly given. It is considered that the first time it appears in its own right is in 1690, in a letter from Leibniz to Huygens, who used the notation $\mathfrak{b}$. Christiaan Huygens (1629–1695) was a Dutch mathematician, astronomer, and physicist. He is, perhaps, best known for his argument that light consists of waves. The letter e was introduced by Euler in a letter to Goldbach in 1731. Leonhard Euler (1707–1783), a Swiss mathematician, was probably the best mathematician in the 18th century and one of the the greatest of all time. Christian Goldbach (1690–1764) was a German mathematician, remembered mostly for "Goldbach's conjecture" (every even integer greater than 2 can be expressed as the sum of two primes). Bernoulli discovered the sequence $b_n = \left(1 + \frac{1}{n}\right)^n$ And the other one is due to Euler. It appeared in his book published in 1748. He also proved there that $\lim a_n = e$, calculated e to 18 decimal places, and gave an incomplete argument that e is not a rational number.

5.9.25 Lemma. Let $n \in \mathbb{N}$, let $a_n = 1 + \frac{1}{1!} + + \frac{1}{2!} + ... + \frac{1}{n!}$ and define $\theta_n = (e - a_n)n!$. Then $0 < \theta_n < 1$.

5.9.26 Theorem. The number e is not a rational number.

5.10 Decimal representation of a real number:

5.10.1 Example. This example illustrates the above theorem on inductive definition of function. Here with our axiomatic treatment of real numbers we want to prove the existence of a decimal representation of a real number and for this we use the greatest integer function $x \to [x]$, whose existence depends on the Archimedian property of real numbers.
Let x be a real number, and suppose for the moment we have a decimal representation of x, that is

$$x = a_0.a_1a_2a_3... = a_0 + \frac{a_1}{10} + \frac{a_2}{10^2} + \frac{a_3}{10^3} + ...,$$

where a_n are integers and $0 \le a_n \le 9$ for $n \ge 1$. Suppose further that this decimal does not end in recurring 9's, i.e. the set of n for which $a_n < 9$ is infinite. Then if m is a positive integer such that $a_n < 9$ and $n \ge m$, we have

$$\sum_{r=0}^{n} \frac{a_r}{10^r} \le a_0 + \sum_{r=1}^{n} \frac{9}{10^r} - \frac{1}{10^m} = a_0 + 1 - \frac{1}{10^n} - \frac{1}{10^m},$$

hence

$$x = \lim_{n\to\infty} \sum_{r=0}^{n} \frac{a_r}{10^r} \le a_0 + 1 - \frac{1}{10^m},$$

therefore $a_0 = [x]$. Further, $x - a_0 == 0.a_1a_2a_3...$ so that $10(x - a_0) = a_1.a_2a_3...$, hence $a_1 = [10(x - a_0)]$. Similarly $10\{10(x - a_0) - a_1\} = a_2.a_3a_4...$, so that

$$a_2 = [10\{10(x - a_0) - a_1\}] = [10^2\{x - a_0 - a_1/10\}],$$

and so on. The integers a_n in the decimal representation therefore satisfy the formula

$$a_0 = [x], a_{n+1} = \left[10^{n+1}\left(x - a_0 - \frac{a_1}{10} - \frac{a_2}{10^2} - .. - \frac{a_n}{10^n}\right)\right]; n \ge 0. \tag{5.3}$$

Thus to prove the existence of a decimal representation of any real number x, we simply define a sequence of integers a_n by the above formula (5.3) that is an example of the inductive definition of a sequence a_n, where a_n depends on n and on all the preceeding terms of the sequence, so that to verify the existence of the sequence a_n, we have to appeal to the full result on the inductive definition.

With this definition of a_n, for any integer $n \ge 1$ we have

$$0 \le 10^{n+1}\left(x - a_0 - \frac{a_1}{10} - \frac{a_2}{10^2} - .. - \frac{a_n}{10^n}\right) - a_{n+1} < 1,$$

so that

$$0 \le \left(x - a_0 - \frac{a_1}{10} - \frac{a_2}{10^2} - .. - \frac{a_n}{10^n} - \frac{a_{n+1}}{10^{n+1}}\right) < \frac{1}{10^{n+1}}, \tag{5.4}$$

Hence if x_n is a sequence given by

$$x_n = a_0 + \frac{a_1}{10} + \frac{a_2}{10^2} + .. + \frac{a_n}{10^n}$$

then

$$0 \le x - x_n < \frac{1}{10^n}$$

Hence $x_n \to x$ as $n \to \infty$, so that the series

$$a_0 + \frac{a_1}{10} + \frac{a_2}{10^2} + .. + \frac{a_n}{10^n} + ... \tag{5.5}$$

converges to the sum x.

Further from (5.4) replacing $n + 1$ by n we have

$$0 \le 10^{n+1}\left(x - a_0 - \frac{a_1}{10} - \frac{a_2}{10^2} - .. - \frac{a_n}{10^n}\right) < 10,$$

so that $0 \leq a_{n+1} < 10$ for $n \geq 0$, hence (5.5) is the decimal representation of x. Moreover, this representation does not end in recurring $9's$. Indeed, if on the contrary $a_n = 9$ for all $n \geq r$, then the series

$$\frac{a_{r+1}}{10^{r+1}} + \frac{a_{r+2}}{10^{r+2}} + .. = \frac{9}{10^{r+1}} + \frac{9}{10^{r+2}} + ..$$

converges to $\frac{1}{10^r}$, and since the sum of this series is $x - x_r$, this contradicts the fact that $x - x_r < \frac{1}{10^r}$. Since we have already shown that any decimal representation $a_0.a_1a_2a_3...$ of x which does not end in recurring 9's necessarily satisfies (5.3), we have thus proved that *any real number has one and only one decimal representation not ending in recurring* 9*'s.*

In the preceeding discussion the integer 10 can be replaced by any integer $p \geq 2$, and to obtain "p-nary" decimal representation of x.

5.11 Problems and Solutions on Chapter 5

Warning! *Never Never Ever read this solution unless you tried problems quite long time and gave up. Doing so may impair your ability to think and solve problems.*

5.11.1 Problem. Show that $\lim_{\theta \to 0} \cos\theta = 1$.

5.11.1.1 Solution. Let $0 < \epsilon \leq 2$. Let O be the centre of a circle of radius 1, and $\overline{BC}$ be the chord perpendicular to radius OA at a distance ϵ from A.(Figure 5.1) Let $\delta = \angle AOB$. It is evident that $1 - \cos\theta < \epsilon$, i.e. $|\cos\theta - 1| < \epsilon$ whenever $|\theta - 0| < \delta$. If $\epsilon > 2$ then $|\cos\theta - 1| < \epsilon$ for all θ. □

5.11.2 Problem. Show that $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$.

5.11.2.1 Solution. We first prove the inequality

$$\cos\theta < \frac{\theta}{\sin\theta} < \sec\theta$$

where $0 < \theta < \pi/2$.

Let AB be an arc of a circle with centre O and radius r (Figure 5.2) and let BC and AD be $\perp$ to OA. Suppose that $\angle AOB = \theta$ such that $0 < \theta < \pi/2$. Since

$$area\triangle OCB < area\ sectorOAB < area\triangle OAD,$$

So, we have

$$\begin{aligned}
&\frac{1}{2}r^2 \sin\theta\cos\theta < \frac{1}{2}r^2\theta < \frac{1}{2}r^2\tan\theta, \\
&\Rightarrow \cos\theta < \frac{\theta}{\sin\theta} < \sec\theta, \text{ dividing by } \frac{1}{2}r^2\sin\theta \\
&\Rightarrow \sec\theta > \frac{\sin\theta}{\theta} > \cos\theta.
\end{aligned}$$

Again, since $\lim_{\theta \to 0} \cos\theta = 1$ and $\lim_{\theta \to 0} \sec\theta = 1$ so, if $\epsilon > 0$ $\exists$ $\delta_1, \delta_2 > 0$ such that $|\cos\theta - 1| < \epsilon$

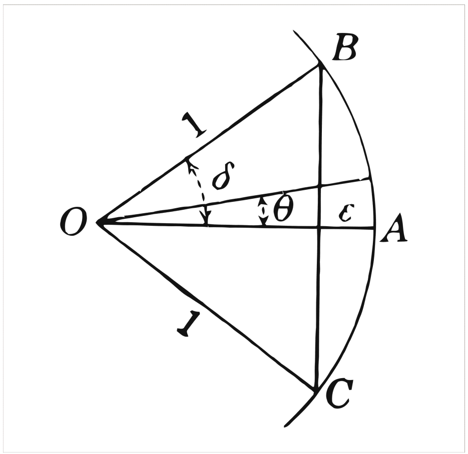

Figure 5.1:

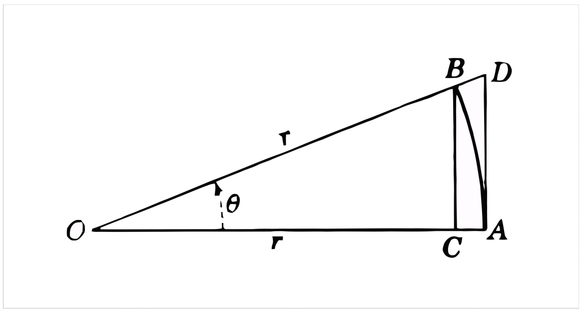

Figure 5.2:

when $|\theta| < \delta_1$ and $|\sec\theta - 1| < \epsilon$ when $|\theta| < \delta_2$. If $\delta = \min\{\delta_1, \delta_2, \pi/2\}$, then

$$\begin{aligned}
&\sec\theta - 1 > \frac{\sin\theta}{\theta} - 1 > \cos\theta - 1 \\
\Rightarrow &\epsilon > \sec\theta - 1 > \frac{\sin\theta}{\theta} - 1 > \cos\theta - 1 > -\epsilon \\
\Rightarrow &\left|\frac{\sin\theta}{\theta} - 1\right| < \epsilon \text{ whenever } |\theta| < \delta.
\end{aligned}$$

Thus $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$. □

5.11.3 Problem. Show that

$$\lim_{x\to x_0} f(x) = y_0 \Leftrightarrow \lim_{x\to x_0} (f(x) - y_0) = 0 \Leftrightarrow \lim_{h\to 0} f(x_0 + h) = y_0$$

5.11.3.1 Solution. Suppose $\lim_{x\to x_0} f(x) = y_0$. Then,

$$\begin{aligned}
&\lim_{x\to x_0} f(x) = y_0 \\
\Leftrightarrow &\forall\ \epsilon > 0\ \exists\ \delta > 0 \text{ such that } x \in B(x_0;\delta) \Rightarrow f(x) \in B(y_0;\epsilon) \\
\Leftrightarrow &x \in B(x_0;\delta) \Rightarrow f(x) - y_0 \in B(0;\epsilon) \\
\Leftrightarrow &\lim_{x\to x_0} (f(x) - y_0) = 0.
\end{aligned}$$

Again, let $x = x_0 + h$, then we get

$$\begin{aligned}
&x_0 + h \in B(x_0;\delta) \Rightarrow f(x_0 + h) \in B(y_0;\epsilon) \\
\Leftrightarrow &h \in B(0;\delta) \Rightarrow f(x_0 + h) \in B(y_0;\epsilon) \\
\Leftrightarrow &\lim_{h\to 0} f(x_0 + h) = y_0.
\end{aligned}$$

Hence the result follows. □

5.11.4 Problem. Find $\lim_{n\to\infty} \left(\sqrt{n^2 + 2n} - \left[\sqrt{n^2 + 2n}\right]\right)$, if it exists.

5.11.4.1 Solution. Hint: Note that $n < \sqrt{n^2 + 2n} < n + 1$.
Thus $\left[\sqrt{n^2 + 2n}\right] = n$ and $\left(\sqrt{n^2 + 2n} - \left[\sqrt{n^2 + 2n}\right]\right) = \left(\sqrt{n^2 + 2n} - n\right) \to 1$. □

5.11.5 Problem. Let $f : S \to \mathbb{R}$ and a is an accumulation point of S. Let (x_n) be a sequence in S and $x_n \neq a$ such that $\lim_{n\to\infty} x_n = a$. Then we have

1. If $\lim_{x\to a} f(x) = l$, then $\lim_{n\to\infty} f(x_n) = l$.
2. Conversely, if for every such sequence the limit $\lim_{n\to\infty} f(x_n)$ exists, then all these sequences have the same limit (y say) and also $\lim_{x\to a} f(x)$ exists and equals y.

5.11.5.1 Solution.

1. For every $\epsilon > 0$, $\exists\ \delta > 0$ such that $x \in B(a;\delta)\cap S$ implies $f(x) \in B(l;\epsilon)$, then as $\lim_{n\to\infty} x_n = a$ so $\exists\ m \in \mathbb{N}$ such that $n \geq m$ implies $x_n \in B(a;\delta) \cap S$ and consequently $f(x_n) \in B(l;\epsilon)$, which shows that $\lim_{n\to\infty} f(x_n) = l$.
2. Let $(p_n), (q_n)$ be two sequences such that $p_n \neq a, q_n \neq a$ for each n, and such that $\lim_{n\to\infty} p_n = a = \lim_{n\to\infty} q_n$. Let $\lim_{n\to\infty} f(p_n) = p, \lim_{n\to\infty} f(q_n) = q$. We prove that $p = q$. If $p \neq q$ we consider the sequence (x_n) defined by

$$x_n = \begin{cases} p_n & \text{if } n \text{ is even,} \\ q_n & \text{if } n \text{ is odd.} \end{cases}$$

Then $\lim_{n\to\infty} x_n = a$ as (x_{2n-1}) and (x_{2n}) both converge to a and $\lim_{n\to\infty} f(x_n)$ exists. Let $r = \frac{1}{2}|p - q|$ then $\exists\ m_1$ such that $n \geq m_1 \Rightarrow f(x_n) \in B(p;r)$ and similarly $\exists\ m_2$ such that $n \geq m_2 \Rightarrow f(x_n) \in B(q;r)$. Thus, if $m = \max\{m_1, m_2\}$ then

$$\begin{aligned}
2r &= |p - q| \\
&= |p - f(x_m) + f(x_m) - q| \\
&\leq |p - f(x_m)| + |f(x_m) - q| \\
&< r + r = 2r,
\end{aligned}$$

shows a contradiction. Thus $p = q = y$ (say). Next we show that $\lim_{x\to a} f(x)$ exists and is equal to y. If not, then $\exists\ \epsilon > 0$ such that for each $n = 1, 2, ..,$ let $x_n \in B(a; \frac{1}{n}) \cap S$ for which $f(x_n) \notin B(y; \epsilon)$. This defines a sequence (x_n) such that $\lim_{n\to\infty} x_n = a, x_n \neq a$. But for this sequence it is not true that $\lim_{n\to\infty} f(x_n) = y$, which is a contradiction. Therefore $\lim_{x\to a} f(x) = y$. □

5.11.6 Problem. (**Cauchy condition for limit of functions**) Let $f : S \to \mathbb{R}$ and a is an accumulation point of S. Then $\exists$ a point $l \in \mathbb{R}$ such that $\lim_{x\to a} f(x) = l$ if and only if $\forall\ \epsilon > 0\ \exists\ \delta > 0$ such that $x, y \in \hat{B}(a; \delta) \cap S$ implies $|f(x) - f(y)| < \epsilon$.

5.11.6.1 Solution. Assume that $\lim_{x\to a} f(x) = l$ and let $\epsilon > 0$. Then $\exists\ \delta > 0$ such that $x \in \hat{B}(a; \delta) \cap S$ implies $|f(x) - l| < \epsilon/2$. If $y \in \hat{B}(a; \delta) \cap S$, then

$$|f(x) - f(y)| \leq |f(x) - l| + |l - f(y)| < \epsilon.$$

Conversely, suppose the condition holds, and let $B(a; \delta)$ be the nbhd. corresponding ϵ. Let (x_n) be a sequence with $x_n \neq a$ and $\lim_{n\to\infty} x_n = a$. Then for some $n_0, n > n_0$ implies $x_n \in \hat{B}(a; \delta)$ and therefore

$$|f(x_n) - f(x_m)| < \epsilon\ \forall\ m, n > n_0.$$

So, by Cauchy condition for sequences $\lim_{n\to\infty} f(x_n)$ exists. Part (2) of the above problem implies that $\lim_{x\to a} f(x)$ exists. □

5.11.7 Problem. Let (x_n) be a sequence in $\mathbb{R}$. If $(x_{2n}), (x_{2n+1})$ converge to the same limit $l \in \mathbb{R}$, then (x_n) converges to l.

5.11.7.1 Solution. Since $(x_{2n}), (x_{2n+1})$ both converge to l, so for every $\epsilon > 0\ \exists\ k_1, k_2 \in \mathbb{N}$ such that $x_{2n} \in B(l; \epsilon)\ \forall\ n \geq k_1$ and $x_{2n+1} \in B(l; \epsilon)\ \forall\ n \geq k_2$. Now, if $k = \max\{k_1, k_2\}$, then $x_{2k}, x_{2(k+1)}, ... \in B(l; \epsilon)$ and $x_{2k+1}, x_{2k+3}, ... \in B(l; \epsilon)$, thus, $x_n \in B(l; \epsilon)\ \forall\ n \geq 2k$. Hence (x_n) converges to l. □

5.11.8 Problem. Prove that $\mathbb{R}$ contains an open set U such that both U and its complement U^C are infinite.

5.11.8.1 Solution. Since $\mathbb{R}$ is infinite, so it contains an infinite set, say

$$S = \{x_1, x_2, x_3, ..., x_n, ...\}.$$

Let $A = \{x_1, x_3, ...\}$ and $B = \{x_2, x_4, ...\}$. Now, if $A' = \emptyset$ then A is closed and take $U = \mathbb{R} \setminus A$ that is infinite as $B \subseteq U$ and $U^C = A$ is infinite. Again, if $A' \neq \emptyset$, so let $a \in A'$ then $\exists$ a subsequence (x_{n_k}) of (x_{2n-1}) that converges to a, hence $D = \{x_{n_k}; k \in \mathbb{N}\} \cup \{a\}$ is a closed set and take $U = \mathbb{R} \setminus D$. Thus $U \supseteq B$ and $U^C = D$ are infinite. □

5.11.9 Problem. Let (x_n) be a sequence in $\mathbb{R}$ with no convergent subsequence. Prove that $\{x_n; n \in \mathbb{N}\}$ is a closed subset of $\mathbb{R}$.

5.11.9.1 Solution. Let $F = \{x_n; n \in \mathbb{N}\}$ and x is a limit point of F then $\exists$ a subsequence that converges to x contradicts our hypothesis. Hence $F' = \emptyset \subseteq F$. Thus $\{x_n; n \in \mathbb{N}\}$ is a closed subset of $\mathbb{R}$. □

5.11.10 Problem. Let E be a set and (x_n) a sequence in $\mathbb{R}$, such that x_i's are not necessarily elements of E. Suppose that $\lim_{n\to\infty} x_n = x$ and that x is an interior point of E. Show that there is an integer N so that $x_n \in E$, $\forall\ n \geq N$.

5.11.10.1 Solution. Since x is an interior point of E, so there exists an $\epsilon > 0$ such that $B(x;\epsilon) \subseteq E$ and $\exists\ m \in \mathbb{N}$ such that $n > m \Rightarrow x_n \in B(x;\epsilon)$. Thus only finite number of elements are outside of $B(x;\epsilon)$, and hence of outside of E. Let $N_0 = \max\{n; x_n \notin E\}$. Thus $n \geq N = N_0 + 1 \Rightarrow x_n \in E$. □

5.11.11 Problem. Let E be a set and (x_n) a sequence in E. Suppose that $\lim_{n\to\infty} x_n = x$ and that x is an isolated point of E. Show that there is an integer N so that $x_n = x,\ \forall\, n \geq N$.

5.11.11.1 Solution. If possible, let for all $n \in \mathbb{N},\ x_n \neq x$. Since $\lim_{n\to\infty} x_n = x$, so for every $\epsilon > 0\ \exists\ m \in \mathbb{N}$ such that $n \geq m \Rightarrow |x_n - x| < \epsilon$ implies that $n \geq m \Rightarrow \hat{B}(x;\epsilon) \cap E \neq \emptyset$ ontradicts that x is isolated point of E. Hence the conclusion follows. □

5.11.11.2 Solution. If possible, let for all $n \in \mathbb{N},\ x_n \neq x$. Since x is an isolated point of E, so there exists an $\epsilon > 0$ such that $B(x;\epsilon) \cap E = \{x\}$ and thus $x_n \in E \Rightarrow |x_n - x| \geq \epsilon\ \forall\ n \in \mathbb{N}$ contradicts that $\lim_{n\to\infty} x_n = x$. Hence the result. □

5.11.11.3 Solution. Since x is an isolated point of E, so there exists an $\epsilon > 0$ such that $B(x;\epsilon) \cap E = \{x\}$ and $\exists\ N \in \mathbb{N}$ such that $n \geq N \Rightarrow x_n \in B(x,\epsilon)$. Therefore $\{x_N, x_{N+1}, ...\} = \{x\} \Rightarrow x_N = x_{N+1} = ... = x$. Hence the conclusion follows. □

5.11.12 Problem. Let E be a set and $(x_n) \in \mathbb{R}^{\mathbb{N}}$, such that x_i's are not necessarily elements of E. Suppose that $\lim_{n\to\infty} x_n = x$ and that $x_{2n} \in E$ and $x_{2n+1} \notin E\ \forall\ n \in \mathbb{N}$. Show that x is a boundary point of E.

5.11.12.1 Solution. Now $\lim_{n\to\infty} x_n = x \Rightarrow\ \forall\ \epsilon > 0\ \exists\ N \in \mathbb{N}$ such that $n \geq N \Rightarrow x_n \in B(x,\epsilon)$. If N is even, then $N+1$ is odd, so by the question $\{x_N, x_{N+2}, ...\} \subseteq E$ and $\{x_{N+1}, x_{N+3}, ...\} \subseteq E^C$ which implies $E \cap B(x,\epsilon)$ and $E^C \cap B(x,\epsilon)$ are both non-empty. Hence x is a boundary point of E. □

5.11.13 Problem. Show that $\lim_{n\to\infty} x_n = x$ holds if and only if every subsequence of (x_n) has a subsequence that converges to x.

5.11.13.1 Solution. If $\lim_{n\to\infty} x_n = x$, then every subsequence must converge to x. So, every subsequence of a subsequence (as being itself a subsequence of (x_n) must converge to x. For the converse, assume that each subsequence of (x_n) has a subsequence that converges to x, but x_n does not converge to x. Then there exists $\epsilon > 0$ such that for any positive integer k there is $n_k > k$ satisfying $|x - x_{n_k}| > \epsilon$. So, there exists a subsequence (x_{n_k}) of (x_n) such that $|x - x_{n_k}| > \epsilon$. Such a subsequence (x_{n_k}) does not contain any subsequence converging to x, which contradicts our hypothesis. Therefore $\lim_{n\to\infty} x_n = x$. □

5.11.14 Problem. Prove that if an ordered field satisfies the completeness theorem, then the Cantor axiom holds.

5.11.14.1 Solution. Hint: Suppose that $[a_1, b_1] \supseteq [a_2, b_2] \supseteq ..$ is a descending chain of closed intervals. Show that $\sup\{a_1, a_2, ...\}$ is contained by all of the intervals.

5.11.15 Problem. Show that every convergent sequence has a minimum or a maximum.

5.11.15.1 Solution. Hint: Show that if the set $A = \{a_n; n \in \mathbb{N}\}$ has no maximum, then the sequence (a_n) has a subsequence $a_{n_k} \to \sup A$.

5.11.16 Problem. Show that $\lim_{x\to 0} x\left[\frac{1}{x}\right] = 1$, where $[.]$ stands for the integral part of the number.

5.11.16.1 Solution. Hint. $\frac{1}{x} - 1 < \left[\frac{1}{x}\right] \leq \frac{1}{x}$ and the sandwich lemma.

5.11.17 Problem. Suppose that $x_n \geq 0 \;\forall\; n \in \mathbb{N}$ and that $\lim_n (-1)^n x_n$ exists. Show that (x_n) converges.

5.11.17.1 Solution. Let $y_n = (-1)^n x_n$. Again $\lim_n y_n$ exists implies both y_{2n-1} and y_{2n} converge to the same limit. Since $y_{2n-1} < 0$ and $y_{2n} > 0$, so, $\lim_n y_n = \lim_n y_{2n-1} \leq 0$ and $\lim_n y_n = \lim_n y_{2n} \geq 0$. Thus $\lim_n (-1)^n x_n = 0$. Hence, $x_{2n-1} \to 0$ and $x_{2n} \to 0$ implies $\lim_n x_n = 0$. $\square$

5.11.18 Problem. Show that if $(x_n) \in \mathbb{R}$ is unbounded, then there exists a subsequence (x_{n_k}) such that $\lim(1/x_{n_k}) = 0$.

5.11.18.1 Solution. Suppose that (x_n) is unbounded above, so, for each $k \in \mathbb{N}$ there exists $x_{n_k} > k$, which implies implies $1/x_{n_k} < 1/k$. Thus, if $\epsilon > 0 \;\exists\; N \in \mathbb{N}$ such that $1/N < \epsilon$. Choose $k > N$, then $1/x_{n_k} < 1/k < 1/N < \epsilon$. Hence $\lim(1/x_{n_k}) = 0$. When it is unbounded below, it can be proved in a similar way. $\square$

5.11.19 Problem. Let $(a_n), (b_n)$ be sequences of positive real numbers such that $a_n > nb_n \;\forall\; n > 1$. Prove that if (a_n) is increasing and (b_n) is unbounded, then the sequence (c_n) given by $c_n = a_{n+1} - a_n$, is also unbounded.

5.11.19.1 Solution. Assume that $\exists\; M > 0$ such that $a_{n+1} - a_n < M \;\forall\; n \in \mathbb{N}$. Summing up these inequalities from 1 to n, we get

$$\begin{aligned} a_{n+1} - a_1 &< nM \\ \text{or, } \frac{a_{n+1}}{n} &< \frac{a_1}{n} + M \end{aligned}$$

shows that $\frac{a_{n+1}}{n}$ is bounded. Now $a_n > nb_n \Rightarrow a_{n+1} > (n+1)b_{n+1} \Rightarrow \frac{a_{n+1}}{n} > \frac{n+1}{n} b_{n+1}$. Since (b_n) is unbounded above, we obtained the sequence $\left(\frac{a_{n+1}}{n}\right)$ is unbounded above, a contradiction. $\square$

5.11.20 Problem. Let $0 < a < \alpha$ be a real numbers and let (x_n) be a sequence defined by

$$x_n = \begin{cases} a & \text{if } n = 1 \\ \dfrac{(\alpha+1)x_{n-1} + \alpha^2}{x_{n-1} + (\alpha+1)} & \text{if } n > 1. \end{cases}$$

Prove that the sequence is convergent and find its limit.

5.11.20.1 Solution. Note that

$$0 < x_2 = \frac{(\alpha+1)x_1 + \alpha^2}{x_1 + (\alpha+1)} < \alpha.$$

Since $0 < x_2 < \alpha$, we obtain $0 < x_3 < \alpha$ and then $0 < x_n < \alpha$ by induction on n. On the other hand

$$x_n - x_{n-1} = \frac{(\alpha+1)x_{n-1} + \alpha^2}{x_{n-1} + (\alpha+1)} - x_{n-1} = \frac{\alpha^2 - x_{n-1}^2}{x_{n-1} + (\alpha+1)} > 0,$$

therefore the sequence is increasing and bounded. It follows that the sequence is convergent and let $\lim_{n\to\infty} x_n = l$. Then

$$\begin{aligned} l &= \frac{(\alpha+1)l + \alpha^2}{l + (\alpha+1)} \\ \Rightarrow & l^2 + l(\alpha+1) = (\alpha+1)l + \alpha^2 \Rightarrow l = \alpha. \quad \square \end{aligned}$$

5.11.21 Problem. Evaluate $\lim_{n\to\infty}(n!e - [n!e])$.

5.11.21.1 Solution. Let $S_n = \sum_{n=1}^{\infty} \frac{1}{n}$. Then, we have $0 < n!e - n!S_n < 1/n$. We conclude that $[n!e] = n!S_n$ and therefore $n!e - [n!e] \to 0$. □

5.11.22 Problem. Show that $\lim_{n\to\infty} n\sin(2\pi en!) = 2\pi$.

5.11.22.1 Solution. Since $\lim_{n\to\infty}(n!e - [n!e]) = 0$, we have

$$\lim_{n\to\infty} \frac{\sin(2n!e - 2[n!e])}{(2n!e - 2[n!e])} = 1 \Rightarrow \lim_{n\to\infty} \frac{\sin(2n!e)}{(n!e - [n!e])} = 2\pi.$$

Note that by the Maclaurin series expansion of e implies $1/(n+1) < en! - [en!] < 1/n$, and so $\lim_{n\to\infty} n(en! - [en!]) = 1$. It follows that

$$n\sin(2\pi en!) = n(en! - [en!])\frac{\sin(2\pi en!)}{en! - [en!]} \to 2\pi. \quad \square$$

5.11.22.2 Solution. Show that, $\theta_n = (e - b_n)n!$, where $b_n = 1 + \frac{1}{1!} + + \frac{1}{2!} + \ldots.. + \frac{1}{n!}$ and $0 < \theta_n < 1$. Therefore

$$\sin(2\pi en!) = \sin[2\pi(\theta_n + b_n n!)] = \sin(2\pi\theta_n),$$

because $b_n n!$ is an integer. Furthermore $\theta_n < \frac{1}{n+1}$, so $\theta_n \to 0$ As a consequence, $\lim_{\theta_n\to 0} \frac{\sin(2\pi\theta_n)}{(2\pi\theta_n)} = 1$. Finally

$$\theta_n = (e - b_n)n! > (b_{n+1} - b_n)n! = \frac{1}{(n+1)!}n! = \frac{1}{n+1}, \text{so}$$

$$\frac{n}{n+1} \le n\theta_n \le 1,$$

and we conclude

$$n\sin(2\pi en!) = n\frac{\sin(2\pi\theta_n)}{2\pi\theta_n}(2\pi\theta_n) \to 2\pi. \quad \square$$

5.11.23 Problem. Compute

$$\lim_{n\to\infty} \left|\sin\left(\pi\sqrt{n^2+n+1}\right)\right|.$$

5.11.23.1 Solution. The function $|\sin x|$ is periodic with period π. Hence

$$\lim_{n\to\infty} \left|\sin\left(\pi\sqrt{n^2+n+1}\right)\right| = \lim_{n\to\infty} \left|\sin\left(\pi\sqrt{n^2+n+1} - n\pi\right)\right|$$

But

$$\begin{aligned}
\lim_{n\to\infty} \left(\sqrt{n^2+n+1} - n\right) &= \lim_{n\to\infty} \frac{n^2+n+1-n^2}{\sqrt{n^2+n+1}+n} \\
&= \lim_{n\to\infty} \frac{n+1}{\sqrt{n^2+n+1}+n} \\
&= \lim_{n\to\infty} \frac{1+\frac{1}{n}}{\sqrt{1+\frac{1}{n}+\frac{1}{n^2}}+1} \\
&= \frac{1}{2}
\end{aligned}$$

It follows that the limit is equal to $\left|\sin\frac{\pi}{2}\right|$, which is 1. □

5.11.24 Problem. Give an example of a sequence that contains subsequences converging to every number in [0,1] (and no other numbers).

5.11.24.1 Solution. The set $[0,1] \cap \mathbb{Q}$. $\square$

5.11.25 Problem. Show that there cannot exist a sequence that contains subsequences converging to every number in (0,1) and no other numbers.

5.11.25.1 Solution. Set of all subsequential limits is closed. $\square$

5.11.26 Problem (General Principle of Convergence:)**.** Let f be a function with domain $[a,\infty)$. Then $\lim_{x\to\infty} f(x)$ exists iff, for every $\epsilon > 0$ there exists $M \in [a,\infty)$ such that $|f(x) - f(y)| < \epsilon \; \forall \; x, y > M$.

5.11.26.1 Solution. Suppose first that $\lim_{x\to\infty} f(x)$ exists and equals L. Let $\epsilon > 0$ be given. Then there exists M such that $|f(x) - L| < \epsilon/2 \; \forall \; x, y > M$. Hence, for all $x, y > M$, $|f(x) - f(y)| \leq |f(x) - L| + |L - f(y)| < \epsilon$.
Conversely, suppose that, for all $\epsilon > 0$ there exists M such that If $|(x) - f(y)| < \epsilon$ for all $x, y > M$. Then the sequence $(f(n))$ is a Cauchy sequence, and so, it has a limit L. Let $\epsilon > 0$ be given. There exists K_1 such that $|f(n) - L| < \epsilon/2$ for all integers $n > K_1$. Also, by the property we are assuming, there exists K_2 such that If $|f(x) - f(y)| < \epsilon/2$ for all $x, y > K_2$. Let $K = \max\{K_1, K_2\}$ and let n be an integer greater than K. Then, for all $x > K$,

$$\begin{aligned} |f(x) - L| &= |f(x) - f(n) + |f(n) - L| \text{ (where } n \in \mathbb{N} \text{ and } n > K) \\ &\leq |f(x) - f(n)| + |f(n) - L| < \epsilon, \end{aligned}$$

and so $\lim_{x\to\infty} f(x) = L$. $\square$

5.11.27 Problem. Distinguish between decimal representation of rational and irrational number. Hence show that $\frac{1}{4}$ is an accumulation point of $\mathbb{R}$.

5.11.27.1 Solution. The decimal representation of a rational number is either terminating or recurring and the decimal representation of an irrational number is non-terminating and non-recurring. For the last part, we shall use the fact that

$$1 = .99999.....\text{and } .999..9(n\text{-times}) = 1 - 10^{-n}$$

Since $\frac{1}{4} = .25 = .24 + .0099999.....$, we show that the sequence $.249, .2499, .24999, .., .25 - 10^{-n-2}, ..$ converges to $\frac{1}{4}$. Let $\epsilon > 0$ then $\exists \; p \in \mathbb{N}$ such that $10^{-p-2} < \epsilon$ and
$\big|.25 - (.25 - 10^{-n-2})\big| = 10^{-p-2} < \epsilon \; \forall \; n \geq p$. Thus we conclude that $(.25 - 10^{-n-2})$ converges to .25. $\square$

5.11.28 Problem. Let $x_n = \frac{1}{1^2} + \frac{1}{2^2} + ... + \frac{1}{n^2}$.

1. Show that for all integers $n \geq 1$, $x_n < 2 - \frac{1}{n} < 2$.
2. Conclude that the sequence (x_n) converges.
3. Show that for an integer n large enough $n^2 < 2^n$.

5.11.28.1 Solution.

1. Since $\frac{1}{n} < \frac{1}{n-1} \Rightarrow \frac{1}{n^2} < \frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n}\ \forall\, n > 1$. Thus,
$$\begin{aligned}\frac{1}{2^2}+\frac{1}{3^2}+...+\frac{1}{n^2} &< \left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+...+\left(\frac{1}{n-1}-\frac{1}{n}\right)\\ &< 1-\frac{1}{n}\\ \Rightarrow \frac{1}{1^2}+\frac{1}{2^2}+...+\frac{1}{n^2} &< 2-\frac{1}{n} \Rightarrow x_n < 2-\frac{1}{n} < 2.\end{aligned}$$
2. By monotone convergence theorem.
3. Use induction. □

5.11.29 Problem. Let $0 < y_1 < x_1$, and
$$x_{n+1} = \frac{x_n + y_n}{2};\ y_{n+1} = \frac{2}{\frac{1}{x_n}+\frac{1}{y_n}}$$
then show that the sequences (x_n) and (y_n) converge to the same limit $\sqrt{x_1 y_1}$.

5.11.29.1 Solution. Since $HM \le AM$, so $y_2 < x_2$. So, $y_1 < y_2 < x_2 < x_1$, and in this way we have $y_{n-1} < y_n < x_n < x_{n-1}$ for $n = 1, 2, ...$ Thus the sequence (x_n) is decreasing and bounded below by y_1 and (y_n) is increasing and bounded above by x_1, hence both sequences converge. Suppose $x_n \to l$ and $y_n \to m$ as $n \to \infty$. Then $l = \frac{1}{2}(l+m) \Rightarrow l = m$. Again, from the above relations, we get $x_{n+1}y_{n+1} = x_n y_n = ... = x_2 y_2 = x_1 y_1$. Hence $\lim_{n\to\infty} x_n y_n = \lim_{n\to\infty} x_n \lim_{n\to\infty} y_n = l^2 = x_1 y_1 \Rightarrow l = \sqrt{x_1 y_1}$. □

5.11.30 Problem. Consider the sequence defined by the following procedure:
step 1: The first 3 terms are $-1, 0, 1$ (we go from -1 to 1 in steps of 1).
step 2: The next 7 terms are $-2, -3/2, -1, -1/2,\ 0,\ 1/2,\ 1,\ 3/2,\ 2$ (we go from -2 to 2 in steps of 1/2).
The next k_n terms are $-n, -n+1/n, ..., -1/n, 0, 1/n, ..., n-1/n, n$ (we go from $-n$ to n in steps of $1/n$).
Find the number of terms k_n in the n-th block. Show that the sequence takes every rational value infinitely many times. Find also an explicit example of a sequence which takes each rational value exactly once.

5.11.30.1 Solution. Hint. For instance 5/7 occurs in the 7-th, 14-th, $7k$-th block of terms, while 20+5/7 occurs in the $7k$-th block when $7k \ge 21$. Generally a fraction p/q in lowest terms occurs in the qk-th blocks when $qk > p/q$. □

5.11.31 Problem (The Collatz Problem). Define a function $\phi : \mathbb{N} \to \mathbb{N}$ by
$$\phi(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,}\\ 3n+1 & \text{if } n \text{ is odd.}\end{cases}$$
Show that for initial values ≤ 30 say, repeated application of ϕ comes back eventually to 1. Does this happen for all initial values?

5.11.31.1 Solution. Unknown. □

5.11.32 Problem. Let

$$x_n = \frac{an+b}{cn+d}, \quad a,b,c,d > 0.$$

What conditions on a,b,c,d are needed to make the sequence (x_n) increasing, or decreasing?

5.11.32.1 Solution.

$$x_{n+1} - x_n = \frac{ad-bc}{(cn+d)(c(n+1)+d)}$$

so the sequence is increasing if and only if $ad - bc > 0$. □

5.11.33 Problem. Show that, if (n_k) is a strictly increasing sequence of positive integers then $n_k \geq k$ for all k.

5.11.33.1 Solution. Here, (n_k) is a strictly increasing sequence of positive integers means the function $n : \mathbb{N} \to \mathbb{N}$ defined by $n(k) = n_k,\ k \in \mathbb{N}$ is a strictly increasing function. So, either $n_1 = 1$ or $n_1 > 1$ implies $n_1 \geq 1$. If $n_1 = 1$ then $n_2 = 2$ or $n_2 > 2$. Thus $n_2 \geq 2$. Let $n_p \geq p$. Then $n_p = p \Rightarrow n_{p+1} \geq p+1$, thus we have completed our induction argument. Hence the result follows. □

5.11.34 Problem. Let (a_n) be any sequence with $a_n > 0$ for all $n \in \mathbb{N}$. Show that

$$\limsup_n \left(\frac{a_1 + a_{n+1}}{a_n}\right)^n \geq e = \lim_n \left(1 + \frac{1}{n}\right)^n$$

5.11.34.1 Solution. If the result is false, then there is some $N \in \mathbb{N}$ such that for all $n \geq N$

$$\left(\frac{a_1 + a_{n+1}}{a_n}\right)^n < \left(1 + \frac{1}{n}\right)^n$$

equivalently

$$\frac{a_1}{n+1} < \frac{a_n}{n} - \frac{a_{n+1}}{n+1} \quad \text{for } n \geq N$$

Adding these inequalities gives

$$a_1 \left(\frac{1}{N+1} + ... + \frac{1}{n}\right) < \frac{a_N}{N} - \frac{a_{n+1}}{n+1} \quad \text{for } n \geq N$$

which is impossible since it gives an upper bound for $\left(\frac{1}{N+1} + ... + \frac{1}{n}\right)$ which we know tends to infinity. □

5.11.35 Problem. If $f(x) < g(x)$ for all $x > 0$ and both $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} g(x)$ exist, then $\lim_{x\to\infty} f(x) < \lim_{x\to\infty} g(x)$. True or false?

5.11.35.1 Solution. False. Counterexample: For the functions

$$f(x) = -\frac{1}{x},\ g(x) = \frac{1}{x}$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = 0. \quad \square$$

5.11.36 Problem. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the two conditions:

1. $f(x+y) + f(x-y) = 2f(x)f(y) \; \forall \; x, y \in \mathbb{R}$.
2. $\lim_{x\to\infty} f(x) = 0$.

5.11.36.1 Solution. Let $a \in \mathbb{R}$ and let $y = x - a$. By (1), $f(2x-a) + f(a) = 2f(x)f(x-a)$. Now, when $x \to \infty$, $f(2x-a), f(x), f(x-a) \to 0$. So $f(a) = 0$. $\square$

5.11.37 Problem. The following definitions of a non-vertical asymptote are equivalent: True or false?

1. The straight line $y = mx + c$ is called a non-vertical asymptote to a curve $y = f(x)$ as x tends to infinity if $\lim_{x\to\infty}(f(x) - (mx+c)) = 0$.
2. A straight line is called a non-vertical asymptote to a curve as x tends to infinity if the curve gets closer and closer to the straight line (as close as we like) as x tends to infinity, but does not touch or cross it.

5.11.37.1 Solution. False. As x tends to infinity the function $y = \sin x/x$ gets closer to the x-axis from above and below and $\lim_{x\to\infty} (\sin x/x)$. According to the first definition the x-axis is the non-vertical asymptote of the function $y = \sin x/x$, but its graph crosses the x-axis infinitely many times, so the definitions (1) and (2) are not equivalent.
Comment: The correct definition is (1). A function's graph can touch or cross a non-vertical asymptote. $\square$

5.11.38 Problem. The tangent line to a curve at a certain point that touches the curve at infinitely many other points cannot be a non-vertical asymptote to this curve. True or false?

5.11.38.1 Solution. False. The tangent line $y = 0$ to the curve $y = (\sin^2 x)/x$ at $x = \pi$ touches the curve at infinitely many other points and is a non-vertical asymptote to this curve. $\square$

5.11.39 Problem. The following definitions of a vertical asymptote are equivalent: True or false?

1. The straight line $x = a$ is called a vertical asymptote to a curve $y = f(x)$, if $\lim_{x\to a+} f(x) = \pm\infty$ or $\lim_{x\to a-} f(x) = \pm\infty$.
2. The straight line $x = a$ is called a vertical asymptote for the curve $y = f(x)$, if there are infinitely many values of $f(x)$ that can be made arbitrarily large or arbitrarily small as x gets closer to a from either side of a.

5.11.39.1 Solution. There are infinitely many values of the function f defined by

$$f(x) = \frac{1}{x}\sin\frac{1}{x}$$

that can be made arbitrarily large or small as x gets closer to 0, but the straight line $x = 0$ is not a vertical asymptote of this curve. The correct definition is (1). $\square$

5.11.40 Problem. If $\lim_{x\to a} f(x)$ exists and $\lim_{x\to a} g(x)$ does not exist because of oscillation of $g(x)$ near $x = a$, then $\lim_{x\to a} (f(x)g(x))$ does not exist. True or false?

5.11.40.1 Solution. False. Counterexample: For the function $f(x) = x, \lim_{x\to 0} x = 0$ and for the function $g(x) = \sin 1/x$ the limit $\lim_{x\to 0} \sin 1/x$ does not exist because of oscillation of $g(x)$ near $x = 0$, but

$$\lim_{x\to 0} (f(x)g(x)) = \lim_{x\to 0} \left(x \sin \frac{1}{x}\right) = 0. \quad \square$$

5.11.41 Problem. If a function f is not bounded in any neighborhood of the point $x = a$, then either $\lim_{x\to a+} |f(x)| = \infty$ or $\lim_{x\to a-} |f(x)| = \infty$. True or false?

5.11.41.1 Solution. False. Counterexample: The function f defined by

$$f(x) = \frac{1}{x} \cos \frac{1}{x}$$

is not bounded in any neighborhood of the point $x = 0$, but neither

$$\lim_{x\to 0+} \left|\frac{1}{x} \cos \frac{1}{x}\right| \text{ nor } \lim_{x\to 0-} \left|\frac{1}{x} \cos \frac{1}{x}\right|$$

exists. $\square$

5.11.42 Problem. If $f : \mathbb{R} \to \mathbb{R}$, then there is at least one point where $\lim_{x\to a} f(x)$ exists. True or false?

5.11.42.1 Solution. Consider the Dirichlet's function $\mathcal{D} : \mathbb{R} \to \mathbb{R}$ defined by

$$\mathcal{D}(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \in \mathbb{Q}^C. \end{cases}$$

It is easy to show that $\lim_{x\to a} \mathfrak{D}(x)$ does not exist for any real number a. In fact, in any deleted neighborhood of an arbitrary point a there exist both rational and irrational points. Therefore, the number 1 cannot be the limit of $\mathfrak{D}(x)$ at a, because for $\epsilon = 1$ and any small δ there exist points x (irrational ones) such that $0 < |x - a| < \delta$, but $|f(x) - 1| = |0 - 1| \geq \epsilon$. Similarly, the number 0 cannot be the limit, because in any δ-nbhd. of a there exist points x (rational ones) such that $0 < |x - a| < \delta$, but $|f(x) - 0| = |1 - 0| \geq \epsilon$. Finally, any real number $A \neq 0$ cannot be the limit of $\mathfrak{D}(x)$, because for $\epsilon = |A|$ in any δ-nbhd. of a there exist points x (irrational ones) such that $0 < |x - a| < \delta$,, but $|f(x) - A| = |0 - A| \geq \epsilon$. Hence, the definition of limit is not satisfied at arbitrary point a, whatever the value of A. $\square$

5.11.43 Problem. If $\lim_{x\to a} f(x)$ does not exist then $\lim_{x\to a} |f(x)|$ does not exist. True or false?

5.11.43.1 Solution. False. Consider the function $D : \mathbb{R} \to \mathbb{R}$ defined by

$$D(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ -1, & \text{if } x \in \mathbb{Q}^C. \end{cases}$$

Evidently, D has no limit at any point just like $\mathfrak{D}$. However, $|D(x)| = 1$ and this function has the limit 1 at any point. $\square$

5.11.44 Problem. If a sequence (y_n) converges to A and a function $f : \mathbb{R} \to \mathbb{R}$ is such that $f(n) = y_n$, $\forall\, n \in \mathbb{N}$, then $\lim_{x\to\infty} f(x) = A$. True or false?

5.11.44.1 Solution. False. Consider the function f defined by $f(x) = \sin \pi x$, and $f(n) = 0 \; \forall \; n \in \mathbb{N}$. The corresponding sequence $y_n = f(n) = \sin n\pi = 0$ converges to 0, but the function f has no limit as $x \to \infty$. □

5.11.45 Problem. If there exists a sequence x_n such that $x_n \to a$ and $\lim_{x_n \to a} f(x_n) = A$, then $\lim_{x \to a} f(x) = A$. True or false?

5.11.45.1 Solution. Consider the Dirichlet's function $\mathfrak{D} : \mathbb{R} \to \mathbb{R}$ defined by

$$\mathfrak{D}(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \in \mathbb{Q}^C. \end{cases}$$

Consider a sequence (x_n) of rationals (say $x_n = 1/n$) that converges to 0, and $\lim_{x_n \to 0} \mathfrak{D}(x_n) = 1$, however $\lim_{x \to 0} \mathfrak{D}(x)$ does not exist. □

5.11.46 Problem. The definition of the limit can be reformulated as follows: the limit of $f(x)$ as x approaches a is A if for every $\epsilon > 0$, there is a $\delta > 0$ such that if $|f(x) - A| < \epsilon$ then $|x - a| < \delta$. True or false?

5.11.46.1 Solution. At first glance, the above "definition" conveys an impression that it has at least some relation to the correct definition, because the inequalities of the correct definition, meaning proximity among function values and among argument values, are involved here. Nevertheless, this statement has nothing to do with the correct definition. In fact, if $f(x)$ is defined on a bounded interval, say (a, b), then the above statement does not imply any restriction on $f(x)$, because by choosing $\delta = b - a$ the required inequality $|x - a| < \delta$ is satisfied for an arbitrary $x \in (a, b)$ and for an arbitrary $\epsilon > 0$. On the other hand, if $f(x)$ is defined on an unbounded domain, say on $\mathbb{R}$, then the above "definition" also fails. In fact, let us consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} e^x, & \text{if } x \in \mathbb{Q} \\ -e^x, & \text{if } x \in \mathbb{Q}^C. \end{cases}$$

Let us choose any $A > 0$ and consider ϵ-nbhd. of $A : |y - A| < \epsilon, \epsilon \leq A/2$. The corresponding δ-nbhd. of the point $a = \ln A$ is defined as $|x - a| < \delta$ with $\delta = \max\{\ln(A+\epsilon) - a, a - \ln(A-\epsilon)\} = a - \ln(A-\epsilon)$. It can be checked now that for all the values of $f(x)$ that satisfied the condition $|f(x) - A| < \epsilon$, it follows that $|x - a| < \delta$ (more precisely, $x \in (\ln(A - \epsilon), \ln(A + \epsilon)) \subseteq (a - \delta, a + \delta)$). However, $f(x)$ has no limit at a, because for any $x \in \mathbb{Q}^C$ (in particular for $x \in \mathbb{Q}^C$ in the neighborhood of $a = \ln A$) one obtains $|f(x) - A| = |e^x - A| > A$. Remark: For an unbounded domain, say $\mathbb{R}$, the above "definition" can also be too restrictive. For example, if $f(x) = \sin x, A = 0$ and $\epsilon = 3$, then the inequality $|x - a| < \delta$ holds for an arbitrary x (and a) whatever δ is. It means that the implication of the "definition" is not true even for some continuous functions (the concept of continuity will be considered in the next chapter). □

5.11.47 Problem. The definition of the limit can be reformulated as follows: the limit of $f(x)$ as x approaches a is A if for every $\delta > 0$, there is an $\epsilon > 0$ such that if $|f(x) - A| < \epsilon$ then $|x - a| < \delta$. True or false?

5.11.47.1 Solution. Actually this condition means that there exists a partial limit A of $f(x)$ as x approaches a (that is, there is a sequence of points x_k such that $\lim_{x_k \to a} f(x_k) = A$). However, it is

not sufficient to guarantee the existence of the general limit. For example, if

$$f(x) = \begin{cases} x^2, & \text{if } x \in \mathbb{Q} \\ 1, & \text{if } x \in \mathbb{Q}^C. \end{cases}$$

Let $a = 0$ and $A = 0$, then by choosing $\epsilon = \delta^2$ for any $\delta < 1$ one guarantees that for all $f(x)$ such that $|f(x)| = |x^2| < \epsilon$ we obtain $|x| < \delta$ for the corresponding values of x. However, the general limit does not exist, because there are two different partial limits: $\lim_{x\to 0, x\in\mathbb{Q}} f(x) = 0$ and $\lim_{x\to 0, x\in\mathbb{Q}^C} f(x) = 1$. $\square$

5.11.1 Remark. If we assume that the inverse function exists, then the above statement is the definition of the limit (equal to a) of the inverse function $x = f^{-1}(y)$ when the argument y approaches A.

5.11.48 Problem. The definition of the limit can be reformulated as follows: the limit of $f(x)$ as x approaches a is A if for every $\delta > 0$, there is an $\epsilon > 0$ such that if $|x - a| < \delta$ then $|f(x) - A| < \epsilon$. True or false?

5.11.48.1 Solution. The above "definition" seems to be rather natural and tempting, because it apparently corresponds to a vague idea about limits: when x approaches a the function values $f(x)$ approach A. However, this similarity is only appearing, because the inversion of the dependence between δ and ϵ (in comparison with the correct definition) leads to possibility to attribute to ϵ arbitrary large values, that destroys the requirement that $f(x)$ should approach A. For example, Dirichlet's function $\mathfrak{D}$, which actually does not have a limit at any point, satisfies the above "definition" with an arbitrary a and $A = 0$ if one chooses $\epsilon = 2$ for every $\delta > 0$. $\square$

5.11.2 Remark. The condition of the statement implies boundedness of $f(x)$ in any deleted neighborhood of the point a.

5.11.49 Problem. If for a specific function $f(x)$ and point a the dependence of δ from ϵ in the definition of limit cannot be expressed in the form $\delta = c\epsilon$, where c is a positive constant, then $f(x)$ does not have a limit at a. True or false?

5.11.49.1 Solution. For $f(x) = \sqrt[3]{x}$ considered in a neighborhood of the point $a = 0$, if $\delta = \epsilon$ then the limit definition is satisfied: $\lim_{x\to 0} \sqrt[3]{x} = 0$. However, there is no constant $c > 0$ such that $\delta = c\epsilon$. Indeed, if $0 < |x| < \delta = c\epsilon$, then we can obtain $|\sqrt[3]{x}| < \sqrt[3]{c\epsilon}$, but for any fixed c the expression in the right-hand side is greater than ϵ, if ϵ is sufficiently small. Therefore, we will not obtain the required evaluation $|\sqrt[3]{x}| < \epsilon$ if $\delta = c\epsilon$. $\square$

5.11.3 Remark. Of course, similar statements with requirement of any other specific law $\delta(\epsilon)$ are also false.

5.11.50 Problem. If for a specific function $f(x)$ and point a there are two different ways to determine $\delta(\epsilon)$ in the definition of limit, then $f(x)$ does not have a limit at a. True or false?

5.11.50.1 Solution. Actually, if the limit definition is satisfied, then always there are infinitely many ways to determine $\delta(\epsilon)$. For example, if $f(x) = x$ and $a = 0$ then an evident choice to satisfy the definition of $\lim_{x\to 0} x = 0$ is $\delta = \epsilon$. But it means that for any constant $0 < c < 1$ the law $\delta = c\epsilon$ is also suitable, as well as $\delta = \min\{\epsilon, \epsilon^2\}$, or $\delta = \min\{\epsilon, \epsilon^3\}$ more exotic $\delta = \ln(1+\epsilon)$ and $\delta = 1 - e^{-\epsilon}$, or many others. $\square$

5.11.51 Problem. If a function f is continuous for all real x and $\lim_{n\to\infty} f(n) = A$ for natural numbers n, then $\lim_{x\to\infty} f(x) = A$. True or false? But the converse is true.

5.11.51.1 Solution. False. Counterexample: For the continuous function $f(x) = \cos(2\pi x)$ the limit $\lim_{n\to\infty}\cos(2n\pi) = 1$ but $\lim_{x\to\infty}\cos(2\pi x)$ does not exist. But

$$\lim_{x\to\infty} f(x) = A \Rightarrow \lim_{n\to\infty} f(n) = A.$$

Here, $\lim_{x\to\infty} f(x) = A$ implies for $\forall\ \epsilon > 0\ \exists\ M > 0$ such that

$$x > M \Rightarrow |f(x) - A| < \epsilon.$$

Let $n_1 \in \mathbb{N}$ such that $n_1 > M$, then $n > n_1 \Rightarrow |f(n) - A| < \epsilon$, shows that $f(n) \to A$. □

5.11.52 Problem. If there exists a sequence (x_n) such that $\lim_{x_n\to a} f(x_n) = A$, then $\lim_{x\to a} f(x) = A$. True or false?

5.11.52.1 Solution. Let us consider Dirichlet's function $\mathfrak{D}$, defined by

$$\mathfrak{D}(x) = \begin{cases} 1, \text{if}\, x \in \mathbb{Q} \\ 0, \text{if}\, x \in \mathbb{Q}^C \end{cases}$$

and $a = 0$. Using the rational points in the form $x_n = \frac{1}{n}; n \in \mathbb{N}$ we form the sequence such that $x_n \to 0$ when $n \to \infty$. Calculating the corresponding limit we have $\lim_{x_n\to 0}\mathfrak{D}(x_n) = \lim_{x_n\to 0} 1 = 1$. However, a general limit does not exist. □

5.11.53 Problem. Suppose a sequence (x_n) of real numbers satisfies

$$4x_{n+1} = x_n^3\ \forall\ n \in \mathbb{N}.$$

For what values of x_1 does the sequence (x_n) converge? For each such x_1 what is $\lim x_n$?

5.11.53.1 Solution. If $l = \lim_{n\to\infty} x_n$ exists, then taking limits as $n \to \infty$ both sides of the expression $4x_{n+1} = x_n^3$ yields $4l = l^3$ That is,

$$l^3 - 4l = l(l+2)(l-2) = 0.$$

Thus if l exists, it must be -2, 0 or 2. Next notice that

$$\begin{aligned} x_{n+1} - x_n &= \frac{1}{4}x_n^3 - x_n \\ &= \frac{1}{4}x_n(x_n+2)(x_n-2). \end{aligned}$$

Therefore

$$x_{n+1} > x_n \text{ if } x_n \in (-2,0) \cup (2,\infty) \tag{A}$$
$$\text{and } x_{n+1} < x_n \text{ if } x_n \in (-\infty,-2) \cup (0,2). \tag{B}$$

Now consider the seven cases: $x_1 < -2, x_1 = -2, -2 < x_1 < 0, x_1 = 0, 0 < x_1 < 2, x_1 = 2$, and $x_1 > 2$. In case $x_1 < -2$, for example, show that $x_n < -2$ for all n. Use this to show that the

sequence (x_n) is decreasing and that it has no limit. The other cases can be treated in a similar fashion. Three of these are trivial: if $x_1 = -2$, 0, or 2, then the resulting sequence is constant (therefore certainly convergent).
Next suppose $x_1 < -2$. Then $x_n < -2$ for every n. [The verification is an easy induction: If $x_n < -2$, then $x_n^3 < -8$, so $x_{n+1} = \frac{1}{4}x_n^3 < -2$]. From this and (B) we see that $x_{n+1} < x_n$ for every n. That is, the sequence (x_n) decreases. Since the only possible limits are -2, 0, and 2, the sequence cannot converge. (It must, in fact, be unbounded.)
The case $x_1 > 2$ is similar. We see easily that $x_n > 2$ for all n and therefore [by (A)] the sequence (x_n) is increasing. Thus it diverges (and is unbounded).
If $-2 < x_1 < 0$, then $-2 < x_n < 0$ for every n. [Again an easy inductive proof: If $-2 < x_n < 0$, then $-8 < x_n^3 < 0$; so $-2 < (1/4)x_n^3 = x_{n+1} < 0$.] From (A) we conclude that (x_n) is increasing. Being bounded above it must converge to some real number l. The only available candidate is $l = 0$.
Similarly, if $0 < x_1 < 2$, then $0 < x_n < 2$ for all n and (x_n) is decreasing. Again the limit is $l = 0$.
We have shown that the sequence (x_n) converges if and only if $x_1 \in [-2, 2]$. If $x_1 \in (-2, 2)$, then $\lim x_n = 0$, if $x_1 = -2$, then $\lim x_n = -2$ and if $x_1 = 2$, then $\lim x_n = 2$. □

5.11.54 Problem. If $Y = (y_p)$ is a subsequence of a sequence $X = (x_n)$ and $Z = (z_k)$ is a subsequence of $Y = (y_p)$ then show that $Z = (z_k)$ is a subsequence of $X = (x_n)$ where $y_p = x_{n_p}$ and $z_k = x_{n_{p_k}} \forall\ k \in \mathbb{N}$, for some increaing functions $n, p : \mathbb{N} \to \mathbb{N}$.

5.11.54.1 Solution. Since (y_p) is a subsequence of (x_n), then n an increasing function $n : \mathbb{N} \to \mathbb{N}$ such that $Y = X \circ n$ and $y_p = Y(p) = (X \circ n)(p) = X(n(p)) = x_{n_p}$ Since (z_k) is a subsequence of (y_p), then p an increasing function $p : \mathbb{N} \to \mathbb{N}$ such that $Z = Y \circ p = (X \circ n) \circ p$ and $z_k = Z(k) = (Y \circ p)(k) = ((X \circ n) \circ p)(k) = (X \circ (n \circ p))(k) = x_{n_{p_k}}$. □

5.11.55 Problem. Suppose $f : \mathbb{R} \to \mathbb{R}$, and (x_n) is a sequence in $\mathbb{R}$, such that $x_{n+1} = f(x_n)$. Then show that

1. f is increasing implies that (x_n) is monotone.
2. f is decreasing implies that $(x_{2n-1}), (x_{2n})$ are monotonic sequences of which one is increasing and the other is decreasing.

5.11.55.1 Solution.

1. Suppose f is increasing, then $m \geq n \Rightarrow f_{|\mathbb{N}}(m) \geq f_{|\mathbb{N}}(n) \Rightarrow x_m \geq x_n$, and $m \leq n \Rightarrow f|_{\mathbb{N}}(m) \leq f|_{\mathbb{N}}(n) \Rightarrow x_m \leq x_n$. Thus f is monotone.
2. Suppose f is decreasing, then $m \geq n \Rightarrow f(m) \leq f(n) \Rightarrow f(f(m)) \geq f(f(n))$. Thus if $f(f(m)) = g(m)$, then g is increasing, so by (i), the sequence $(g(x_n))$ is monotonic. Now $g(x_n) = f(f(x_n)) = f(x_{n+1}) = x_{n+2}$. Thus x_{2n-1} and x_{2n} are monotonic. If $x_1 \leq x_3$ then $f(x_1) \geq f(x_3) \Rightarrow x_2 \geq x_4$ and if $x_1 \geq x_3$ then $f(x_1) \leq f(x_3) \Rightarrow x_2 \leq x_4$, and so on. □

5.11.56 Problem. The sequence (a_n) is monotone and it has a convergent subsequence. Does it imply that (a_n) is convergent?

5.11.56.1 Solution. Suppose that we have a subsequence $a_{n_k} \to a$. Now because of the monotonicity $\forall\ n > n_k \Rightarrow |a_n - a| \leq |a_{n_k} - a|$, therefore $a_n \to a$. □

5.11.57 Problem. If at least one of the following conditions is fulfilled:

1. the set $\{n; x_n = l\}$ is infinite;

2. for every $\epsilon > 0$, the set $(l-\epsilon, l+\epsilon) \cap \{x_n; n \in \mathbb{N}\}$ is infinite.

Then prove that the point l is a cluster point for the sequence (x_n).

5.11.57.1 Solution. If the set $M_1 = \{n; x_n = l\}$ is infinite, then its elements can be ordered into a monotonically increasing sequence which diverges to ∞. So we can put $x_{n_k} = l$ for $k \in \mathbb{N}$ and $n_1 < n_2 < .. < n_k < ...$ This means that for every $m \in \mathbb{N}$ there exist $n_k \in M_1$, such that $n_k > m$. Then, for arbitrary $\epsilon > 0$, we have

$$|x_{n_k} - l| = |l - l| = 0 < \epsilon.$$

We obtain that from condition (1) it follows that l is a cluster point for the sequence (x_n).
Again, if for every $\epsilon > 0$, the set $(l-\epsilon, l+\epsilon) \cap \{x_n; n \in \mathbb{N}\}$ is infinite, then the set $M_2 = \{n; |x_n - l| < \epsilon\}$ can be ordered into a monotonically increasing sequence $n_1 < n_2 < .. < n_k < ....$ This means that, for every $m \in \mathbb{N}$, there exists $k \in \mathbb{N}$, such that $n_k \in M_2$ and $n_k > m$. So for arbitrary $\epsilon > 0$ we have

$$|x_{n_k} - l| < \epsilon.$$

We obtain that from condition (2) it follows that l is a cluster point for the sequence (x_n). □

5.11.57.2 Solution. Let us suppose that l is a cluster point for the sequence (x_n) and that the condition (l) is not fulfilled (this means that the sequence (x_n) has no stationary subsequence x_{n_k} with the property $x_{n_k} = l$, for every $n \in \mathbb{N}$. We have to show that then condition (2) is fulfilled. Let us suppose that this is not true, namely that there exists an $\epsilon > 0$ such that the set $(l-\epsilon, l+\epsilon) \cap \{x_n; n \in \mathbb{N}\}$ is finite. If n_1 is the smallest natural number such that $n > n_1 \Rightarrow x_n \neq l$, then the supposition means that the finite set $(l-\epsilon, l+\epsilon) \cap \{x_n; n \in \mathbb{N}\}$ is either empty, or there exists a natural number $n_2 > n_1$ such that for every $n > n_2 \Rightarrow x_n \notin (l-\epsilon, l+\epsilon)$. Hence this means that l is not a cluster point for the sequence (x_n), a contradiction. □

5.11.4 Note. A real number l is a cluster point of the sequence (x_n) iff either the sequence (x_n) has a stationary subsequence whose every element is equal to l, or every interval containing l has an infinite number of terms of the sequence (x_n) (or both).

5.11.58 Problem. Let us suppose that the sequence (x_n) has two cluster points a and b and assume $\liminf x_n = -\infty$ and $\limsup x_n = \infty$. Prove that then there exist subsequences $(n_l), (n_k), (n_p)$ and (n_q) of the set $\mathbb{N}$ such that

$$\lim_{l\to\infty} x_{n_l} = -\infty, \ \lim_{k\to\infty} x_{n_k} = a,$$
$$\lim_{p\to\infty} x_{n_p} = b, \ \lim_{q\to\infty} x_{n_q} = \infty.$$

5.11.58.1 Solution. We shall construct only two subsequences of (x_n) converging to a and ∞ respectively. First let

$$M = \{n \in \mathbb{N}; x_n = a\}.$$

If the set M is infinite, then it can be written in a unique way as an increasing sequence (n_k) which diverges to ∞. Clearly, (x_{n_k}) is a stationary subsequence of (x_n) which converges to a.
If, however, the set M is finite or empty, then there exists an $n_0 \in \mathbb{N}$ such that

$$\forall\, n \in \mathbb{N} \geq n_0 \Rightarrow x_n \neq a.$$

Let $n_1 = n_0+1$ and $\epsilon_1 = |x_{n_1} - a|/2$. Since a is a cluster point of (x_n), there exists a natural number $n_2 > n_1$ such that

$$x_{n_2} \in (a - \epsilon_1, a + \epsilon_1).$$

Taking $\epsilon_2 = |x_{n_2} - a|/2$, there exists a term $x_{n_3} \in (a - \epsilon_2, a + \epsilon_2)$ etc. Continuing this procedure ad infinitum, we obtain a subsequence (x_{n_k}) of the sequence (x_n) which converges to the point a. A subsequence (x_{n_q}) which diverges to ∞ can be obtained from the fact that the sequence (x_n) is not bounded from above. Namely, denote by n_1 the smallest natural number such that $x_{n_1} > 1$. Next, let n_2 for the smallest integer greater than $x_{n_1} + n_1$. Continuing in this manner we get a monotonically increasing sequence n_q which diverges to infinity. This sequence is the set of indices of a monotonically increasing subsequence x_{n_q} which diverges to ∞. □

5.11.59 Problem. Show that,

1. if (x_n) be a sequence and $\{x_n; n \in \mathbb{N}\}$ is a finite set, then (x_n) has a constant subsequence.
2. every sequence (x_n) has a monotone subsequence.

5.11.59.1 Solution.

1. Suppose $\{x_n; n \in \mathbb{N}\}$ is a finite set. Then $\{x_n; n \in \mathbb{N}\} = \{x_1, x_2, ...x_k\} = A$ (say). Since $X : \mathbb{N} \to A$ is onto, we define the set $A_i = \{n \in \mathbb{N}; X(n) = x_i, 1 \leq i \leq k\}$. Clearly $\mathbb{N} = A_1 \cup A_2 \cup .. \cup A_k$. Since $\mathbb{N}$ is infinite, then at least one of the sets A_i must be infinite. Let p be the smallest positive integer for which A_p is the infinite subset of $\mathbb{N}$. Now by WOP, we can write $A_p = \{n_1, n_2, n_3, ..\}$ where $n_1 < n_2 < n_3 < ..$ Now define $\alpha : \mathbb{N} \to \mathbb{N}$ by $\alpha(i) = n_i \ \forall\ i \in \mathbb{N}$, it is clear that α is increasing and hence $X \circ \alpha$ is a subsequence of X for $(X \circ \alpha)(\mathbb{N}) = X(\alpha(\mathbb{N})) = X(A_p) = \{x_p\}$.

2. Let $X = (x_1, x_2, ...x_n, ..) \in \mathbb{R}^{\mathbb{N}}$. If $\{x_n; n \in \mathbb{N}\}$ is a finite set, then (x_n) has a constant subsequence, by the above. Thus we may suppose that $B = \{x_n; n \in \mathbb{N}\}$ is infinite. If B is bounded, then $\exists$ a non-empty subset $A \subseteq B$ such that $\sup A \notin A$ or $\inf A \notin A$. If B is unbounded, then B cannot have both $\sup B$ and $\inf B$, in this case we take A as B.

   (a) Case (1) $\sup A \notin A$
   Let i_1 be the smallest integer such that $x_{i_1} \in A$. Since none of the real numbers x_n, where $1 \leq n \leq i_1$, can be the supremum of A, define i_2 as the smallest integer greater than i_1 such that $x_{i_2} \in A$ and $x_{i_1} < x_{i_2}$. If x_{i_k} has been defined for $1 \leq n \leq k$, and satisfying $x_{i_1} < x_{i_2} < ... < x_{i_k}$, define i_{k+1} as the smallest integer greater than i_k such that $x_{i_{k+1}} \in A$ and $x_{i_k} < x_{i_{k+1}}$. We have thus defined inductively an increasing sequence $\alpha : \mathbb{N} \to \mathbb{N}$ and $\alpha(n) = i_n \forall\ n \in \mathbb{N}$ and it is easy to check that $x \circ \alpha$ is an increasing subsequence of $x = (x_n)$.

   (b) Case(2) $\inf A \notin A$ and

   (c) Case(3) $A = B$ can be done similarly.

5.11.60 Problem. Let K be a finite nonempty set of real numbers. Construct a sequence (x_n) in K such that the set of all cluster points of (x_n) is exactly K.

5.11.60.1 Solution.

Hint. If $K = \{x_1, x_2, ..., x_n\}$, consider $S = \{x_1, x_2, ..., x_n, x_1, x_2, ..., x_n, x_1, x_2, ..., x_n, ...\}$. □

5.11.61 Problem. Show that a bounded sequence in $\mathbb{R}$ converges if and only if, all of its convergent subsequences converge to the same limit.

5.11.61.1 Solution. One implication is obvious. Assume now that a bounded sequence (x_n) does not converge, so, there exists a subsequence (x_{n_k}) that converges to some x_0. Since (x_n) does not converge to x_0, there exists $\epsilon > 0$ such that, for all $n \in \mathbb{N}$, we can find $m > n$ with $|x_m - x_0| \geq \epsilon$. Proceeding recursively, we find a subsequence x_{n_j} of such that $|x_{n_j} - x_0| \geq \epsilon \; \forall \; j \in \mathbb{N}$. If we apply again theorem (Any bounded sequence in $\mathbb{R}$ has a convergent subsequence.), this subsequence has a further subsequence that converges (to some point y_0 that satisfies, certainly, $|y_0 - x_0| \geq \epsilon$), and we reach a contradiction. $\square$

5.11.62 Problem. Let $A \subseteq \mathbb{R}$. Then prove that the following are equivalent:

1. $\overline{A} = A \cup A'$(where A' is the derived set of A).
2. $\overline{A} = \{x \in \mathbb{R};$ every nbhd. U of x, $U \cap A \neq \emptyset\}$.
3. $x \in \overline{A}$ iff $\forall \; n \in \mathbb{N},\; B\left(x, \frac{1}{n}\right) \cap A \neq \emptyset$.
4. $x \in \overline{A}$ iff there exists a sequence (x_n) in A such that $\lim_{n\to\infty} x_n = x$.
5. $x \in \overline{A}$ iff $\inf_{t\in A} |t - x| = 0$.

5.11.62.1 Solution.

1. (1) implies (2)
   Let $x \in A \cup A'$. Then $x \in A$ or $x \in A'$. Let U be any nbhd. of x, then $x \in U$ or $U \setminus \{x\} \cap A \neq \emptyset$ implies $U \cap A \neq \emptyset$. Conversely, if $\exists$ a nbhd. of U of x such that $U \cap A = \emptyset$ then neither $x \in A$ nor $x \in A'$, i.e. $x \notin A \cup A'$.

2. (2) implies (3)
   Since $\forall \; n \in \mathbb{N},\; B(x; 1/n)$ is nbhd of x, so by (1) $B(x; 1/n) \cap A \neq \emptyset$.

3. (3) implies (2)
   Let U be any nbhd of x. Then $\exists \; \epsilon > 0$, so $B(x; \epsilon) \subseteq U$, and $\exists \; n \in \mathbb{N}$ such that such that $1/n < \epsilon$, so $B(x; 1/n) \subseteq B(x; \epsilon) \subseteq U$. Thus $\emptyset \neq B(x; 1/n) \cap A \subseteq U \cap A \Rightarrow U \cap A \neq \emptyset$ for every nbhd. U of x.

4. (4) implies (5)
   Let $\epsilon > 0$. Since $x_n \in A \; \forall \; n \in \mathbb{N}$, we see that $\inf_{t\in A} |t - x| \leq |x_n - x| \; \forall \; n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = x$ implies $\exists \; n_0 \in \mathbb{N}$ such that $n \geq n_0 \Rightarrow |x_n - x| < \epsilon$, then $\inf_{t\in A} |t - x| \leq |x_n - x| < \epsilon \Rightarrow \inf_{t\in A} |t - x| = 0$.

5. (5) implies (3)
   Suppose $\inf_{t\in A} |t - x| = 0$. So for each $n \in \mathbb{N}$; $\exists \; x_n \in A$ such that $|x_n - x| < \frac{1}{n}$ i.e. $x_n \in B(x; \frac{1}{n}) \Rightarrow \; B\left(x, \frac{1}{n}\right) \cap A \neq \emptyset$.

6. (3) implies (4)
   Let $\epsilon > 0$. Then $\exists \; p \in \mathbb{N}$ such that $\frac{1}{p} < \epsilon$. Since $B\left(x, \frac{1}{p}\right) \cap A \neq \emptyset$. Choose $x_p \in B\left(x, \frac{1}{p}\right) \cap A$. Then for $n \geq p$, $B\left(x, \frac{1}{p}\right) \cap A \neq \emptyset \Rightarrow |x_n - x| < \frac{1}{p} < \epsilon$. Thus $\lim_{n\to\infty} x_n = x$.

5.11.63 Problem (**Cluster point or subsequential limit** of a sequence.)**.** or A point x is said to be a **cluster point** of a sequence (x_n) iff it satisfies one of the following four equivalent properties:

1. x is the limit of a subsequence (x_{n_k}) of the sequence (x_n).
2. $\forall\, \epsilon > 0$ and $\forall\, n \in \mathbb{N}\, \exists\, m > n$ such that $|x_m - x| < \epsilon$.
3. $\forall\, \epsilon > 0$ and $\forall\, m \in \mathbb{N}$, x belongs to the closure of the set $A_m = \{x_n; n \geq m\}$.
4. Every nbhd. of x contains infinitely many points of (x_n).

5.11.63.1 Solution.

1. (1) implies (2)
   Let $\epsilon > 0$ and $p \in \mathbb{N}$. Since x is the limit of (x_{n_k}) of (x_n), so $\exists\, q \in \mathbb{N}$ such that $k \geq q \Rightarrow |x_{n_k} - x| < \epsilon$. Since n_k is increasing, so there exists $r \in \mathbb{N}$ such that $r > \max\{p, q\}$. Thus $m = n_r > n_p > p \Rightarrow |x_m - x| < \epsilon$.
2. (2) implies (3)
   Let $\epsilon > 0$, and $p \in \mathbb{N}$, then $A_p = \{x_p, x_{p+1}, ...\}$ and there exists $m \geq p \Rightarrow |x_m - x| < \epsilon$. In that case, $A_p \cap (x - \epsilon, x + \epsilon) \neq \emptyset$. Thus $x \in \overline{A_p}$.
3. (3) implies (1)
   We know that for every $\epsilon = 1/k$ and for every $n \geq 1$, there exists a number n_k such that $|x_{n_k} - x| < 1/k$. We define the subsequence n_k by taking to be the smallest integer strictly greater than n_{k-1} and verifying $|x_{n_k} - x| < 1/k$. The subsequence x_{n_k} converges to x, since for every $\epsilon > 0$, there exists $k_0 \geq 1/\epsilon$ such that $|x_{n_k} - x| < \epsilon$ when $k > k_0$.
4. See problem 5.11.57. □

5.11.64 Problem. The set of all subsequential limits of a sequence (x_n) in $\mathbb{R}$, is a closed subset of $\mathbb{R}$.

5.11.64.1 Solution. Let S be the set of all subsequential limits of (x_n). If y is an accumulation point of S, we must show that $y \in S$, and hence that some subsequence of (x_n) converges to y. Let $\delta > 0$, since y is an accumulation point of S, there exists $z \in S \cap N(y; \delta); z \neq y$. But $z \in S$ implies that z is a subsequential limit. Again, $z \in N(y; \delta)$ implies $\exists\, \epsilon$ such that $N(z; \epsilon) \subset N(y; \delta)$, choose $x_{n_1} \in N(z; \epsilon) \subset N(y; \delta)$, so $|x_{n_1} - y| < \delta$. Choose n_2 by taking to be the smallest integer strictly greater than n_1 such that $x_{n_2} \in N(y; \delta/2)$. Now suppose $x_{n_1}, x_{n_2}, .., x_{n_{k-1}}$ have been choosen such that $|z - y| < \delta/2^k$ and hence there exists $n_k > n_{k-1}$ such that $|z - x_{n_k}| < \delta/2^k$. Therefore, for each $k = 1, 2, ...$ we have

$$|x_{n_k} - y| \leq |x_{n_k} - z| + |z - y| < \delta/2^{k-1} < \delta,$$

thus the subsequence (x_{n_k}) converges to y. □

5.11.65 Problem. Consider the functions

$$f(x) = \begin{cases} \sin x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{Q}^C \end{cases} \text{ and}$$

$$g(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{Q}^C. \end{cases}$$

Prove that $f(x) \sim g(x)$, as $x \to 0$.

5.11.65.1 Solution. Let us put $\psi(x) = \dfrac{\sin x}{x}, x \neq 0$. Then

$$f(x) = \psi(x).g(x) \text{ and } \lim_{x\to 0} \psi(x) = 1.$$

Thus $f(x) \sim g(x)$, as $x \to 0$, but $\lim_{x\to 0} \frac{f(x)}{g(x)}$ does not exist.

5.11.66 Problem. Are the following asymptotic equalities correct?

1. $2020x + x\cos x = O(x),\ x \to \infty$;
2. $O(2020x + x\cos x) = x,\ x \to \infty$;
3. $O(x + x\cos x) = x,\ x \to \infty$;
4. $\sqrt{x^2+3} = O\left(\dfrac{1}{x}\right),\ x \to \infty$;

5.11.66.1 Solution.

1. Since it is true that $|2020x + x\cos x| \leq C|x|, \quad x \in \mathbb{R}$ (for example for $x \geq 1$, one can take $C = 2021$), it follows that the given statement is true.
2. The statement is true, for $x \geq 1$ implies
$$|x| \leq |x + (2019 + \cos x)x| = 1.|2020x + x\cos x|.$$
3. The statement is not true, because for every $x_0 > 0$ and for every $C > 0$, there exists $x_1 > x_0$ such that
$$1 + \cos x_1 < \frac{1}{C} \text{ or } x_1 > C(x_1 + \cos x_1).$$
4. The statement is true, for
$$\left|\sqrt{x^2+3} - x\right| < 3.\frac{1}{x}; x \geq 1$$

5.11.67 Problem. If the sequences (x_n) and (y_n) satisfy $x_n > 0, y_n > 0$, then prove that

1. $O(1) + O(1) = O(1)$.
2. $o(1) + o(1) = o(1)$.
3. $o(1) = O(1)$.
4. $O(x_n) + O(y_n) = O(x_n + y_n)$.
5. $o(x_n) + o(y_n) = o(x_n + y_n)$.

where $(x_n) = O(1)$ means that the sequence (x_n) is bounded and $(x_n) = o(1)$ means that the sequence (x_n) tends to 0.

5.11.67.1 Solution.

1. The sum of two bounded sequences is a bounded sequence.
2. The sum of two sequences converging to zero is a sequence converging to zero too.

3. A convergent sequence is a bounded one.

4. Let $u_n = O(x_n)$ and $v_n = O(y_n)$, $\exists\ P, Q > 0$ and $\exists\ n_0 \in \mathbb{N}$ such that $n \geq n_0 \Rightarrow |u_n| \leq Px_n, |v_n| \leq Qy_n$. Thus

$$n \geq n_0 \Rightarrow |u_n + v_n| \leq |u_n| + |v_n| \leq Px_n + Qy_n \leq L(x_n + y_n);\ L = \max\{P, Q\}.$$

This means that $O(x_n) + O(y_n) = O(x_n + y_n)$.

5. The expressions $p_n = o(x_n)$ and $q_n = o(x_n)$ means

$$\lim_{n\to\infty} \frac{p_n}{x_n} = 0 \text{ and } \lim_{n\to\infty} \frac{q_n}{y_n} = 0$$
$$\text{then } \left|\frac{p_n}{x_n + y_n}\right| \leq \left|\frac{p_n}{x_n}\right| \text{ and } \left|\frac{q_n}{x_n + y_n}\right| \leq \left|\frac{q_n}{y_n}\right|.$$

And, we have

$$\lim_{n\to\infty} \frac{p_n + q_n}{x_n + y_n} = \lim_{n\to\infty} \left(\frac{p_n}{x_n + y_n} + \frac{q_n}{x_n + y_n}\right) = 0.$$

So we obtain $o(x_n) + o(y_n) = o(x_n + y_n)$. □

5.11.68 Problem. Let f be a positive on some neighbourhood of the point x_0. Prove the following asymptotic relations when $x \to x_0$.

1. $O(O(f)) = O(f)$;

2. $o(O(f)) = o(f)$;

3. $O(o(f)) = o(f)$;

4. $o(f) + O(f) = O(f)$.

5.11.68.1 Solution.

1. Let $g = O(f)$ and $h = O(g)$. Then, by definition there exist constants $K_1 > 0$ and $K_2 > 0$ such that in some nbhd. U of x_0 such that for $x \neq x_0$

$$|g(x)| \leq K_1 f(x) \text{ and } |h(x)| \leq K_2 g(x).$$

Then
$$|h(x)| \leq K_1 K_2 f(x)$$

for $x \in U \setminus \{x_0\}$, which proves the statement.

2. Left to the reader.

3. Left to the reader.

4. Let $g = o(f)$ and $h = O(f)$. Then by definitions there exist a function ϕ, a constant $K > 0$ and a nbhd. U of x_0 such that $g(x) = \phi(x).f(x), \lim_{x\to x_0} \phi(x) = 0$ and $h(x) \le Kf(x), x \in U\setminus\{x_0\}$. Then the sum of the functions g and h on the set $U \setminus \{x_0\}$ can be written as

$$g(x) + h(x) = \phi(x)f(x) + h(x).$$

Since $\phi(x)$ tends to zero as $x \to x_0$, there exists a nbhd. $U_1 \subseteq U$ of the point x_0, such that $|\phi(x)| \le 1$ for $x \in U_1 \setminus \{x_0\}$,. Thus

$$\forall\ x \in U_1 \setminus \{x_0\},\ |g(x) + h(x)| \le 1.f(x) + K.f(x) \le K_1.f(x)$$

where $K_1 = 1 + K$. □

5.11.69 Problem. Show that if the functions f and g have the properties $f(x) \ne 0$ and $g(x) \ne 0$ for $x \ne x_0$, then as $x \to x_0$ such that

$$f(x) \sim g(x) \Leftrightarrow g(x) - f(x) = o(g).$$

5.11.69.1 Solution. From the equality $\lim_{x\to x_0} \frac{f(x)}{g(x)} = 1$, it follows that

$$\lim_{x\to x_0} \left(1 - \frac{f(x)}{g(x)}\right) = 0,$$

from where we obtain $\lim_{x\to x_0} \left(\dfrac{g(x) - f(x)}{g(x)}\right) = 0$, which implies

$$g(x) - f(x) = o(g) \text{ as } x \to x_0. \quad \square$$

5.11.70 Problem. Let us suppose that $f(x) \sim f_1(x)$ and $g(x) \sim g_1(x)$, when $x \to x_0$. If $\lim_{x\to x_0} \frac{f_1(x)}{g_1(x)}$ exists, then $\lim_{x\to x_0} \frac{f(x)}{g(x)}$ exists and satisfies

$$\lim_{x\to x_0} \frac{f_1(x)}{g_1(x)} = \lim_{x\to x_0} \frac{f(x)}{g(x)}.$$

5.11.70.1 Solution. From the above example it follows that $f(x) = f_1(x) + o(f_1(x))$ and $g(x) = g_1(x) + o(g_1(x))$ when $x \to x_0$. Therefore we can write

$$\begin{aligned}
\lim_{x\to x_0} \frac{f(x)}{g(x)} &= \frac{f_1(x) + o(f_1(x))}{g_1(x) + o(g_1(x))} \\
&= \lim_{x\to x_0} \frac{f_1(x)}{g_1(x)}. \lim_{x\to x_0} \frac{1 + \dfrac{o(f_1(x))}{f(x)}}{1 + \dfrac{o(g_1(x))}{g(x)}} \\
&= \lim_{x\to x_0} \frac{f_1(x)}{g_1(x)}. \quad \square
\end{aligned}$$

5.11.71 Problem. Give examples of:

1. a function $f : \mathbb{R} \to \mathbb{R}$ which is unbounded in every open interval.

2. a function $f : [0, 1] \to \mathbb{R}$ such that

$$S = \{a \in [0, 1]; \lim_{x\to a} f(x) \text{ does not exist}\}.$$

5.11.71.1 Solution.

1. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 0$ when x is irrational, and $f(x) = n$ when x is rational and $x = \frac{m}{n}$ with $(m, n) = 1$. Now, we prove that f is unbounded in every open interval (a, b). Let $K > 0$. So $\exists\ p, q \in \mathbb{N}$ such that $2^p > K$, and $2^q(b - a) > 1$, then $c = \max\{p, q\}$ implies $2^c > K$, and $2^c(b - a) > 1$ implies that $\exists\ t \in \mathbb{N}$ such that $2^c a < t < 2^c b$ shows that $a < \frac{t}{2^c} < b$, hence $f\left(\frac{t}{2^c}\right) = 2^c > K$ shows that f is unbounded in the open interval (a, b).
2. Define $f : [0, 1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} \frac{1}{n}, & \text{if } x \in \left(\frac{1}{n+1}, \frac{1}{n}\right] \text{ for } n \in \mathbb{N}, \\ 0, & \text{when } x = 0. \end{cases}$$

We see that $\lim_{x\to\frac{1}{n}+} f(x) = \frac{1}{n-1}$ and $\lim_{x\to\frac{1}{n}-} f(x) = \frac{1}{n}$, hence $\lim_{x\to\frac{1}{n}} f(x)$ does not exist and the set $S = \left\{\frac{1}{n}; n \in \mathbb{N}\right\}$ is countably infinite. $\square$

5.11.72 Problem. Prove that $\lim_{x\to-\infty} \left(1 + \frac{1}{x}\right)^x = e$.

5.11.72.1 Solution. Let us put $x = -t$. Then $t \to \infty$ as $x \to -\infty$. Hence

$$\left(1 + \frac{1}{x}\right)^x = \left(1 + \frac{1}{-t}\right)^{-t}.$$

Now $\left(1 + \frac{1}{-t}\right)^{-t} = \left(\frac{t-1}{t}\right)^{-t} = \left(\frac{t}{t-1}\right)^t = \left(1 + \frac{1}{t-1}\right)^t = \left(1 + \frac{1}{t-1}\right)^{t-1}\left(1 + \frac{1}{t-1}\right) \to e$ as $t \to \infty$.

5.11.73 Problem. Show that $f(x) = O(|x - x_0|^2)$ as $x \to x_o$ implies $f(x) = o(|x - x_0|)$ as $x \to x_o$, but give an example to show that the converse is not true.

5.11.73.1 Solution. By the definition of "big"O, $\exists\ M > 0, n \in \mathbb{N}$ such that

$$\frac{|f(x)|}{|x - x_0|^2} \le M \text{ as } |x - x_0| < \frac{1}{n},$$

for some $n > 0$, which implies

$$\frac{|f(x)|}{|x - x_0|} \le \frac{|x - x_0||f(x)|}{|x - x_0|^2} \le M|x - x_0|\ ,$$

for all x such that $|x - x_0| < \dfrac{1}{n}$, which implies $f(x) = o(|x - x_0|)$ as $x \to x_o$.
For the converse, consider $f(x) = |x - x_0|^{2/3}$. $\square$

5.11.74 Problem. Define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x, & \text{if } x < 0 \\ 1, & \text{if } x \ge 0. \end{cases}$$

Find an open set G and a closed set E such that the set $\{x \in \mathbb{R}; f(x) \in G\}$ is not open and the set $\{x \in \mathbb{R}; f(x) \in E\}$ is not closed.

5.11.74.1 Solution. Consider an open set $G = (0,2)$, then $f^{-1}(G) = [0,\infty)$ is not open and $E = [-1,0]$ is closed and $f^{-1}(E) = [-1,0)$ is not closed. $\square$

5.11.75 Problem. Let $x \in \mathbb{R}, n \in \mathbb{N}$, then show that

$$[x] + \left[x + \frac{1}{n}\right] + ... + \left[x + \frac{n-1}{n}\right] = [nx],$$

where $[x]$ denotes the greatest integer function of x.

5.11.75.1 Solution. Let $f(x) = [x] + \left[x + \frac{1}{n}\right] + ... + \left[x + \frac{n-1}{n}\right] - [nx]$, then show that $f(x+\frac{1}{n}) = f(x)$ and hence $f(x) = f(x - \frac{k}{n}); \forall\ k \in \mathbb{Z}$. Now $k = [nx]$ is taken to show that $f(x - \frac{k}{n}) = 0$. $\square$

5.11.76 Problem. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ such that

$$f^{-1}(0,1) = \bigcup_{n\in\mathbb{Z}} (2n, 2n+1)$$

and find a bijection $g : [0,4] \to [0,2]$ so that

$$g^{-1}(0,1) = (0,1) \cup (2,3); g^{-1}(1,2) = (1,2) \cup (3,4).$$

5.11.76.1 Solution.

1. $f(x) = \begin{cases} x - 2n & \text{if } x \in (2n, 2n+1) \\ 0 & \text{if } x \in [2n-1, 2n] \end{cases}.$

2. $g(x) = \begin{cases} \frac{x}{2} & \text{if } x \in [0,1) \\ \frac{1}{2}(x+1) & \text{if } x \in [1,2) \\ \frac{1}{2}(x-1) & \text{if } x \in [2,3) \\ \frac{x}{2} & \text{if } x \in [3,4]. \end{cases}$ $\square$

5.11.77 Problem. Let $f : [a,b] \to \mathbb{R}$ be monotone. Prove that

1. $\lim_{x\to c+} f(x)$ and $\lim_{x\to c-} f(x)$ exist for any $c \in (a,b)$.
2. $\lim_{x\to a+} f(x)$ and $\lim_{x\to b-} f(x)$ exist.

5.11.77.1 Solution.

1. Without loss of generality, we may assume that f is increasing. Let $c \in (a,b)$. Suppose that

$$U(c) = \inf\{f(x); x \in (c,b)\} \text{ and}$$
$$L(c) = \sup\{f(x); x \in (a,c)\}.$$

Note that for any $x \in (c,b)$, we have $f(c) \leq f(x)$. Hence the set $\{f(x); x \in (c,b)\}$ is bounded below by $f(c)$ which implies the existence of $U(c)$ and forces the inequality $f(c) \leq U(c)$. Similarly for any $x \in (a,c)$, we have $f(x) \leq f(c)$. Hence the set $\{f(x); x \in (a,c)\}$ is bounded above by $f(c)$ which implies the existence of $L(c)$ and forces the inequality $L(c) \leq f(c)$. So we have

$$L(c) \leq f(c) \leq U(c).$$

First we claim $\lim_{x\to c+} f(x) = U(c)$. Let $\epsilon > 0$. Then by definition of $U(c)$, there exists $x_0 \in (c,b)$ such that $U(c) < f(x_0) < U(c)+\epsilon$. Let $\delta = x_0 - c$. Then $\delta > 0$ since $c < x_0$. Let $x \in (c, c+\delta) = (c, x_0)$, then we have $U(c) \leq f(x) \leq f(x_0) < U(c)+\epsilon$. In particular we have $|f(x) - U(c)| < \epsilon$. which completes the proof of our claim.
In a similar way we can prove that $\lim_{x\to c-} f(x)$ exist for any $c \in (a,b)$.

2. For the last part of this problem similar ideas as described above will show $\lim_{x\to a+} f(x) = U(a)$ and $\lim_{x\to b-} f(x) = L(b)$. Note that $L(a)$ and $U(b)$ do not exist. □

5.11.78 Problem. Let $f : [a,b] \to \mathbb{R}$ be monotone. Prove that the set

$$D = \{x; x \in [a,b] \text{ and } f \text{ does not have a limit at x}\}$$

is countable. What happens at the points x that are not in D?

5.11.78.1 Solution. We have seen in the previous problems that for any $c \in (a,b)$, then $\lim_{x\to c+} f(x)$ and $\lim_{x\to c-} f(x)$ exist. In particular, we have

$$\lim_{x\to c-} f(x) \leq f(c) \leq \lim_{x\to c+} f(x).$$

In particular, $c \in D \cap (a,b)$ if and only if

$$\lim_{x\to c-} f(x) < \lim_{x\to c+} f(x).$$

So if $x \in D$, then we must have

$$\lim_{x\to c-} f(x) = f(c) = \lim_{x\to c+} f(x),$$

or that $\lim_{x\to c} f(x) = f(c)$. On the other hand, let $\epsilon > 0$, and consider the set

$$D_\epsilon = \left\{ c \in D \cap (a,b);\ \lim_{x\to c+} f(x) - \lim_{x\to c-} f(x) > \epsilon \right\}.$$

We claim that D_ϵ is finite. Let $x_i \in D_\epsilon, i = 1,2,..,n$, such that $x_1 < x_2 < .... < x_n$. Then it is easy to check that

$$f(a) \leq \lim_{x\to x_1-} f(x) < \lim_{x\to x_1+} f(x) < .. < \lim_{x\to x_n-} f(x) < \lim_{x\to x_n+} f(x) \leq f(b).$$

This will then imply

$$\begin{aligned} &\sum_{i=1}^{n} \left(\lim_{x\to x_i+} f(x) - \lim_{x\to x_i-} f(x) \right) \leq f(b) - f(a) \\ \Rightarrow\ & n\epsilon \leq f(b) - f(a). \end{aligned}$$

Since the number n is bounded above, then D_ϵ must be finite. Finally we get

$$D = \bigcup_{n\geq 1} D_{\frac{1}{n}}$$

which implies that D is a countable union of finite sets. Hence D is countable. Note that if $f : \mathbb{R} \to \mathbb{R}$ is monotone, then the set

$$D = \{x; x \in [a, b] \text{ and } f \text{ does not have a limit at x}\}$$

is countable. Indeed, We have

$$D = \bigcup_{n \geq 1} D \cap (-n, n)$$

Since $D \cap (-n, n)$ is countable, then D is a countable union of countable sets. So D is countable. $\square$

5.11.79 Problem. We say f is **asymptotic** to g and write $f \sim g$ if $\frac{f(x)}{g(x)} \to 1$ as $x \to \infty$. We write $f = O(g)$ (pronounced "Big Oh") if $g(x) > 0$ for sufficiently large $x \in \mathbb{R}$ and $\frac{f(x)}{g(x)}$ is bounded for sufficiently large x. We write $f = o(g)$ (pronounced "Little Oh") if $\frac{f(x)}{g(x)} \to 0$ as $x \to \infty$. Prove the following:

1. $x^2 + x \sim x^2$.
2. $e^{\sqrt{\ln x}} = o(x)$.

5.11.79.1 Solution.

1. $\lim_{x\to\infty} \frac{x^2+x}{x^2} = \lim_{x\to\infty} \left(1 + \frac{1}{x}\right) = 0$.
2. We see that $\frac{e^{\sqrt{\ln x}}}{x} = e^{\sqrt{\ln x} - \ln x}$. Now, because $\lim_{x\to\infty} \ln x = \infty$ and $\lim_{x\to\infty} \frac{\sqrt{\ln x}}{\ln x} = 0$ we have that for large x, $\sqrt{\ln x} < \frac{\sqrt{\ln x}}{2}$. Hence, for similarly large x, this gives $\frac{e^{\sqrt{\ln x}}}{x} \leq \frac{1}{x}\left[e^{\frac{\ln x}{2}}\right] = \frac{1}{\sqrt{x}} \to 0$. Thus $e^{\sqrt{\ln x}} = o(x)$.

5.11.80 Problem. Prove that if $f = o(g)$ and if $g(x) \to \infty$ as $x \to \infty$, then $e^f = o(e^g)$ as $x \to \infty$.

5.11.80.1 Solution. Assume $g(x) > 0$. Since $g(x) \to \infty$, so $f(x) = o(g(x))$ as $x \to \infty$, there is $k > 0$ such that $|f(x)| < g(x)/2$ for $x > k$. Therefore, $f(x) - g(x) < -g(x)/2$ for $x > k$ and $\lim_{x\to\infty}(f(x) - g(x)) = -\infty$. Now notice that $\frac{e^{f(x)}}{e^{g(x)}} = e^{f(x)-g(x)}$, and thus $e^{f(x)-g(x)} \to 0$ as $x \to \infty$. Therefore $e^{f(x)} = o(e^{g(x)})$ as $x \to \infty$, as claimed.

5.11.81 Problem. Let $a_1, a_2, ..., a_n$ be positive real numbers. Prove that

$$\lim_{x\to 0} \left(\frac{a_1^x + a_2^x + ... + a_n^x}{n}\right)^{\frac{1}{x}} = \sqrt[n]{a_1 a_2 .....a_n}$$

5.11.81.1 Solution. First, note that

$$\lim_{x\to 0} \frac{a^x - 1}{x} = \ln a.$$

Indeed, the left-hand side can be recognized as the derivative of the exponential at 0, or we can argue as follows: let $a^x = 1 + t$, with $t \to 0$. Then $x = \dfrac{\ln(1+t)}{\ln a}$ and the limit becomes

$$\lim_{t\to 0} \frac{t \ln a}{\ln(1+t)} = \lim_{t\to 0} \frac{\ln a}{\ln(1+t)^{\frac{1}{t}}} = \frac{\ln a}{\ln e} = \ln a$$

Let us return to the problem. Because the limit is of the form 1^∞, it is standard to write it as

$$\begin{aligned}
&\lim_{x\to 0}\left(1+\frac{a_1^x+a_2^x+....+a_n^x-n}{n}\right)^{\frac{1}{x}}\\
&=\lim_{x\to 0}(1+p)^{\frac{1}{p}\frac{p}{x}} \text{ where } p=\frac{a_1^x+a_2^x+....+a_n^x-n}{n}\\
&=\lim_{x\to 0}\exp\left(\frac{p}{x}\right)\\
&=\lim_{x\to 0}\exp\frac{1}{n}\left(\frac{a_1^x+a_2^x+....+a_n^x-n}{x}\right)\\
&=\exp\left[\frac{1}{n}\lim_{x\to 0}\left(\frac{a_1^x-1}{x}+\frac{a_2^x-1}{x}+..+\frac{a_n^x-1}{x}\right)\right]\\
&=\exp\left[\frac{1}{n}\left(\ln a_1+\ln a_2+..+\ln a_n\right)\right]\\
&=\sqrt[n]{a_1a_2.....a_n}. \quad \square
\end{aligned}$$

5.11.82 Problem. Let $a\in(0,1)$ be a real number and $f:\mathbb{R}\to\mathbb{R}$ a function that satisfies the following conditions:

1. $\lim_{x\to\infty} f(x)=0$,
2. $\lim_{x\to\infty}\dfrac{f(x)-f(ax)}{x}=0$.

Show that $\lim_{x\to\infty}\dfrac{f(x)}{x}=0$.

5.11.82.1 Solution. The second condition reads, for any >0, there exists >0 such that if $x\in(-\delta,\delta)$ then $|f(x)-f(ax)|<|x|$. Applying the triangle inequality, we find that for all positive integers n and all $x\in(-\delta,\delta)$,

$$\begin{aligned}
|f(x)-f(a^nx)|&\le|f(x)-f(ax)|+|f(ax)-f(a^2x)|+..+|f(a^{n-1}x)-f(a^nx)|\\
&<\epsilon|x|(1+a+a^2+...+a^{n-1})=\epsilon\frac{1-a^n}{1-a}\le\epsilon\frac{1}{1-a}|x|.
\end{aligned}$$

Taking the limit as $n\to\infty$, we obtain

$$|f(x)|\le\epsilon\frac{1}{1-a}|x|.$$

Since $\epsilon>0$ is arbitrary, this proves that $\lim_{x\to\infty}\dfrac{f(x)}{x}=0$. $\square$

5.11.83 Problem. For two positive integers m and n, compute

$$\lim_{x\to 0}\frac{\sqrt[m]{\cos x}-\sqrt[n]{\cos x}}{x^2}$$

5.11.83.1 Solution. Without loss of generality, we may assume that $m > n$. Let $p = \cos^m x$ and $q = \cos^n x$ Write the limit as

$$\begin{aligned}
&\lim_{x\to 0} \frac{\sqrt[mn]{\cos^n x} - \sqrt[mn]{\cos^m x}}{x^2} \\
&= \lim_{x\to 0} \frac{q^{\frac{1}{mn}} - p^{\frac{1}{mn}}}{x^2} \\
&= \lim_{x\to 0} \frac{q-p}{x^2\left(\sqrt[mn]{q^{mn-1}} + .. + \sqrt[mn]{p^{mn-1}}\right)} \\
&= \lim_{x\to 0} \frac{q-p}{mnx^2} = \lim_{x\to 0} \frac{\cos^n x(1-\cos^{m-n} x)}{mnx^2} \\
&= \lim_{x\to 0} \frac{(1-\cos^{m-n} x)}{mnx^2} \\
&= \lim_{x\to 0} \frac{(1-\cos x)(1+\cos x + ... + \cos^{m-n-1} x)}{mnx^2} \\
&= \frac{m-n}{mn} \lim_{x\to 0} \frac{(1-\cos x)}{x^2} = \frac{m-n}{mn}. \quad \square
\end{aligned}$$

5.11.84 Problem. Let $f : (0,\infty) \to (0,\infty)$ be an increasing function with $\lim_{t\to\infty} \dfrac{f(2t)}{f(t)} = 1$. Prove that $\lim_{t\to\infty} \dfrac{f(mt)}{f(t)} = 1$ for any $m > 0$.

5.11.84.1 Solution. We can assume that $m > 1$; otherwise, we can write the fraction and change t to $\frac{t}{m}$ and there is an integer n such that $m < 2^n$. Because f is increasing, $f(t) < f(mt) < f(2^n t)$. We obtain

$$1 < \frac{f(mt)}{f(t)} < \frac{f(2^n t)}{f(t)} \text{ and}$$

$$\frac{f(2^n t)}{f(t)} = \frac{f(2^n t)}{f(2^{n-1} t)} \frac{f(2^{n-1} t)}{f(2^{n-2} t)} \cdots\cdots\cdots \frac{f(2t)}{f(t)}$$

Thus $\lim_{t\to\infty} \dfrac{f(2^n t)}{f(t)} = 1$. Hence, by Sandwich theorem $\lim_{t\to\infty} \dfrac{f(mt)}{f(t)} = 1$. $\square$

5.11.85 Problem. (28th W.L. Putnam Mathematics Competition, 1967) Let $f(x) = \sum_{k=1}^n a_k \sin kx$, with $a_1, a_2, .., a_n \in \mathbb{R}, n \geq 1$. Prove that if $f(x) \leq |\sin x| \ \forall\, x \in \mathbb{R}$, then

$$\left|\sum_{k=1}^n k a_k\right| \leq 1.$$

5.11.85.1 Solution. We see that $f'(x) = \sum_{k=1}^n k a_k \cos kx$ and $f'(0) = \sum_{k=1}^n k a_k$. Now, using the result $\lim_{x\to 0} \frac{f(x)-f(0)}{x} = f'(0)$. We get

$$\begin{aligned}
\left|\sum_{k=1}^n k a_k\right| &= |f'(0)| = \lim_{x\to 0} \left|\frac{f(x)-f(0)}{x}\right| \\
&= \lim_{x\to 0} \left|\frac{f(x)}{x}\right| = \lim_{x\to 0} \left|\frac{f(x)}{\sin x}\right| \left|\frac{\sin x}{x}\right| \leq 1. \quad \square
\end{aligned}$$

5.11.86 Problem. Assume that $f : [a,b] \to \mathbb{R}$ is increasing, and define $g : (a,b) \to \mathbb{R}$ by $g(x) = \sup\{f(y); y < x\}$. Prove that $g(c) \leq f(c)$. Also prove that if $g(c) \neq f(c)$, then f does not have a limit at c.

5.11.86.1 Solution. Let $c \in (a,b)$, then $\forall y < c \Rightarrow f(y) < f(c)$. So the set $S = \{f(y); y < c\}$ is bounded above by $f(c)$, hence $\sup S \leq f(c)$. i.e. $g(c) \leq f(c)$. Again if $g(c) \neq f(c)$ then $g(c) < f(c)$.We show that f does not have a limit at c. Now for each $n \in \mathbb{N}$, we get sequences u_n and v_n such that $c - 1/n < u_n < c < v_n < c + 1/n$, then $f(u_n) \leq g(c) < f(c) \leq f(v_n).f(v_n) - f(u_n) \geq f(c) - g(c) > 0$. From the above we see that u_n is increasing and v_n is decreasing, and both converging to c, but $f(v_n) - f(u_n)$ does not converge to 0. Hence f does not have a limit at c. □

5.11.87 Problem. Find all functions f mapping non-negative integers into non-negative integers and such that

$$f(f(n)) + f(n) = 2n + 6 \ \forall \ n \in \mathbb{N}. \tag{1}$$

5.11.87.1 Solution. Choose and fix an integer $n \geq 0$. Consider the following sequence of non-negative integers:

$$a_0 = n, a_1 = f(n), a_2 = f(f(n)), a_k = f^k(n), ..., \tag{2}$$

superscript denoting iteration. In equation (1) set $f^k(n)$ in place of n; the result is

$$a_{k+2} + a_{k+1} = 2a_k + 6.$$

Subtracting $2a_{k+1}$ from both sides we get $a_{k+2} - a_{k+1} = 2(a_k - a_{k+1} + 6)$; that is, $r_{k+1} + 2r_k - 6 = 0$, where $r_k = a_{k+1} - a_k$. Write $r_k = x_k + 2$; the equation becomes $x_{k+1} + 2x_k = 0$. All these relations hold for $k = 0, 1, 2, ...$ The last equation obviously implies the explicit formula $x_k = (-2)^k x_0$. Consequently $x_k = (-2)^k x_0$ for $k = 0, 1, 2, ...$ By telescoping, we obtain for every integer

$$\begin{aligned} a_m &= a_0 + \sum_{k=0}^{m-1} (a_{k+1} - a_k) \\ &= a_0 + \sum_{k=0}^{m-1} r_k \\ &= a_0 + 2m + \sum_{k=0}^{m-1} (-2)^k x_0 \\ &= a_0 + 2m + \frac{1 - (-2)^m}{3} x_0. \end{aligned} \tag{3}$$

Recall that all a_m's are supposed to be non-negative. The exponential growth of $|(-2)^m x_0|$ can be in no way matched by the linear term $2m$, unless $x_0 = 0$. (To be more precise: if $x_0 > 0$ then the expression obtained in (3) is negative for large even m; and if $x_0 < 0$ then it is negative for large odd m.) Therefore x_0 must be 0, whence $r_0 = 2$; i.e. $a_1 - a_0 = 2$. This in view of definition (1) means that $f(n) - n = 2$. □

5.11.88 Problem.

1. Let the sequence (X_n) be the sequence of arithmetic means of a sequence (x_n). Prove that
$$\liminf_{n\to\infty} x_n \le \liminf_{n\to\infty} X_n \le \limsup_{n\to\infty} X_n \le \limsup_{n\to\infty} x_n.$$
2. (Cauchy's first limit theorem) If (x_n) converges, then show that
$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} X_n$$
3. (Cauchy's second limit theorem) If x_n is positive for all $n \in \mathbb{N}$ and (x_n) converges, then show that
$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} Y_n$$
where $Y_n = \sqrt[n]{x_1 x_2 ... x_n}$.
4. Give an example to show that (X_n) could converge even if (x_n) diverges.

5.11.88.1 Solution.

1. We shall prove only that
$$\limsup_{n\to\infty} X_n \le \limsup_{n\to\infty} x_n.$$
Let us denote by $x = \limsup_{n\to\infty} x_n$ and assume that $x \ge 0$. Then for every $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for every $n > n_0 \Rightarrow\ x_n > x + \epsilon$. So for $n \ge n_0$, we have
$$\begin{aligned} X_n &= \frac{x_1 + x_2 + ... + x_{n_0} + ... + x_n}{n} \\ &< \frac{x_1 + x_2 + ... + x_{n_0}}{n} + \frac{(n - n_0)(x + \epsilon)}{n} \le \frac{B}{n} + x + \epsilon \end{aligned}$$
where $B = x_1 + x_2 + ... + x_{n_0}$. Now for $\epsilon > 0\ \exists\ n_1 \in \mathbb{N}$, such that $\dfrac{|B|}{n} < \epsilon\ \forall\ n > n_1$. Let $n_2 = \max\{n_0, n_1\}$ then we get $\forall\ n > n_2 \Rightarrow X_n < x + 2\epsilon$ which shows
$$\limsup_{n\to\infty} X_n \le x = \limsup_{n\to\infty} x_n.$$
The case $x < 0$ is similar and left to the reader.
2. From the condition $\lim_{n\to\infty} x_n = x$, it follows that
$$(\forall\ \epsilon > 0)(\exists\ n_0 \in \mathbb{N})(\forall\ n \in \mathbb{N})\ n \ge n_0 \Rightarrow |x_n - x| < \epsilon/2.$$
Then we can write
$$\begin{aligned} |X_n - x| &= \left|\frac{x_1 + x_2 + ... + x_{n_0} + ... + x_n}{n} - x\right| \\ &\le \frac{1}{n}(|x_1 - x| + |x_2 - x| + ... + |x_{n_0 - 1} - x| + |x_{n_0} - x| ... + |x_n - x|) \\ &< \frac{A}{n} + \frac{n - n_0}{n}.\frac{\epsilon}{2} < \frac{A}{n} + \frac{\epsilon}{2} \end{aligned}$$

where $A = |x_1 - x| + |x_2 - x| + ... + |x_{n_0-1} - x|$ and there exists a natural number $n_1 = n_1(\epsilon)$ such that

$$n > n_1 \Rightarrow \frac{A}{n} < \frac{\epsilon}{2}.$$

Let $n_2 = \max\{n_1, n_2\}$. Then we obtain

$$(\forall\ \epsilon > 0)(\exists\ n_2 \in \mathbb{N})(\forall\ n \in \mathbb{N})\ n \geq n_2 \Rightarrow |X_n - x| < \epsilon.$$

3. Since the logarithmic function is continuous, hence from $\lim_{n\to\infty} x_n = x$, it follows that $\lim_{n\to\infty} \ln x_n = \ln \lim_{n\to\infty} x_n = \ln x$. Hence we have

$$\ln Y_n = \frac{1}{n}(\ln x_1 + \ln x_2 + ... + \ln x_n),$$

and we can apply the previous result to obtain $\lim_{n\to\infty} \ln Y_n = \ln x$, hence

$$\lim_{n\to\infty} Y_n = x.$$

Remark: The opposite does not necessarily hold. Find an example!

4. Consider $x_n = (-1)^{n+1}$, then $X_n = \begin{cases} 0, & \text{if } n \text{ is even} \\ \frac{1}{n}, & \text{if } n \text{ is odd.} \end{cases}$ □

5.11.89 Problem. Let the sequence (p_n) of positive numbers such that

$$\lim_{n\to\infty} \sum_{k=1}^{n} p_k = \infty$$

and let (x_n) be a sequence of real numbers. Let

$$X_n = \frac{p_1x_1 + p_2x_2 + ... + p_nx_n}{p_1 + p_2 + ... + p_n}$$

(X_n is the generalised arithmetic means of the sequence (x_n)). Prove the following:

$$\liminf_{n\to\infty} x_n \leq \liminf_{n\to\infty} X_n \leq \limsup_{n\to\infty} X_n \leq \limsup_{n\to\infty} x_n.$$

5.11.89.1 Solution. Same as above. Thus if the sequence (x_n) converges, then its sequence of generalized arithmetic means converges to the same limit. □

5.11.90 Problem. Define

$$(1)\ \overline{\lim}_{x\to a} f(x) = \inf_{\delta>0} \sup_{x\in\hat{N}(a;\delta)} f(x).$$

$$(2)\ \underline{\lim}_{x\to a} f(x) = \sup_{\delta>0} \inf_{x\in\hat{N}(a;\delta)} f(x).$$

where $\hat{N}(a;\delta) = N(a;\delta) \setminus \{a\}$.

1. Show that $\overline{\lim}_{x\to a} f(x) \leq L$, iff given $\epsilon > 0$, $\exists\ \delta > 0$ such that $\forall\ x \in \hat{N}(a,\delta) \Rightarrow f(x) \leq L + \epsilon$.

2. Show that $\overline{\lim}_{x\to a} f(x) \geq L$, iff given $\epsilon > 0$ and $\delta > 0, \exists\, x \in \hat{N}(a,\delta)$ and $f(x) \geq L - \epsilon$.

3. Show that $\underline{\lim}_{x\to a} f(x) \leq \overline{\lim}_{x\to a} f(x)$,
and if "equality" holds then $\lim_{x\to a} f(x)$ exists.

4. If $\overline{\lim}_{x\to a} f(x) = L$, and (x_n) is a sequence with $x_n \neq a$ such that $\lim_{n\to\infty} x_n = a$ and then $\overline{\lim}_{n\to\infty} f(x_n) \leq L$.

5. If $\overline{\lim}_{x\to a} f(x) = L$,, then there is a sequence (x_n) with $x_n \neq a$ such that $\lim_{n\to\infty} x_n = a$ and $\overline{\lim}_{n\to\infty} f(x_n) = L$.

6. $\lim_{x\to a} f(x) = L$ iff $\lim_{n\to\infty} f(x_n) = L$ for every sequence (x_n) with $x_n \neq a$ and $\lim_{n\to\infty} x_n = a$.

5.11.90.1 Solution.

1. Suppose $\overline{\lim}_{x\to a} f(x) \leq L$. Given $\epsilon > 0$, $\exists\, \delta > 0$ such that $\sup_{x\in\hat{N}(a;\delta)} f(x) < L + \epsilon$. Thus for all x with $0 < |x-a| < \delta, f(x) \leq L + \epsilon$. Conversely, for any $\epsilon > 0$, there exists $\delta > 0$ such that $f(x) \leq L + \epsilon$ for all x with $0 < |x-a| < \delta$. Then for each n, there exists $\delta_n > 0$, such that $f(x) \leq L + 1/n$ for all x with $0 < |x-a| < \delta_n$, so $\sup_{0<|x-a|<\delta_n} f(x) \leq L + 1/n$. Thus
$$\overline{\lim}_{x\to a} f(x) \leq \inf_{\delta>0} \sup_{0<|x-a|<\delta} f(x) \leq \inf_{n} \sup_{0<|x-a|<\delta_n} f(x) \leq L + 1/n$$
for all n, so $\overline{\lim}_{x\to a} f(x) \leq L$.

2. Suppose $\overline{\lim}_{x\to a} f(x) \geq L$. Given $\epsilon > 0$, and $\delta > 0$, $\sup_{x\in\hat{N}(a;\delta)} > L - \epsilon$. So, $\exists\, x$ such that $0 < |x-a| < \delta, f(x) \geq L - \epsilon$. Conversely, suppose that $\epsilon > 0$, and $\delta > 0$, $\exists\, x$ such that $f(x) \geq L - \epsilon$. Then for each n, there exists x_n, such that $0 < |x_n - a| < \delta, f(x_n) \geq L - 1/n$. Thus for each $\delta > 0$, $\sup_{0<|x-a|<\delta} f(x) > 1/n$ for all n, so $\sup_{x\in\hat{N}(a;\delta)} f(x) \geq L$. Hence, $\overline{\lim}_{x\to a} f(x) = \inf_{\delta>0} \sup_{x\in\hat{N}(a;\delta)} f(x) \geq L$.

3. For any $\delta_1, \delta_2 > 0$, if $\delta_1 < \delta_2$, then
$$\inf_{x\in\hat{N}(a;\delta_2)} f(x) \leq \inf_{x\in\hat{N}(a;\delta_1)} f(x) \leq \sup_{x\in\hat{N}(a;\delta_1)} f(x) \leq \sup_{x\in\hat{N}(a;\delta_2)} f(x).$$
In particular,
$$\inf_{x\in\hat{N}(a;\delta_1)} f(x) \leq \sup_{x\in\hat{N}(a;\delta_2)} f(x) \text{ and } \inf_{x\in\hat{N}(a;\delta_2)} f(x) \leq \sup_{x\in\hat{N}(a;\delta_1)} f(x).$$
Hence
$$\sup_{\delta>0} \inf_{x\in\hat{N}(a;\delta)} f(x) \leq \sup_{x\in\hat{N}(a;\delta_0)} f(x) \text{ for any } \delta_0 > 0$$
So, $\sup_{\delta>0} \inf_{x\in\hat{N}(a;\delta)} f(x) \leq \inf_{\delta>0} \sup_{x\in\hat{N}(a;\delta)} f(x)$.
Thus $\underline{\lim}_{x\to a} f(x) \leq \overline{\lim}_{x\to a} f(x)$.
Suppose that $\underline{\lim}_{x\to a} f(x) = \overline{\lim}_{x\to a} f(x) = L$. Given $\epsilon > 0$, there exists $\delta_1 > 0$ such that $\sup_{x\in\hat{N}(a;\delta_1)} f(x) < L + \epsilon$, i.e. $f(x) < L + \epsilon$ whenever $0 < |x-y| < \delta_1$, also there exists $\delta_2 > 0$ such that $\inf_{x\in\hat{N}(a;\delta_2)} f(x) > L - \epsilon$, i.e. $f(x) > L - \epsilon$ whenever $0 < |x-y| < \delta_2$.

Let $\delta = \min\{\delta_1, \delta_2\}$. Then, when $0 < |x-a| < \delta \Rightarrow |f(x) - L| < \epsilon$ so $\lim_{x\to a} f(x)$ exists. Conversely, let $\lim_{x\to a} f(x) = L$. Then, given $\epsilon > 0$, there exists $\delta > 0$ such that $0 < |x-a| < \delta \Rightarrow |f(x) - L| < \epsilon$. By part (1), $\overline{\lim}_{x\to a} f(x) \le L$. Similarly, $\underline{\lim}_{x\to a} f(x) \ge L$, i.e. $\underline{\lim}_{x\to a} f(x) \ge L \ge \overline{\lim}_{x\to a} f(x) \Rightarrow \underline{\lim}_{x\to a} f(x) = \overline{\lim}_{x\to a} f(x)$.

4. Suppose $\overline{\lim}_{x\to a} f(x) = L$ and (x_n) is a sequence with $x_n \ne a$ such that $a = \lim_n x_n$. For any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $0 < |x_n - a| < \delta$ for $n \ge N$. Thus for any $\delta > 0, \inf_N \sup_{n\ge N} f(x_n) \le \sup_{n\ge N} f(x_n) \le \sup_{x\in \hat{N}(a;\delta_0)} f(x)$ so $\inf_N \sup_{n\ge N} f(x_n) \le \inf_{\delta>0} \sup_{x\in\hat{N}(a;\delta)} f(x)$ i.e. $\overline{\lim}_n f(x_n) \le \overline{\lim}_{x\to a} f(x) = L$.

5. Suppose $\overline{\lim}_{x\to a} f(x) = L$. By part (a), for each $n \in \mathbb{N}$, there exists $\delta_n > 0$ such that $f(x) < L + 1/n$ whenever $0 < |x-a| < \delta_n$. By part (2), there exists x_n such that $0 < |x_n - a| < \min(\delta_n, 1/n)$ and $f(x_n) > L - 1/n$. Thus $0 < |x_n - a| < 1/n$ and $|f(x_n) - L| < 1/n$ for each n. i.e. (xn) is a sequence with $x_n \ne a$ such that $a = \lim x_n$ and $L = \lim f(x_n)$.

6. Suppose $l = \lim_{x\to a} f(x)$ and let and (x_n) is a sequence with $x_n \ne a$ such that $a = \lim_n x_n$. Given $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - l| < \epsilon$ whenever $0 < |x-a| < \delta$. Also there exists $N \in \mathbb{N}$ such that $0 < |x_n - a| < \delta$ for $n \ge N$. Thus for $n \ge N, |f(x_n) - l| < \epsilon$. i.e. $l = \lim f(x_n)$. Conversely, suppose $l \ne \lim_{x\to a} f(x)$. Then there exists $\epsilon > 0$ such that for each n there exists x_n with $0 < |x_n - a| < 1/n$ and $|f(x_n) - l| \ge \epsilon$. Thus (x_n) is a sequence with $x_n \ne a$ and $a = \lim x_n$ but $l \ne \lim f(x_n)$. □

5.11.5 Theorem. Stolz-Cesaro theorem: Let (a_n) be a sequence in $\mathbb{R}$ and (b_n) a strictly monotone and divergent sequence. Then

$$\lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = l$$

implies

$$\lim_{n\to\infty} \frac{a_n}{b_n} = l.$$

5.11.6 Remark. Generally, the converse of the above theorem does not hold. However partial converse holds by the following example.

5.11.91 Problem. Let (a_n) be a sequence in $\mathbb{R}$ and (b_n) a strictly monotone and divergent sequence. Suppose

$$\lim_{n\to\infty} \frac{b_n}{b_{n+1}} = b(\ne 1).$$

Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = l.$$

implies

$$\lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = l.$$

5.11.91.1 Solution. Now observe that

$$\frac{a_{n+1} - a_n}{b_{n+1} - b_n} = \frac{\frac{a_{n+1}}{b_{n+1}} - \frac{a_n}{b_n}\frac{b_n}{b_{n+1}}}{1 - \frac{b_n}{b_{n+1}}} \to \frac{l - lb}{1 - b} = l$$

5.11.92 Problem. Let (x_n) be a sequence in $\mathbb{R}$, and $x_n \to x$, then $\lim_{n\to\infty} \frac{\sum x_n}{n} = x$

5.11.92.1 Solution. Take $b_n = n$ and $a_n = \sum_{k=1}^{n} x_k$ Then $\frac{a_{n+1}-a_n}{b_{n+1}-b_n} = x_{n+1} \to x$.

5.11.7 Remark. Generally, the converse of the above problem does not hold.

5.11.8 Example. Consider the sequence (a_n) in $\mathbb{R}$ given by

$$a_n = \frac{1+(-1)^n}{2}$$

Obviously, the sequence does not converge. At the same time

$$\frac{\sum x_n}{n} = \begin{cases} \frac{1}{2}, & \text{if } n \text{ is even} \\ \frac{n-1}{2n} & \text{if } n \text{ is odd.} \end{cases}$$

Hence $\lim_{n\to\infty} \frac{\sum x_n}{n} = \frac{1}{2}$.

5.11.93 Problem. Let (a_n) be a sequence in $\mathbb{R}$, $a_n > 0\ \forall\ n \in \mathbb{N}$.

1. If
$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = a,$$
then
$$\lim_{n\to\infty} \sqrt[n]{a_n} = a(a > 0),$$
2. If
$$\lim_{n\to\infty} a_n = a, (a > 0)$$
then
$$\lim_{n\to\infty} \sqrt[n]{a_1 a_2 \dots a_n} = a(a > 0),$$

5.11.93.1 Solution.

1. Take $x_n = \ln \sqrt[n]{a_n}$, $y_n = n$, and apply Stolz-Cesaro theorem.
2. $x_n = \sqrt[n]{a_1 a_2 \dots a_n}$, $y_n = n$. □

5.11.9 Remark. Generally, the converses of the above problem do not hold. In order to see directly that the converse of (1) is false it is enough to consider two distinct positive numbers p, q and the sequence (a_n) defined by

$$a_n = \begin{cases} p^k q^{k-1}, & \text{if } n = 2k-1 \\ p^k q^k, & \text{if } n = 2k. \end{cases}$$

Then $a_n \to \sqrt{pq}$. At the same time the sequence $(\frac{a_{n+1}}{a_n})$ has no limit. In order to see directly that the converse of (2), consider as before two distinct positive numbers p, q and the sequence (a_n) defined by

$$a_n = \begin{cases} p, & \text{if } n = 2k-1 \\ q, & \text{if } n = 2k \end{cases}$$

Then $\sqrt[n]{a_1 a_2 \dots a_n} \to \sqrt{pq}$. and the sequence (a_n) is divergent.

5.11.94 Problem (The Stolz theorem)**.** (a discrete analogue of L'Hopital's rule). Let the sequence (x_n) of positive numbers satisfy the following two conditions

$$\lim_{n\to\infty} x_n = \infty \text{ and } x_{n+1} > x_n.$$

If there exists a sequence (y_n) such that

$$\lim_{n\to\infty} \frac{y_n - y_{n-1}}{x_n - x_{n-1}} \text{ exists,}$$

then prove that

$$\lim_{n\to\infty} \frac{y_n}{x_n} = \lim_{n\to\infty} \frac{y_n - y_{n-1}}{x_n - x_{n-1}}.$$

5.11.94.1 Solution. Using the conditions on the sequence (x_n), we can construct the following two sequences

$$x_n = p_1 + p_2 + ... + p_n \text{ and } y_n = p_1 a_1 + p_2 a_2 + ... + p_n a_n,$$

where p_i's are positive real numbers and (a_n) is a sequence in $\mathbb{R}$. So we have

$$\frac{y_n - y_{n-1}}{x_n - x_{n-1}} = \frac{p_n a_n}{p_n} = a_n.$$

If $\lim a_n = \ell$ exists, then from the previous exercise it follows that the sequence of generalized arithmetic means converges to the same limit. □

5.11.95 Problem. The converse of Stoltz theorem is not true. In other words, if

$$\lim_{n\to\infty} \frac{y_n}{x_n} = \ell, \text{ then } \lim_{n\to\infty} \frac{y_n - y_{n-1}}{x_n - x_{n-1}}$$

then may not exist.

5.11.95.1 Solution. If we take, for example,

$$x_n = n \text{ and } y_n = \sum_{k=1}^{n} \frac{k\pi}{3}, \ n \in \mathbb{N},$$

$$\text{then } \lim_{n\to\infty} \frac{y_n}{x_n} = \lim_{n\to\infty} \frac{\sin\frac{\pi}{3} + \sin\frac{2\pi}{3} + ... + \sin\frac{n\pi}{3}}{n} = 0.$$

However, the limit $\lim_{n\to\infty} \dfrac{y_n - y_{n-1}}{x_n - x_{n-1}}$ does not exist. For

$$\liminf_{n\to\infty} \frac{y_n - y_{n-1}}{x_n - x_{n-1}} = \liminf_{n\to\infty} \sin\frac{n\pi}{3} = -\frac{\sqrt{3}}{2},$$

$$\limsup_{n\to\infty} \frac{y_n - y_{n-1}}{x_n - x_{n-1}} = \limsup_{n\to\infty} \sin\frac{n\pi}{3} = \frac{\sqrt{3}}{2}. \quad \square$$

5.11.96 Problem. Determine the following limits $\lim_{n\to\infty} t_n$ by using the Stolz theorem,

1. $t_n = \dfrac{1^k + 2^k + ... + n^k}{n^{k+1}}, \ k \in \mathbb{N}.$
2. $t_n = \dfrac{1^p + 3^p + ... + (2n+1)^p}{n^{p+1}}, \ p \in \mathbb{Q}.$
3. $t_n = \dfrac{1^k + 2^k + ... + n^k}{n^k} - \dfrac{n}{k+1}, \ k \in \mathbb{N}.$
4. $t_n = \dfrac{\ln n}{n^{1/k}}, \ k \in \mathbb{N}$
5. $t_n = \dfrac{1 + \frac{1}{\sqrt{2}} + ... + \frac{1}{\sqrt{n}}}{n}$

5.11.96.1 Solution.

1. $y_n = 1^k + 2^k + ... + n^k$ and $x_n = n^{k+1}$ then find $\lim_{n\to\infty} \dfrac{y_n - y_{n-1}}{x_n - x_{n-1}}$ and the value is $\dfrac{1}{k+1}$.
2. the value is $\dfrac{2^p}{p+1}$.
3. $y_n = (k+1)(1^k + 2^k + ... + n^k) - n^{k+1}, \ x_n = (k+1)n^k$ and the value is $\dfrac{1}{2}$.
4. $y_n = \ln n, \ x_n = n^{1/k}$ and the value is 0.
5. $y_n = 1 + \dfrac{1}{\sqrt{2}} + ... + \dfrac{1}{\sqrt{n}}, \ x_n = n$ and the value is 0. □

5.11.97 Problem. Let (x_n) be a sequence. Prove that

$$\lim_{n\to\infty}(x_n - x_{n-1}) = x \Rightarrow \lim_{n\to\infty} \frac{x_n}{n} = x.$$

5.11.97.1 Solution. Let us denote $x_n - x_{n-1}$ by $y_n, x_0 = 0$. Then $\lim_{n\to\infty} y_n = \lim_{n\to\infty}(x_n - x_{n-1}) = x$ and let us calculate the sequence of arithmetic means of the sequence (y_n) and we have

$$A_n = \frac{y_1 + y_2 + ... + y_n}{n} = \frac{\sum_{i=1}^n (x_i - x_{i-1})}{n} = \frac{x_n}{n}$$

it follows from Cauchy's first limit theorem that $\lim_{n\to\infty} \dfrac{x_n}{n} = x$. □

5.11.98 Problem. Let (x_n) be a sequence. Prove that

$$\lim_{n\to\infty} \frac{x_n}{x_{n-1}} = x \Rightarrow \lim_{n\to\infty} \sqrt[n]{x_n} = x.$$

5.11.98.1 Solution. Let us denote x_n/x_{n-1} by $y_n, x_0 = 1$. Then from

$$G_n = \sqrt[n]{y_1 y_2 ... y_n} = \sqrt[n]{(x_1/x_0)(x_2/x_1).....(x_n/x_{n-1})} = \sqrt[n]{x_n}$$

it follows from Cauchy's second limit theorem that $\lim_{n\to\infty} G_n = \lim_{n\to\infty} \sqrt[n]{x_n} = x$ □

5.11.99 Problem. Let $a_1 = \sqrt{2}$ and

$$a_{n+1} = \sqrt{2 + \sqrt{a_n}}, \forall\, n \geq 1.$$

Show that (a_n) is convergent and find its limit.

5.11.99.1 Solution. By induction, we can prove that $0 < a_k < 2$ and, now

$$a_{n+1}^2 - a_n^2 = \sqrt{a_n} - \sqrt{a_{n-1}}$$

shows that (a_n). Hence (a_n) is convergent. Denoting by l its limit, we get $l = \sqrt{2 + \sqrt{l}}$, which, using Cardan's formula, leads to

$$l = \frac{1}{3}\left(\sqrt[3]{\frac{1}{2}\left(79 + 3\sqrt{249}\right)} + \sqrt[3]{\frac{1}{2}\left(79 - 3\sqrt{249}\right)} - 1\right). \quad \square$$

5.11.100 Problem. Let (a_n) and (b_n) be two sequences and (c_n) be a defined by

$$c_n = \frac{a_1 b_n + a_2 b_{n-1} + ... + a_n b_1}{n}$$

1. If $\lim_{n\to\infty} a_n = 0$ and $|b_n| < B\ \forall\, n \in \mathbb{N}$, then $\lim_{n\to\infty} c_n = 0$.

2. If $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} a_n = a$, then $\lim_{n\to\infty} c_n = ab$.

5.11.100.1 Solution.

1. From the implication

$$\lim_{n\to\infty} a_n = 0 \Rightarrow \lim_{n\to\infty} |a_n| = 0$$

and by Stoltz theorem, it follows

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} |a_n| = 0 \Rightarrow \lim_{n\to\infty} |a_n| = 0.$$

Since the sequence (b_n) is bounded by B, we have

$$|c_n| \leq \frac{|a_1| + |a_2| + ... + |a_n|}{n} B \text{ and } \lim_{n\to\infty} |c_n| \leq \lim_{n\to\infty} \frac{B}{n}\sum_{k=1}^{n} |a_n| = 0.$$

This means that $\lim_{n\to\infty} c_n = 0$.

2. Let us put $x_n = a_n - a$ then $\lim_{n\to\infty} x_n = 0$ and

$$\begin{aligned} c_n &= \frac{(x_1 + a)b_n + (x_2 + a)b_{n-1} + ... + (x_n + a)b_1}{n} \\ &= \frac{x_1 b_n + x_2 b_{n-1} + ... + x_n b_1}{n} + a\frac{b_n + b_{n-1} + ... + b_1}{n} = p_n + q_n. \end{aligned}$$

From (1) it follows that

$$\lim_{n\to\infty} p_n = \lim_{n\to\infty} \frac{x_1 b_n + x_2 b_{n-1} + ... + x_n b_1}{n} = 0$$

$$\text{while } \lim_{n\to\infty} q_n = \lim_{n\to\infty} a.\frac{b_n + b_{n-1} + ... + b_1}{n} = ab.$$

Therefore $\lim_{n\to\infty} c_n = ab$. □

5.11.101 Problem. Let (a_n) and (b_n) be sequences of positive real numbers such that

$$\lim_{n\to\infty} a_n^n = a, \ \lim_{n\to\infty} b_n^n = b; \ a, b > 0,$$

and suppose that the positive numbers p and q satisfy $p + q = 1$. Compute

$$\lim_{n\to\infty} (pa_n + qb_n)^n.$$

5.11.101.1 Solution. Hint: $x_n^n \to x$ iff $n(x_n - 1) \to \log x$. □

5.11.102 Problem. If $0 < c_1 < 1$ and $c_{n+1} = c_n(1 - c_n)$, then

$$\lim_{n\to\infty} \frac{n(1 - nc_n)}{\log n} = 1.$$

5.11.102.1 Solution. Hint: $c_{n+1}^{-1} = c_n^{-1} + (1 - c_n)^{-1}$. □

5.11.103 Problem. Define a sequence (x_n) by

$$x_1 = a, \ x_{n+1} = \ln \frac{e^{x_n} - 1}{x_n}, n > 1.$$

Show that $1 + x_1 + x_1x_2 + x_1x_2x_3 + ... = e^a$.

5.11.103.1 Solution. Hint. Observe that $x_n e^{x_{n+1}} = e^{x_n} - 1$ □

5.11.104 Problem. Let $s_1 = \log a, s_n = \sum_{k=1}^{n-1} \log(a - s_n)$ for $n \geq 2$. Show that $\lim_{n\to\infty} s_n = a - 1$.

5.11.104.1 Solution. Hint. Consider the function $x + \log(a - x)$. □

5.11.105 Problem. Find the limits of

$$\lim_{n\to\infty} (2k^{\frac{1}{n}} - 1)^n \text{ and } \lim_{n\to\infty} (2n^{\frac{1}{n}} - 1)^n / n^2$$

5.11.105.1 Solution. Answer: k^2 and 1. □

5.11.106 Problem. A monotone decreasing sequence (a_n) cannot oscillate: correct or justify.

5.11.106.1 Solution. Note that, if $l \in \mathbb{R}$, then $a_n \to l$ as $n \to \infty$ means (a_n) has a finite limit. If l is ∞ or $-\infty$, then (a_n) has an infinite limit. Also if (a_n) has no limit (finite or infinite), then (a_n) is said to **oscillate boundedly or unboundedly** according as it is bounded or unbounded.
So if a monotone decreasing sequence (a_n) is bounded, it converges to $\inf_n a_n$ and if it is unbounded $\lim_{n\to\infty} a_n = -\infty$. Thus it cannot oscillate. □

5.11.107 Problem. The sequence (u_n) is determined by the equations

$$u_1 = 0, u_2 = 1, u_{n+1} + u_{n-1} = \frac{4u_n}{u_n^2 + 1}(n \geq 1)$$

Prove the following results:

1. $(u_n u_{n+1} - 1)^2 + (u_n - u_{n+1})^2 = 2$;
2. u_n is a rational number;
3. $1 + 4u_n^2 - u_n^4$ is the square of a rational number.

5.11.107.1 Solution.

1. Using the given relation show that

$$(u_n u_{n+1} - 1)^2 + (u_n - u_{n+1})^2 = (u_{n-1}u_n - 1)^2 + (u_{n-1} - u_n)^2.$$

   Sucessive reductions finally give

$$(u_n u_{n+1} - 1)^2 + (u_n - u_{n+1})^2 = (u_1 u_2 - 1)^2 + (u_1 - u_2)^2 = 2.$$

2. If u_n and u_{n-1} are rational, then

$$u_{n+1} = \frac{4u_n}{u_n^2 + 1} - u_{n-1}$$

   is rational, since it is formed from the former by the operations of addition, subtraction, multiplication, and division. Now u_1 and u_2 are rational: therefore so is u_3 , hence u_4 and so on.

3. Use (i). □

5.11.108 Problem. Prove that the sequence (a_n) where $a_n = \sin n$ is not convergent.

5.11.108.1 Solution. Suppose to the contrary that the $\lim_n \sin n$ exists and equal to l, say. Then $0 = \lim_n \sin(n+2) - \lim_n \sin n = \lim_n(\sin(n+2) - \sin n) = \lim_n 2\cos(n+1)\sin 1$, so it follows that $\lim_n \cos n = 0$. Again $0 = \lim_n \cos(n+2) - \lim_n \cos n = \lim_n(\cos(n+2) - \cos n) = -\lim_n 2\sin(n+1)\sin 1$, From here, we see that $\lim_n \sin n = 0$. This is a contradiction because $\sin^2 n + \cos^2 n = 1$ implies that $1 = \lim_n(\sin^2 n + \cos^2 n) = \lim_n(\sin^2 n) + \lim_n(\cos^2 n) = 0$. Therefore, the sequence $a_n = \sin n$ does not have a limit. □

5.11.109 Problem. Determine the limit points of the set $\{\cos n\}$ of the sequence $(\cos n)$.

5.11.109.1 Solution. Observe that $\cos(2m\pi + n) = \cos n, m, n \in \mathbb{Z}$ and the set $G = \{2m\pi + n; m, n \in \mathbb{Z}\}$ is a group under addition. Since π is an irrational number, so the group G is dense in $\mathbb{R}$. Let $y \in [-1, 1]$, since the function $\cos : \mathbb{R} \to [-1, 1]$ is continuous, so by IVP $\exists\ x \in \mathbb{R}$ such that $y = \cos x$. Hence $\exists\ m_1, n_1 \in \mathbb{Z}$ satisfying $x < 2m_1\pi + n_1 < x + \epsilon$. Thus by using the inequality $|\cos x - \cos y| \leq |x - y|$, we get

$$|y - \cos n_1| = |\cos x - \cos(2m_1\pi + n_1)| \leq 2m_1\pi + n_1 - x < \epsilon.$$

The above argument shows, there exist non-negative integers n_k with $|y - \cos n_k| < \epsilon/2^{k-1}$ i.e. $\cos n_k \to y$ as $k \to \infty$. From this, it easily follows that every point of $[-1, 1]$ must be a limit point of $\{\cos n\}$. □

5.11.110 Problem. Show that $\lim_n \frac{n^k}{a^n} = 0$, where $a > 1$.

5.11.110.1 Solution. Let $b = \sqrt[k]{a}$. Clearly $b > 1$ and $0 < \frac{n^k}{a^n} = \left(\frac{n}{\sqrt[k]{a^n}}\right)^k = \left(\frac{n}{b^n}\right)^k$. We shall show that $\frac{n}{b^n} \to 0$. If we write $b = 1 + (b-1)$, then

$$b^n = \sum_{i=0}^{n} \binom{n}{i} (b-1)^i.$$

Since $b > 1$, all terms in this sum are positive, so

$$b^n > \binom{n}{2}(b-1)^2 = \frac{n(n-1)}{2}(b-1)^2 \tag{1}$$

Now

$$0 < \frac{n}{b^n} < \frac{2n}{n(n-1)(b-1)^2} = \frac{2}{(n-1)(b-1)^2} \to 0$$

Thus, $\frac{n}{b^n} \to 0$ implies that $a_n \to 0$. □

5.11.111 Problem. Show that $\lim_{n\to\infty} \frac{\ln n}{n} = 0$.

5.11.111.1 Solution. When n is large enough, we have $\frac{1}{b^n} < \frac{n}{b^n} < 1$. Let $\epsilon > 0$ and $b = e^\epsilon$. Then

$$\frac{1}{e^{n\epsilon}} < \frac{n}{e^{n\epsilon}} < 1 \Rightarrow 1 < n < e^{n\epsilon} \Rightarrow 0 < \ln n < n\epsilon.$$

That is, $0 < \frac{\ln n}{n} < \epsilon$, shows $\lim_{n\to\infty} \frac{\ln n}{n} = 0$. □

5.11.112 Problem.

1. Show that $\lim_n \sqrt[n]{n} = 1$.
2. Show that $\lim_n \frac{1}{\sqrt[n]{n!}} = 0$.

5.11.112.1 Solution.

1. Let $b = \sqrt[n]{n}$ and applying (1) in the result of 5.11.110.1, we have

   $$n > \frac{n(n-1)}{2}(\sqrt[n]{n} - 1)^2$$

   It follows that

   $$\sqrt[n]{n} - 1 < \sqrt{\frac{2}{n-1}} \to 0.$$

   Hence $\lim_n \sqrt[n]{n} = 1$.

2. For any fixed $k > 1, k \in \mathbb{N}$, we show that $\frac{1}{\sqrt[n]{n!}} < \frac{k}{n}$, i.e. $n! > \left(\frac{n}{k}\right)^n$ using induction. When $n = 1$, we have $1 < k$, so suppose it is true for $n = m$. Using hypothesis, $(m+1)! = (m+1)m! > (m+1)\left(\frac{m}{k}\right)^m = \left(\frac{m+1}{k}\right)^{m+1} \frac{k}{(1+\frac{1}{m})^m} > \left(\frac{m+1}{k}\right)^{m+1}$. Therefore $\frac{1}{\sqrt[n]{n!}} < \frac{k}{n}$ for all $n \in \mathbb{N}$. Consequently

   $$0 < \frac{1}{\sqrt[n]{n!}} < \frac{k}{n} \to 0.$$

   Hence $\lim_n \frac{1}{\sqrt[n]{n!}} = 0$. □

5.11.113 Problem. Let $\lim_n a_n = \infty$. Prove that $\lim_n \left(1+\frac{1}{a_n}\right)^{a_n} = e$.

5.11.113.1 Solution. We observe that, if n_k is any increasing sequence of positive integers, then $\lim_k \left(1+\frac{1}{n_k+1}\right)^{n_k} = \lim_k \left(1+\frac{1}{n_k}\right)^{n_k+1} = \lim_k \left(1+\frac{1}{n_k}\right)^{n_k} = e$. Next we establish the inequality

$$\left(1+\frac{1}{\lfloor a_n\rfloor+1}\right)^{\lfloor a_n\rfloor} \le \left(1+\frac{1}{\lfloor a_n\rfloor}\right)^{\lfloor a_n\rfloor+1} \le \left(1+\frac{1}{\lfloor a_n\rfloor}\right)^{\lfloor a_n\rfloor}.$$

Since $\lfloor a_n\rfloor + 1 \ge a_n$ and $\lfloor a_n\rfloor \le a_n$, so

$$\left(1+\frac{1}{\lfloor a_n\rfloor+1}\right)^{\lfloor a_n\rfloor} \le \left(1+\frac{1}{a_n}\right)^{\lfloor a_n\rfloor} \le \left(1+\frac{1}{a_n}\right)^{a_n}.$$

and

$$\left(1+\frac{1}{a_n}\right)^{a_n} \le \left(1+\frac{1}{\lfloor a_n\rfloor}\right)^{a_n} \le \left(1+\frac{1}{\lfloor a_n\rfloor}\right)^{\lfloor a_n\rfloor+1}.$$

Hence the result. Note that $\lfloor x\rfloor$ denotes the integral part of x. □

5.11.114 Problem. Show that if (r_n) is a sequence of positive real numbers which satisfies $r_{m+n} \le r_m r_n \ \forall\ m, n \in \mathbb{N}$, then $(r_n^{\frac{1}{n}})$ converges to its glb.

5.11.114.1 Solution. Let r be be the glb of (r_n). Let $\epsilon > 0$ then $\exists\ m \in \mathbb{N}$ such that $(r_m^{\frac{1}{m}}) < r+\epsilon$. Now for any $n \in \mathbb{N}$, by division algorithm, we have $n = km + t$ for integers k, t with $0 \le t < m$. Then $r_n \le r_{km} r_t \le r_m^k r_t \le (r+\epsilon)^{km} r_t$, and $r_n^{\frac{1}{n}} \le (r+\epsilon)^{km/n} r_t^{1/n}$. But both km/n and $r_t^{1/n} \to 1$ as $n \to \infty$, so $r_n^{\frac{1}{n}} \le r+\epsilon$ for sufficiently large n, as required. □

5.11.115 Problem. Suppose that the terms of the sequence a_n satisfy the inequalities $0 \le a_{n+m} \le a_n + a_m$. Prove that the sequence (a_n/n) converges to its glb or diverges properly to $-\infty$.

5.11.115.1 Solution. It is sufficient to consider the case where the greatest lower bound α is finite. Assume $\epsilon > 0$ so $\exists\ m \in \mathbb{N}$ such that $a_m/m < \alpha + \epsilon$. Again, any number n can be written in the form $n = qm + r$ where r is an integer, $0 \le r \le m-1$. We define $a_0 = 0$. Then we have

$$\begin{aligned} a_n &= a_{qm+r} \le a_m + \le a_m + \ldots.. + \le a_m + a_r, \\ \Rightarrow a_n/n &= \frac{a_{qm+r}}{qm+r} \le \frac{qa_m + a_r}{qm+r} = \frac{a_m}{m}\frac{qm}{qm+r} + \frac{a_r}{n}. \\ \Rightarrow \alpha &\le \frac{a_n}{n} < (\alpha+\epsilon)\frac{qm}{qm+r} + \frac{a_r}{n} = (\alpha+\epsilon)\left(1-\frac{r}{n}\right) + \frac{a_r}{n}. \end{aligned}$$

Thus $\dfrac{a_n}{n} \to \alpha$. □

5.11.116 Problem. Consider the sequence (a_n), such that

$$a_m + a_n - 1 < a_{m+n} < a_m + a_n + 1.$$

Then

$$\lim_{n\to\infty} \frac{a_n}{n}$$

exists; is finite (say ω) and we have

$$\omega n - 1 < a_n < \omega n + 1.$$

5.11.116.1 Solution. From the relations, we get

$$a_{m+n} + 1 < (a_m + 1) + (a_n + 1),\ 1 - a_{m+n} < (1 - a_m) + (1 - a_n)$$

Again, by the previous problem,

$$\lim_{n\to\infty} \frac{a_n + 1}{n} = \omega,\ \lim_{n\to\infty} \frac{1 - a_n}{n} = -\omega,$$

where $-\omega$ being a lower bound, cannot be ∞, and so $\omega \neq -\infty$. We conclude further that

$$\frac{a_n + 1}{n} \geq \omega,\ \frac{1 - a_n}{n} \geq -\omega.$$

Which shows the result. □

5.11.117 Problem. A real sequence (a_n) satisfies the following conditions:

$$a_1 > 1,\ a_1 + ... + a_n < 2a_n \text{ for } n \geq 2.$$

Show that there exists $q > 1$ such that $a_n > q^n\ \forall\ n \in \mathbb{N}$.

5.11.117.1 Solution. Hint: Prove by induction that $a_n > 2^{n-2}a_1$. □

5.11.118 Problem. A real sequence (a_n) satisfies the following conditions:

$$a_{n+1} \geq a_n \text{ and } a_{mn} \geq m a_n\ \forall\ m, n \in \mathbb{N} \text{ and } \sup_n (a_n/n) < \infty.$$

Does the sequence (a_n/n) converge?

5.11.118.1 Solution. Yes and $\lim_{n\to\infty}(a_n/n) = \sup_n(a_n/n)$. □

5.11.119 Problem. A real sequence (a_n) satisfies the following condition:

$$\lim_{n\to\infty} a_n \sum_{i=1}^{n} a_n^2 = 1.$$

Then $\lim_{n\to\infty}(3n)^{1/3}a_n = 1$

5.11.119.1 Solution. Consider the sequence $s_n^3 - s_{n-1}^3$, where $s_n = \sum_{k=1}^{n} a_k^2$. □

5.11.120 Problem.

1. Is the sequence (x_n) where $x_n = (-1)^n$ eventually or frequently in the set $A = \{1\}$.
2. Which definition is stronger? Does eventually imply frequently or Does frequently imply eventually?
3. Give an alternate rephrasing of definition of limit of a sequence using either frequently or eventually.
4. Suppose an infinite number of terms of a sequence (x_n) are equal to 2. Is (x_n) necessarily eventually in the interval (1.9, 2.1)? Is it frequently in (1.9, 2.1)?

5.11.120.1 Solution.

1. The sequence (x_n) frequently in the set $A = \{1\}$.
2. Eventually in A is stronger than frequently in A.
3. A sequence (x_n) in A converges to a limit l iff it is eventually in a set $A \subseteq \mathbb{R}$.
4. No. Yes. $\square$

5.11.121 Problem. Consider the sequence (x_n), defined by $x_0 = 0, x_1 = 1$

$$x_{n+2} = 3x_{n+1} - 2x_n$$

for $n = 1, 2, ...$ Define $y_n = x_n^2 + 2^{n+2}$. Show that y_n is the square of an odd integer, for every $n \in \mathbb{N}$.

5.11.121.1 Solution. The recursion formula for x_{n+2} can be rewritten as $x_{n+2} - x_{n+1} = 2x_{n+1} - 2x_n$. Now, substituting $x_{n+1} - x_n = t_n$ we get $t_{n+1} = 2t_n$. Since $t_0 = 1$, we infer $t_k = 2^k$, i.e. $x_{k+1} - x_k = 2^k$, hence

$$\sum_{k=0}^{n-1}(x_{k+1} - x_k) = 1 + 2^1 + 2^2 + ... + 2^{n-1} = 2^n - 1.$$

Thus, $y_n = x_n^2 + 2^{n+2} = (2^n - 1)^2 + 2^{n+2} = (2^n + 1)^2$, as required. $\square$

5.11.122 Problem. The sequence (x_n) is defined by $x_1 = 1, x_2 = 2$

$$x_{n+2} = \frac{x_{n+1}^2 + 1}{x_n}$$

for $n = 1, 2, ...$ Show that each x_n is an integer.

5.11.122.1 Solution. According to the definition,
$x_{n+1}^2 = x_{n+2}x_n - 1$, so $x_n^2 = x_{n+1}x_{n-1} - 1$. By subtraction, we get

$$\begin{aligned} &x_{n+1}^2 - x_n^2 = x_{n+2}x_n - x_{n+1}x_{n-1} \\ \Rightarrow &x_{n+1}(x_{n+1} + x_{n-1}) = x_n(x_n + x_{n+2}) \\ \Rightarrow &\frac{x_{n+1} + x_{n-1}}{x_n} = \frac{x_{n+2} + x_n}{x_{n+1}} \end{aligned}$$

This shows that the sequence $\frac{x_{n+1}+x_{n-1}}{x_n}$ is constant which begins with $\frac{x_3+x_1}{x_2} = \frac{5+1}{2} = 3$. Equivalently,

$$x_{n+1} = 3x_n - x_{n-1} \ \forall \ n \in \mathbb{N}.$$

Since $x_1 = 1, x_2 = 2$, this forces that all the x_n's are integrs. $\square$

5.11.123 Problem. Define a sequence (a_n) by $a_1 = \alpha, a_{n+1} = a_n^2 - 2a_n + 2$, where $1 < \alpha < 2$. Show that

1. $\forall \ n \in \mathbb{N} \ 1 < a_n < 2$,
2. (a_n) is decreasing,

3. $a_n \to a$ say, and $a = 1$ or 2,
4. $\forall\, n \in \mathbb{N},\; a_n < \alpha$, and $a < \alpha$, and
5. $a_n \to 1$.

5.11.123.1 Solution.

1. $a_{n+1} - 1 = (a_n - 1)^2 \Rightarrow a_{n+1} > 1 \;\forall\, n \in \mathbb{N}$
   $a_2 - a_1 = a_1^2 - 3a_1 + 2 = \alpha^2 - 3\alpha + 2 = (\alpha - 2)(\alpha - 1) < 0 \Rightarrow a_2 < a_1$.
   again, since $a_{n+1} = a_n^2 - 2a_n + 2 \Rightarrow a_{n+1} - 1 = (a_n - 1)^2$ so
   $$\begin{aligned} a_{n+1} - a_n &= (a_n - 1)^2 - (a_{n-1} - 1)^2 \\ &= (a_n + a_{n-1} - 2)(a_n - a_{n-1}) \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \Rightarrow a_3 - a_2 &= (a_2 - 1)^2 - (a_1 - 1)^2 \\ &= (a_2 + a_1 - 2)(a_2 - a_1) < 0. \end{aligned}$$
   Hence $1 < .. < a_{n+1} < a_n < ... < a_3 < a_2 < a_1 = \alpha < 2$, i.e. (a_n) is decreasing.
2. done in (1).
3. Suppose $a_n \to a$, then $a = \inf a_n$, so
   $$\begin{aligned} &\lim_n a_{n+1} = \lim_n (a_n^2 - 2a_n + 2) \\ \Rightarrow & a = a^2 - 2a + 2 \\ \Rightarrow & a^2 - 3a + 2 = 0 \\ \Rightarrow & a = 2 \text{ or } a = 1. \end{aligned}$$
4. Since (a_n) is decreasing, and $\sup a_n = 2$ thus $a = \lim_{p\to\infty} a_n = 1$.
5. done in (4). $\square$

5.11.124 Problem. The sequence (x_n) is defined by $x_1 = 1/2, x_n = \frac{2n-3}{2n} x_{n-1} \;\forall\, n \in \mathbb{N}$.
Prove the inequality
$$x_1 + x_2 + ... + x_n < 1,\; \forall\, n \in \mathbb{N}.$$

5.11.124.1 Solution. Consider the auxiliary sequence $y_n = (2n-1)x_n$ that defines the analogous formula for y_n given by
$$y_n = (2n-1)\frac{2n-3}{2n}x_{n-1} = \frac{(2n-1)(2n-3)}{2n}.\frac{y_{n-1}}{2(n-1)-1},$$
i.e.
$$y_n = \frac{(2n-1)}{2n}y_{n-1},\; \text{for } n = 2, 3, ...$$
This formula is valid also for $n = 1$, if we set $y_0 = 1$.
Now, $y_{n-1} - y_n = \dfrac{2n}{2n-1}y_n - y_n = \dfrac{y_n}{2n-1} = x_n$, for $n = 1, 2, 3, ...$ and therefore $x_1 + x_2 + ... + x_n = (y_0 - y_1) + (y_1 - y_2) + ... + (y_{n-1} - y_n) = y_0 - y_n = 1 - y_n < 1$ as required. $\square$

5.11.125 Problem. Two sequences of integers $a_l, a_2, a_3, ...$ and $b_l, b_2, b_3, ...$ are defined uniquely by the equality $(2+\sqrt{3})^n = a_n + b_n\sqrt{3}$. Compute $\lim_n(a_n/b_n)$.

5.11.125.1 Solution. Expanding $(2+\sqrt{3})^n$ and $(2-\sqrt{3})^n$ binomially, we obtain in both expressions the same coefficients of terms involving the even powers of $\sqrt{3}$, whereas those involving the odd powers of $\sqrt{3}$ differ in sign. Therefore the equality $(2+\sqrt{3})^n = a_n + b_n\sqrt{3}$ implies

$$(2-\sqrt{3})^n = a_n - b_n\sqrt{3}.$$

Since $0 < 2-\sqrt{3} < 1$, and $b_n \geq 1 \; \forall \; n$ so, $(2-\sqrt{3})^n \to 0$ and $1/b_n$ is bounded, hence

$$\frac{a_n}{b_n} = \frac{b_n\sqrt{3} + (2-\sqrt{3})^n}{b_n} = \sqrt{3} + \frac{(2-\sqrt{3})^n}{b_n} \to \sqrt{3}.$$

5.11.126 Problem. For each positive integer n, let

$$\begin{aligned} a_n &= 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + ... + \frac{1}{\sqrt{n}} - 2\sqrt{n} \\ b_n &= 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + ... + \frac{1}{\sqrt{n}} - 2\sqrt{n+1} \end{aligned}$$

Prove that the sequences (a_n) and (b_n) converge.

5.11.126.1 Solution. Observe that

$$\begin{aligned} a_{n+1} - a_n &= \frac{1}{\sqrt{n+1}} - 2\sqrt{n+1} + 2\sqrt{n} \\ &= \frac{1}{\sqrt{n+1}} - 2\frac{1}{\sqrt{n+1}+\sqrt{n}} \\ &= \frac{\sqrt{n+1}+\sqrt{n}-2\sqrt{n+1}}{\left(\sqrt{n+1}+\sqrt{n}\right)\sqrt{n+1}} \\ &= \frac{\sqrt{n}-\sqrt{n+1}}{\left(\sqrt{n+1}+\sqrt{n}\right)\sqrt{n+1}} < 0 \\ \text{and } \; b_{n+1} - b_n &= \frac{1}{\sqrt{n+1}} - 2\sqrt{n+2} + 2\sqrt{n+1} > 0. \end{aligned}$$

Thus a_n is a strictly decreasing sequence and b_n is an strictly increasing sequence. Now, we see that $a_n - b_n > 0$ and

$$b_1 < b_2 < ... < b_n < .... < a_n < .... < a_2 < a_1.$$

Thus by monotone convergence theorem, the sequences (a_n) and (b_n) converge. □

5.11.127 Problem. True or false?(Justify.)

1. Suppose (a_{n_k}) and (a_{n_p}) are two subsequences of a sequence (a_n), and they converge to a and b respectively, then $a = b$.

2. If (a_n) diverges then every subsequence diverges.

3. If (a_n) is unbounded then every subsequence is unbounded.

4. Give an example of a sequence (a_n) such that a_{n+1} is not necessarily smaller than a_n, but (a_n) converges.

5.11.127.1 Solution.

1. False: Consider the sequence (a_n) defined by $a_n = (-1)^n; n \in \mathbb{N}$, then a_{2n-1} and a_{2n} both converges to -1 and 1 respectively, but $-1 \neq 1$.

2. False: Consider the sequence (a_n) defined by

$$a_n = \begin{cases} n, & \text{if } n \text{ is odd.} \\ \frac{1}{n}, & \text{if } n \text{ is even.} \end{cases}$$

3. False: Consider the sequence (a_n) defined by $a_n = n^{(-1)^n}$.

4. Consider

$$a_n = \begin{cases} 1, & \text{if } n \text{ is odd} \\ 1 - \frac{1}{n}, & \text{if } n \text{ is even.} \end{cases}$$

□

5.11.128 Problem. Let (a_n) and (b_n) be two sequences and suppose that the set $\{n \in \mathbb{N}; a_n \neq b_n\}$ is finite. Prove that the sequences either both converge or both diverge.

5.11.128.1 Solution. Let $m = \max\{n \in \mathbb{N}; a_n \neq b_n\}$, then for $n > m, a_n = b_n$, i.e. they are ultimately identical. Hence the sequences either both converge or both diverge. □

5.11.129 Problem. Suppose that a sequence has a finite range. Under what conditions will the sequence converge?

5.11.129.1 Solution. Let $X = (x_n)$ be a sequence with finite range $\{x_1, x_2, .., x_k\} = A$ (say). Let $t = \min_{i \neq j}\{|x_i - x_j|\}$, then for $t > \epsilon > 0 \ \exists\ n_0 \in \mathbb{N}$ such that $|x_m - x_n| < \epsilon \ \forall\ m, n > n_0 \Rightarrow x_m = x_n = x_p$ (say) for some $x_p \in A$ i.e. $X = (x_n)$ is ultimately a constant sequence. □

5.11.130 Problem. Let (a_n) and (b_n) be two sequences and suppose that (r_n) is a sequence of positive numbers that converges to 0. Let $0 < |a_k - b_k| < r_k$ for each positive integer k.

1. Give an example to show that the sequences (a_n) and (b_n) may not converge.
2. If (a_n) converges to L, then (b_n) converges to L.

5.11.130.1 Solution.

1. $a_n = (-1)^n \left(1 + \frac{1}{n}\right),\ b_n = (-1)^n$.

2. Let $\epsilon > 0$, then there exists n_0 such that $n > n_0 \Rightarrow |x_n - L| < \epsilon/2$. Again, $r_k \to 0 \Rightarrow \exists\ n_1 \in \mathbb{N}$ such that $r_n < \epsilon/2 \ \forall\ n > n_1$. Let $n_2 = \max\{n_0, n_1\}$. Hence, $0 < |a_k - b_k| < r_k$ for each positive integer k implies $|b_k - L| = |b_k - a_k - L + a_k| \leq |b_k - a_k| + |a_k - L| \leq r_k + +|a_k - L| < \epsilon/2 + \epsilon/2 = \epsilon$, whenever $k > \max\{n_0, n_1\}$. Hence (b_n) converges to L. □

5.11.131 Problem. If (a_n) converges to a and $a > 0$. Then show that there exist positive number m and a positive integer q such that $a_n > m,\ \forall\ n \geq q$.

5.11.131.1 Solution. $a_n \to a \Rightarrow \forall \epsilon > 0 \, \exists n_0 \in \mathbb{N}$ such that $n \geq n_0 \Rightarrow a_n \in (a - \epsilon, a + \epsilon)$. Now choose $m = a - \epsilon$ and $q = n_0$, the result follows. □

5.11.132 Problem. Let (x_n) be a sequence of distinct elements in $\mathbb{R}$, and suppose that $x_i \to x$. Let f be a one-to-one map of the set of x_i's into itself. Prove that $f(x_i) \to x$.

5.11.132.1 Solution. Since $x_n \to x$, so, for $\epsilon > 0 \, \exists N \in \mathbb{N}$ such that $n \geq N \Rightarrow x_n \in B(x; \epsilon)$. Again, let $m = \max\{i; f(x_i) \in \{x_1, x_2, ..., x_N\}\}$ and $M = \max\{N, m\}$. Thus $n \geq M \Rightarrow f(x_n) \in \{x_N, x_{N+1}, x_{N+2}, ...\} \subseteq B(x; \epsilon)$. Hence $f(x_n) \to x$. □

5.11.133 Problem. Let (a_n) be a null sequence in $\mathbb{R}$, let $\sigma : \mathbb{N} \to \mathbb{N}$ be injective and define $b_n = a_{\sigma(n)}$ (in other words, the n-th term of the new sequence is the $\sigma(n)$-th term of the old sequence). Prove that (b_n) is also a null sequence; in particular, every rearrangement of a null sequence is null.

5.11.133.1 Solution. Here (a_n) is a null sequence in $\mathbb{R}$ implies $\forall \epsilon > 0 \, \exists N \in \mathbb{N}$ such that $n \geq N \Rightarrow |a_n| < \epsilon$. Let $M = \max\{m \in \mathbb{N}; \sigma(m) \leq N\}$, then $m \geq M \Rightarrow \sigma(m) \geq N$ and then $|a_{\sigma(m)}| < \epsilon \Rightarrow |b_m| < \epsilon$, which shows that (b_n) is also a null sequence. □

5.11.134 Problem. If (a_n) converges to a and $\alpha : \mathbb{N} \to \mathbb{N}$ is a bijection. Then show that $a_{\alpha(n)}$ also converges to a.

5.11.134.1 Solution. Since (a_n) converges to a, so $(a_n - a)$ is a null sequence. Then proceed as above. □

5.11.135 Problem. Let $x_0 = a, x_1 = b$. Find $\lim x_n$ in the following

1. $x_n = \left(1 - \frac{1}{n}\right) x_{n-1} + \frac{1}{n} x_{n-2} \; (n = 2, 3, ...)$
2. $x_{n+1} = \left(1 - \frac{1}{2n}\right) x_n + \frac{1}{2n} x_{n-1} \; (n = 1, 2, 3, ...)$

5.11.135.1 Solution.

1.

$$\begin{aligned}
x_n &= \left(1 - \frac{1}{n}\right) x_{n-1} + \frac{1}{n} x_{n-2} \\
\Rightarrow x_n - x_{n-1} &= -\frac{1}{n} \left(x_{n-1} - x_{n-2}\right) \\
\Rightarrow x_{n-1} - x_{n-2} &= -\frac{1}{n-1} \left(x_{n-2} - x_{n-3}\right) \\
\Rightarrow x_{n-2} - x_{n-3} &= -\frac{1}{n-2} \left(x_{n-3} - x_{n-4}\right) \\
&\ldots\ldots\ldots\ldots\ldots\ldots \\
\Rightarrow x_4 - x_3 &= -\frac{1}{4} \left(x_3 - x_2\right) \\
\Rightarrow x_3 - x_2 &= -\frac{1}{3} \left(x_2 - x_1\right) \\
\Rightarrow x_2 - x_1 &= -\frac{1}{2} \left(x_1 - x_0\right)
\end{aligned}$$

on multiplication, we get $x_n - x_{n-1} = (-1)^{n-1} \frac{1}{n!} \left(x_1 - x_0\right)$.
Hence $|x_{n+p} - x_n| \leq |x_{n+p} - x_{n+p-1}| + |x_{n+p-1} - x_{n+p-2}| + ... + |x_{n+1} - x_n|$, then proceed to show that (x_n) is a Cauchy sequence.

2. Left to the reader. $\square$

5.11.136 Problem. Give an example of sequences (x_n) and (y_n) such that $\{x_n : n \in \mathbb{N}\} \subseteq \{y_n : n \in \mathbb{N}\}$ but (x_n) is not a subsequence of (y_n).

5.11.136.1 Solution. Consider the sequences $X = \left(1, \frac{1}{2}, \frac{1}{3}, 1, 1, 1, ...\right)$ and
$Y = \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, .., \frac{1}{n}, ..\right)$. $\square$

5.11.137 Problem. Give an example of a sequence such that $\mathbb{Z}$ is exactly the set of all subsequential limits.

5.11.137.1 Solution. Consider the sequence
$S = (0, -1, 0, 1, -2, -1, 0, 1, 2, -3, -2, -1, 0, 1, 2, 3, ..)$ $\square$

5.11.138 Problem. Find a sequence (x_n) with the property: for every positive integer p, there exists a subsequence of (x_n) that converges to p.

5.11.138.1 Solution. Hint: Consider the sequence $(1, 1, 2, 1, 2, 3, 1, 2, 3, 4, ..)$. $\square$

5.11.139 Problem. Give an example of a sequence such that set of subsequential limits is the point 0 along with the set $\left\{\frac{1}{n}; n \in \mathbb{Z}\right\}$.

5.11.139.1 Solution. Consider the set $S = \left\{\frac{1}{m} + \frac{1}{n}; m, n \in \mathbb{Z} \setminus \{0\}\right\}$ and arrange them in a sequence. $\square$

5.11.140 Problem. Explain why there is no sequence whose set of subsequential limits is $\left\{\frac{1}{n}; n \in \mathbb{Z}\right\}$.

5.11.140.1 Solution. We can show the set of all subsequential limits is a closed set, but the given set is not closed. $\square$

5.11.141 Problem. Prove the following generalization of the property of density of $\mathbb{Q}$ in $\mathbb{R}$: Let (a_n) be a sequence of positive numbers whose infimum is 0. Then the set $\{ma_n; m \in \mathbb{Z}, n \in \mathbb{N}\}$ is dense in $\mathbb{R}$. Infer from this fact that $\mathbb{R} \setminus \mathbb{Q}$ is also dense in $\mathbb{R}$.

5.11.141.1 Solution. Let $0 < a < b$. Since $b - a > 0$ and $\inf a_n = 0$, so $\exists\ a_m$ such that $a_m < a$ and $a_m < b - a$. Now, we claim that, there exists $k \in \mathbb{N}$ such that $a < ka_m < b$. If possible, let $\forall\ n \in \mathbb{N},\ na_m \notin (a, b)$, then let $k = \sup\{n; na_m < a\}$, hence

$$\begin{aligned}
&ka_m < a \text{ and } (k+1)a_m > b \\
\Rightarrow &a_m > b - ka_m > b - a,
\end{aligned}$$

a contradiction. In particular, for the sequence $\left(\frac{1}{n}\right)$, let $A = \left\{\frac{1}{n}; n \in \mathbb{N}\right\}$, then observe that $\mathbb{Q} = A\mathbb{Z}$. $\square$

5.11.142 Problem. Let $\mathbb{N}$ denote the set of all positive integers. Find the set of limit points of $S = \left\{\sqrt{a} - \sqrt{b};\ a, b \in \mathbb{N}\right\}$ and prove that S is dense in $\mathbb{R}$.

5.11.142.1 Solution. For a, b and $n \in \mathbb{N}$ taking $x = an^2, y = bn^2 \Rightarrow\ \sqrt{x} - \sqrt{y} = n(\sqrt{a} - \sqrt{b}) \in S$, so, the set contains all integral multiples of each of its members. Note also that

$$\begin{aligned}
&\left(\sqrt{n+1} - \sqrt{n}\right)\left(\sqrt{n+1} + \sqrt{n}\right) = 1 \\
\Rightarrow &0 < \sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{2\sqrt{n}}.
\end{aligned}$$

Let $\epsilon > 0$. Thus it is sufficient to show that $(0, \epsilon) \cap S \neq \emptyset$. Now $\epsilon > 0$ implies $\exists\ N \in \mathbb{N}$ such that $0 < \dfrac{1}{2\sqrt{N}} < \epsilon \Rightarrow\ \sqrt{N+1} - \sqrt{N} < \epsilon$, shows the desired result. $\square$

5.11.143 Problem. (Generalization of the result of the above problem:) If (a_n) and (b_n) are unbounded monotonic increasing sequences of real numbers and $\lim_{n\to\infty}(a_{n+1} - a_n) = 0$, then the set $D = \{a_m - b_n;\ m, n \in \mathbb{N}\}$ is dense in $\mathbb{R}$.

5.11.143.1 Solution. Let $r \in \mathbb{R}$ and $\epsilon > 0$. Then $\exists\ k \in \mathbb{N}$ such that

$$a_{n+1} - a_n < \epsilon\ \forall\ n \geq k.$$

Since (b_n) is an unbounded monotonic increasing sequence, so $\exists\ m \in \mathbb{N}$ such that $b_m \geq a_k - r$ i.e. $b_m + r \geq a_k$. Since a_k is increasing, so it must lie between a_p and a_{p+1} for some $p \geq k$ and thus $a_p \leq b_m + r < a_{p+1}$. This gives $|(a_p - b_m) - r| < \epsilon$. Hence $a_p - b_m \in (r - \epsilon, r + \epsilon)$, and required result follows.
In the above problem $a_n = b_n = \sqrt{n}$ satisfies the assumption of the problem. $\square$

5.11.144 Problem. Let $a_n = 1 + \frac{1}{2} + ... + \frac{1}{n}$ and $b_n = \ln n$, then show that the set $\{a_m - b_n;\ m, n \in \mathbb{N}\}$ is dense in $\mathbb{R}$.

5.11.144.1 Solution. Apply the result of the previous problem. $\square$

5.11.145 Problem. For a sequence of real numbers (x_n) establish the following:

1. If $x_{n+1} - x_n \to x \in \mathbb{R}$ then $x_n/n \to x$.
2. If (x_n) is bounded and $2x_n \leq x_{n+1} + x_{n-1}$ holds for all $n = 2, 3, ...$ then
$$x_{n+1} - x_n \to 0.$$

5.11.145.1 Solution.

1. Suppose $y_n = x_{n+1} - x_n \to x$, then by Cauchy's limit theorem
$$\frac{1}{n}\sum_{i=1}^{n} y_i \to x \Rightarrow \frac{1}{n}(x_{n+1} - x_1) \to x.$$
Since $x_1/n \to 0$, we have $\frac{1}{n}x_{n+1} \to x$, we note that
$$\frac{x_n}{n} = \frac{x_n}{n-1}\frac{n-1}{n} \to x.1 = x.$$
2. The condition $2x_n \leq x_{n+1} + x_{n-1}$ can be rewritten as $x_n - x_{n-1} \leq x_{n+1} - x_n$ for each $n =$ 2, 3,..., which implies that the bounded sequence $(x_{n+1} - x_n)$ is an increasing sequence, and hence convergent. Let $x_{n+1} - x_n \to x$ in $\mathbb{R}$. By part (1), we have $\frac{x_n}{n} \to x$. But, since (x_n) is a bounded sequence, $\frac{x_n}{n} \to 0$. Therefore, $x = 0$, and so $x_{n+1} - x_n \to 0$. $\square$

5.11.146 Problem. Consider a sequence (a_n) of real numbers such that $a_n \to l$ (finite) as $n \to \infty$, then prove that $a_n - a_{n-1} \to 0$ as $n \to \infty$.

5.11.146.1 Solution. Let $\epsilon > 0$, then $\exists\, N \in \mathbb{N}$ such that $|a_n - l| < \epsilon/2$, whenever $n \geq N+1$, hence

$$|a_n - a_{n-1}| = |a_n - l + l - a_{n-1}| \leq |a_n - l| + |l - a_{n-1}| < \epsilon/2 + \epsilon/2 = \epsilon,$$

whenever $n \geq N$. Thus $a_n - a_{n-1} \to 0$ as $n \to \infty$.

5.11.147 Problem. Consider a sequence (a_n) real numbers such that $a_n + a_{n+1} \to \mu$ and $a_n + a_{n+2} \to \mu'$ as $n \to \infty$, where μ, μ' are real numbers, then prove that $\mu = \mu'$ and $a_n \to \frac{1}{2}\mu$.

5.11.147.1 Solution. Given that, $a_n + a_{n+1} \to \mu$ and $a_n + a_{n+2} \to \mu'$. So,

$$\begin{aligned}
&2a_n = (a_n + a_{n+1}) + (a_n + a_{n+2}) - (a_{n+1} + a_{n+2})\\
\Rightarrow &\lim_{n\to\infty} 2a_n = \lim_{n\to\infty}(a_n + a_{n+1}) + \lim_{n\to\infty}(a_n + a_{n+2}) - \lim_{n\to\infty}(a_{n+1} + a_{n+2})\\
\Rightarrow &2\lim_{n\to\infty} a_n = \mu + \mu' - \mu = \mu'\\
\Rightarrow &\lim_{n\to\infty} a_n = \frac{1}{2}\mu'
\end{aligned}$$

Since $\lim_{n\to\infty}(a_{n+1} + a_{n+2}) = \mu$, hence $\mu'/2 + \mu'/2 = \mu \Rightarrow \mu = \mu'$. □

5.11.148 Problem. (1988 Nanchang City Math Competition) Let (a_n) be a sequence of real numbers such that $a_n - a_{n-2} \to 0$. Prove that

1. $\dfrac{a_n}{n} \to 0$.
2. $\lim_{n\to\infty} \dfrac{a_n - a_{n-1}}{n} = 0$.
3. Give an example of a sequence satisfying the conditions of the above problem.

5.11.148.1 Solution.

1. We take two subsequences (a_{2k+1}) and (a_{2k}) of (a_n). We apply Stoltz's theorem to the sequences (a_{2k+1}) with $(2k+1)$ and (a_{2k}) with $(2k)$ Then

$$\lim_{k\to\infty} \frac{a_{2k+1} - a_{2k-1}}{(2k+1) - (2k-1)} = 0 \text{ and}$$
$$\lim_{k\to\infty} \frac{a_{2k} - a_{2k-2}}{2} = 0$$
$$\text{and thus } \lim_{k\to\infty} \frac{a_{2k+1}}{2k+1} = 0, \quad \lim_{k\to\infty} \frac{a_{2k}}{2k} = 0.$$

Therefore, the two subsequences $\left(\frac{a_{2k+1}}{2k+1}\right), \left(\frac{a_{2k}}{2k}\right)$ of $\left(\frac{a_n}{n}\right)$, one odd and other even converges to 0. Hence $\lim_{n\to\infty} \frac{a_n}{n} = 0$.

2. For $\epsilon > 0$, then $\exists\, k \in \mathbb{N}$ such that $|a_n - a_{n-2}| < \epsilon\ \forall\, n \geq k$. Observe that for $n \geq k$,

$$\begin{aligned}
a_n - a_{n-1} = (a_n - a_{n-2}) - (a_{n-1} - a_{n-3}) + (a_{n-2} - a_{n-4}) - \ldots.\\
+ \{(a_{k+2} - a_k) - (a_{k+1} - a_{k-1})\}.
\end{aligned}$$

Thus $|a_n - a_{n-1}| \leq (n-k)\epsilon + |a_{k+1} - a_{k-1}|$ and so $\lim_{n\to\infty} \dfrac{a_n - a_{n-1}}{n} = 0$.

3. Consider the sequence $a_n = (-1)^n$. $\square$

5.11.149 Problem. Prove that limit point of the range $\{x_n; n \in \mathbb{N}\}$ of a sequence (x_n) is a subsequential limit of the sequence. But a subsequential limit may not be the limit point of $\{x_n; n \in \mathbb{N}\}$. Give an example.

5.11.149.1 Solution. Let l be a limit point of $A = \{x_n; n \in \mathbb{N}\}$ and $\epsilon > 0$, then $A \cap \hat{B}(l;\epsilon) \neq \emptyset$. Let $T = \{n \in \mathbb{N}; x_n \in \hat{B}(l;\epsilon)\}$. By well ordering principle, $\inf T = n_1$ (say) and in this way we can choose $n_i = \inf\{T \setminus \{n_1, .., n_{i-1}\}\}$ and $x_{n_i} \in A \cap \hat{B}(l;\epsilon/2^i)$ for every $i \in \mathbb{N}$, thus (x_{n_i}) is a subsequence of (x_n). Again, $x_{n_k} \in \hat{B}(l;\epsilon/2^i)$ for $k \geq i$, i.e. $|x_{n_k} - l| < \epsilon/2^i < \epsilon \; \forall \; k \geq i$, shows that $(x_{n_i}) \to l$. Example: $x_n = (-1)^n$. $\square$

5.11.150 Problem. Let (x_n) be a sequence in $\mathbb{R}$ such that $x_{n+1} - x_n \to 0$. Prove that the set of limits of its convergent subsequences is the interval with end points $\underline{\lim} x_n$ and $\overline{\lim} x_n$.

5.11.150.1 Solution. Suppose that $\underline{\lim} x_n = l$ and $\overline{\lim} x_n = L$. Since $x_{n+1} - x_n \to 0$, so for all $\epsilon_1 > 0$, there exists $k \in \mathbb{N}$ such that

$$-\epsilon_1 < x_{n+1} - x_n < \epsilon_1 \; \forall \; n \geq k. \tag{1}$$

It is clear that $l < L$. Let $S = \{x_n; n \in \mathbb{N}\}$. We shall show that $S' = [l, L]$. Let $\lambda \in [l, L]$ be an isolated point of S, then $\exists \; \epsilon, \; \epsilon_1 > \epsilon > 0$ such that $(\lambda - \epsilon, \lambda + \epsilon)$ contains no points of S. Let $T = \{n \in \mathbb{N}; x_n \geq \lambda + \epsilon\}$. Now, let $n \in T$. We claim $n + 1 \in T$, otherwise $x_{n+1} \leq \lambda - \epsilon$, which implies

$$\begin{aligned} &x_{n+1} \leq \lambda - \epsilon \text{ and } x_n \geq \lambda + \epsilon \\ \Rightarrow &x_{n+1} - x_n \leq -2\epsilon < -\epsilon < -\epsilon_1 \end{aligned}$$

contradicts (1). Thus $n + 1 \in T$. Hence, at least one nbhd. of l contains only finite number of points of S and l cannot be a subsequential limit of (x_n), a contradiction that l is subsequential limit of (x_n), Thus λ is a limit point of S.

5.11.150.2 Solution. (Arnab Roy,VI-th semester.) Suppose that $\underline{\lim} x_n = l$ and $\overline{\lim} x_n = L$. Then there exist functions $p, q : \mathbb{N} \to \mathbb{N}$ such that $p_k > q_k \; \forall \; k \in \mathbb{N}$ and $x_{p_k} \to L, x_{q_k} \to l$. Clearly $l < L$, and let $l < \lambda < L$. Choose $\epsilon > 0$ such that

$$l < l + \epsilon < \lambda - \epsilon < \lambda + \epsilon < L - \epsilon < L.$$

Hence $\exists \; t \in \mathbb{N}$ such that $k \geq t$ implies

$$\begin{aligned} &l - \epsilon < x_{q_k} < l + \epsilon, \; L - \epsilon < x_{p_k} < L + \epsilon, \\ &\text{also } \exists \; N \in \mathbb{N} \text{ such that } n \geq N \Rightarrow \; -\epsilon < x_{n+1} - x_n < \epsilon. \end{aligned}$$

Again, we can take t so large that $p_t, q_t \geq N$. Clearly

$$x_{p_t} > L - \epsilon > \lambda + \epsilon \text{ and } x_{q_t} < l + \epsilon < \lambda - \epsilon, \text{ so } x_{p_t}, x_{q_t} \notin (\lambda - \epsilon, \lambda + \epsilon). \tag{1}$$

Now, we claim that $\exists \; n \in \mathbb{N}$ with $q_t < n < p_t$ such that $x_n \in (\lambda - \epsilon, \lambda + \epsilon)$. If possible, let

$$x_n \notin (\lambda - \epsilon, \lambda + \epsilon), \; \forall \; q_t \leq n \leq p_t. \tag{2}$$

Suppose that

$$T = \{n; q_t \le n \le p_t, x_n \le \lambda - \epsilon\}.$$

We see that $T \neq \emptyset$ as $x_{q_t} \notin (\lambda - \epsilon, \lambda + \epsilon)$, as $x_{p_t} > \lambda + \epsilon$ so $p_t \notin T$. Again, we claim that, if $n \in T$ then $n+1 \in T$. If not, then

$$\begin{aligned} x_n \le \lambda - \epsilon &\text{ and } \lambda + \epsilon \le x_{n+1} \\ \Rightarrow & x_{n+1} - x_n \ge 2\epsilon > \epsilon, \end{aligned}$$

a contradiction to our assumption. Hence $n+1 \in T$.
Then after finite number of steps, we can say $p_t \in T$ which contradicts the fact $x_{p_t} > \lambda + \epsilon$. Thus (2) is false. Hence there exists $n_\epsilon, q_t \le n_\epsilon \le p_t$ with $x_{n_\epsilon} \in (\lambda - \epsilon, \lambda + \epsilon)$. This is true for all sufficiently large m where $\frac{1}{m} < \epsilon < \frac{1}{2}\min\{\lambda - l, L - \lambda\}$. Now, let m_1 be the smallest such m, then $\forall\ m \ge m_1$, we get $x_{n_m} \in \left(\lambda - \frac{1}{m}, \lambda + \frac{1}{m}\right)$. Now, we construct a subsequence of (x_n) in the following way,

$$\forall\ i \in \mathbb{N}, n_i = n_{m_1} + i - 1, \text{ then } x_{n_i} \in \left(\lambda - \frac{1}{n_{m_1} + i - 1}, \lambda + \frac{1}{n_{m_1} + i - 1}\right).$$

Hence $(x_{n_i}) \to \lambda$. □

5.11.151 Problem. Suppose that the sequence (x_n) of real numbers is such that $x_{n+1} + x_n \to 0$. Prove that the set of limits of convergent subsequences of this sequence is either infinite or contains at most two points.

5.11.151.1 Solution. (Arnab Roy,VI-th Semester) It is given that $x_{n+1} + x_n \to 0$. So

$$x_{n+2} - x_n = (x_{n+2} + x_{n+1}) - (x_{n+1} + x_n) \to 0. \tag{A}$$

Assume that the sequence (x_n) has finitely many subsequential limits. Now, consider the two subsequences $(y_n), (z_n)$ with $y_n = x_{2n-1}, z_n = x_{2n}$. Thus by (A), we get $y_{n+1} - y_n \to 0$. and $z_{n+1} - z_n \to 0$.
Suppose that (y_n) is not convergent, then (y_n) has at least two subsequential limits l and L (say) and $l < L$. Hence, (A) and the previous problem shows that (y_n) has infinitely many subsequential limits, i.e. $[l, L]$. Since (y_n) is a subsequence of (x_n) so, (x_n) will have infinitely many subsequential limits which contradicts our assumption. Threrefore (y_n) in convergent. Similarly (z_n) is convergent. Let $y_n \to m, z_n \to M$. Now, for any subsequence (x_{n_k}) of (x_n) either the indices are eventually odd or even or infinitely many are odd and even. Thus (x_{n_k}) either converges to m, M or does not converge at all. Hence (x_n) can have at most two subsequential limits. □

5.11.152 Problem. If $x_{n+1} - \frac{1}{2}x_n \to 0$ then prove that $x_n \to 0$.

5.11.152.1 Solution. Let $y_n = x_{n+1} - \frac{1}{2}x_n$, then

$$\frac{y_1}{2^{n-1}} = \frac{1}{2^{n-1}}\left(x_2 - \frac{1}{2}x_1\right) = \frac{x_2}{2^{n-1}} - \frac{x_1}{2^n}$$
$$\frac{y_2}{2^{n-2}} = \frac{1}{2^{n-2}}\left(x_3 - \frac{1}{2}x_2\right) = \frac{x_3}{2^{n-2}} - \frac{x_2}{2^{n-1}}$$
$$\ldots\ldots\ldots$$
$$\frac{y_k}{2^{n-k}} = \frac{1}{2^{n-k}}\left(x_{k+1} - \frac{1}{2}x_k\right) = \frac{x_{k+1}}{2^{n-k}} - \frac{x_k}{2^{n-k+1}}$$

$$\text{summing up, we get } \sum_{k=1}^{n}\frac{y_k}{2^{n-k}} = x_{n+1} - \frac{x_1}{2^n}$$

$$\text{that is, } x_{n+1} = \sum_{k=1}^{n}\frac{y_k}{2^{n-k}} = \sum_{k=1}^{m}\frac{y_k}{2^{n-k}} + \sum_{k=m+1}^{n}\frac{y_k}{2^{n-k}} + \frac{x_1}{2^n}.$$

Since $y_k \to 0$, for large m the second sum is small for all $n > m$. For fixed m the first sum is arbitrarily small if n is sufficiently large. Thus $\lim_{n\to\infty} x_{n+1} = 0$. □

5.11.153 Problem. Define $a_1 = 1, a_2 = 7$ and $a_{n+2} = \dfrac{a_{n+1}^2 - 1}{a_n}$ for positive integer n. Prove that $9a_na_{n+1} + 1$ is a perfect square for every positive integer n.

5.11.153.1 Solution. Observe that

$$\begin{aligned}
&a_{n+1}^2 - a_n^2 = a_{n+2}a_n - a_{n+1}a_{n-1}\\
\Rightarrow &a_{n+1}^2 + a_{n+1}a_{n-1} = a_n^2 + a_{n+2}a_n\\
\Rightarrow &a_{n+1}\left(a_{n+1} + a_{n-1}\right) = a_n\left(a_{n+2} + a_n\right)\\
\Rightarrow &\frac{(a_{n+1} + a_{n-1})}{a_n} = \frac{(a_{n+2} + a_n)}{a_{n+1}}\ \forall\ n = 2, 3, \ldots.
\end{aligned}$$

Let (x_n) be a sequence defined by $x_n = \frac{(a_{n+2}+a_n)}{a_{n+1}}$, we see that $x_n = x_{n+1} = x_{n+2} = \ldots..$ Thus x_n is constant and equals to k (say), so by $a_3 + a_1 = ka_2$ and $a_3a_1 = a_2^2 - 1$, we get $a_3 = 48$ and $k = 7$. Hence, we get the relation $a_{n+2} + a_n = 7a_{n+1}$. Next, writing out the first few terms of $9a_na_{n+1} + 1$ will suggest that $9a_na_{n+1} + 1 = (a_n + a_{n+1})^2$. The case $n = 1$ is true as $9.7 + 1 = (1 + 7)^2$. Suppose this is true for $n = k$. Using the recurrence relations, we get

$$a_{k+1}^2 - 1 = a_{k+2}a_k = (7a_{k+1} - a_k)a_k = 7a_{k+1}a_k - a_k^2$$
$$2a_{k+1}^2 - 2 = 14a_{k+1}a_k - 2a_k^2 \tag{5.6}$$

we get the case $n = k + 1$ as follows:

$$\begin{aligned}
9a_{k+1}a_{k+2} + 1 &= 9a_{k+1}(7a_{k+1} - a_k) + 1\\
&= 63a_{k+1}^2 - 9a_{k+1}a_k + 1\\
&= 63a_{k+1}^2 - \left(a_{k+1} + a_k\right)^2 + 2\\
&= 62a_{k+1}^2 - 2a_{k+1}a_k - a_k^2 + 2\\
&= 64a_{k+1}^2 - 16a_{k+1}a_k + a_k^2 \quad \text{by (5.6)}\\
&= \left(8a_{k+1} - a_k\right)^2 = \left(a_{k+1} + a_{k+2}\right)^2.
\end{aligned}$$

Thus the relation is true for $n = k + 1$. Hence the result follows. $\square$

5.11.154 Problem.

1. Give an example of a bounded (unbounded) divergent sequence (x_n) such that $x_{n+1} - x_n \to 0$.

2. Does there exist a bounded (unbounded) sequence (x_n) such that $x_{n+1} - x_n \to 0$, but the sequence $\left(\dfrac{x_1 + ... + x_n}{n}\right)$ does not have a limit?

5.11.154.1 Solution.

1. (a) Consider the sequence (x_n) defined by $x_n = \sin \log n$. Clearly (x_n) is bounded. Now

$$\begin{aligned} x_{n+1} - x_n &= \sin \log (n+1) - \sin \log n \\ &= 2 \cos \left(\frac{\log (n+1) + \log n}{2}\right) \sin \left(\frac{\log (n+1) - \log n}{2}\right) \\ &= 2 \cos \left(\frac{\log n(n+1)}{2}\right) \sin \left(\frac{\log (1 + 1/n)}{2}\right) \to 0 \text{ as } n \to \infty. \end{aligned}$$

   (b) Consider the sequence (x_n) defined by $x_n = \sum_{r=1}^{n} \frac{1}{r}$, and $x_{n+1} - x_n = \frac{1}{n+1} \to 0$.

2. (a) Left to the reader.

   (b) Consider the sequence (x_n) defined by $x_n = \sum_{r=1}^{n} \frac{1}{r}$, and

$$\begin{aligned} \sum_{r=1}^{n} x_r &= 1 + \left(1 + \frac{1}{2}\right) + .. + \left(1 + \frac{1}{2} + .. + \frac{1}{n}\right) \\ &= n.1 + (n-1)\frac{1}{2} + .. + (n - r + 1)\frac{1}{r} + .. + \frac{1}{n} \\ &= \sum_{r=1}^{n} (n - r + 1)\frac{1}{r} = \sum_{r=1}^{n} \frac{n+1}{r} - n \\ &= (n+1)x_n - n \\ \Rightarrow \quad \frac{1}{n}\sum_{r=1}^{n} x_r &= x_n + \frac{x_n}{n} - 1 \end{aligned}$$

   which does not tend to a finite limit, as $\frac{x_n}{n} \to 0$. $\square$

5.11.155 Problem.

1. Suppose that $(a_1 + ... + a_n)/n \to a$ and $(a_1^2 + ... + a_n^2)/n \to b$. Prove that $a^2 \le b \le a$ and that b can take any values between a^2 and a.

2. If (a_n) is a sequence in (0,1), show that

$$\frac{1}{n}\sum_{k=1}^{n} a_k \to 0 \Longleftrightarrow \frac{1}{n}\sum_{k=1}^{n} a_k^2 \to 0.$$

3. If (a_n) is a sequence in $\mathbb{R}$, show that

$$\frac{1}{n}\sum_{k=1}^{n} a_k^2 \to 0 \Rightarrow \frac{1}{n}\sum_{k=1}^{n} a_k \to 0.$$

4. Let $0 < a_n < 1, \forall\, n \in \mathbb{N}$. Show by examples that the existence of one of the limits

$$\lim_{n\to\infty} \frac{a_1 + .. + a_n}{n} \text{ or } \lim_{n\to\infty} \frac{a_1^2 + .. + a_n^2}{n}$$

does not imply the existence of the other.

5.11.155.1 Solution.

1. Applying Cauchy-Schwartz inequality to the following sets of numbers $\{a_1, a_2, ..., a_n\}$ and $\{1, 1, ..., 1\}$, we get

$$\sum_{k=1}^{n} a_k.1 \le \sqrt{\sum_{k=1}^{n} a_k^2 \sqrt{1^2 + 1^2 + ... + 1^2}}$$
$$\Rightarrow \frac{1}{n}\sum_{k=1}^{n} a_k \le \sqrt{\frac{1}{n}\sum_{k=1}^{n} a_k^2}.$$

Thus $a^2 \le b$.
Again, since (a_n) is a sequence in (0,1), so $a_k^2 \le a_k \ \forall\, k \in \mathbb{N}$ and hence

$$\frac{1}{n}\sum_{k=1}^{n} a_k^2 \le \frac{1}{n}\sum_{k=1}^{n} a_k$$
$$\Rightarrow b \le a.$$

Thus $a^2 \le b \le a$.

2. Applying the result of the above, the desired result follows.

3. We prove more general statement: If p is a positive integer and (a_n) is a sequence in $\mathbb{R}$, then

$$\frac{1}{n}\sum_{k=1}^{n} a_k^{2p} \to 0 \Rightarrow \frac{1}{n}\sum_{k=1}^{n} a_k \to 0.$$

We recall the inequality

$$\left(\frac{a_1 + a_2 + ... + a_n}{n}\right)^{2p} \le \frac{a_1^{2p} + a_2^{2p} + ... + a_n^{2p}}{n}.$$

It follows that

$$\left|\frac{a_1 + a_2 + ... + a_n}{n}\right| \le \sqrt[2p]{\frac{a_1^{2p} + a_2^{2p} + ... + a_n^{2p}}{n}} \to 0.$$

Hence the conclusion follows for $p = 1$.
The converse is not true. Take $a_n = (-1)^n$ and observe that

$$\frac{a_1 + a_2 + ... + a_n}{n} = \begin{cases} 0 & \text{if } n \text{ is even} \\ -\frac{1}{n} & \text{if } n \text{ is odd.} \end{cases}$$

Hence

$$\lim_{n\to\infty} \frac{a_1 + a_2 + ... + a_n}{n} = 0$$

but

$$\lim_{n\to\infty} \frac{a_1^2 + a_2^2 + ... + a_n^2}{n} = 1.$$

4. Left to the reader. □

5.11.156 Problem.

1. Let (x_n) be a sequence in $\mathbb{R}$. Assume that subsequences $(x_{2n}), (x_{2n+1})$ and (x_{3n}) are convergent. Show that (x_n) is convergent. But if assume that only two of these three subsequences are convergent, the result is no longer true.

2. Give an example of a sequence (x_n) that is not convergent but has the property that for every integer $k \geq 2$ the subsequence x_{kn} is convergent.

5.11.156.1 Solution.

1. Suppose $x_{2n} \to a, x_{2n+1} \to b$ and $x_{3n} \to c$. Now (x_{6n}) is a subsequence of both (x_{2n}) and (x_{3n}), hence (x_{6n}) is convergent and converges to a and $a = c$, again (x_{6n-3}) is a subsequence of both (x_{2n+1}) and (x_{3n}), hence (x_{6n-3}) is convergent and converges to b and $b = c$,thus $a = b = c$.

   (a) Consider $x_n = (-1)^n$. $x_{2k} \to 1, x_{2k+1} \to -1$ but x_{3k} does not converge.

   (b) Consider $x_n = \begin{cases} 0, & \text{if } n \text{ is prime,} \\ 1, & \text{if } n \text{ is composite.} \end{cases}$
   $x_{3k} \to 1, x_{2k} \to 1$ but x_{2k+1} does not converge.

   (c) Consider $x_n = \begin{cases} 0, & \text{if } n = 2^k, k = 0, 1, .., \\ 1, & \text{otherwise.} \end{cases}$
   $x_{3k} \to 1, x_{2k+1} \to 1$ but x_{2k} does not exist.

2. Consider $x_n = \begin{cases} 0, & \text{if } n \text{ is prime,} \\ 1, & \text{if } n \text{ is composite.} \end{cases}$
   Then every subsequence $(x_{kn}), k > 1, n > 1$ is a constant sequence and therefore it is convergent. □

5.11.157 Problem. Prove that the statement (i) $\lim_{h\to 0} |f(a+h) - f(a)| = 0$ implies the statement (ii) $\lim_{h\to 0} |f(a+h) - f(a-h)| = 0$. Give an example of a function f for which the converse is not true.

5.11.157.1 Solution. Now, since $\lim_{h\to 0}|f(a+h)-f(a)|=0$, so for every $\epsilon>0$ $\exists$ $r>0$ such that $|f(a+h)-f(a)|<\epsilon/2$ whenever $|h|<\epsilon$, hence

$$\begin{aligned}|f(a+h)-f(a-h)| &= |f(a+h)-f(a)-f(a-h)+f(a)|\\ &\le |f(a+h)-f(a)|+|f(a-h)-f(a)|\\ &< \epsilon/2+\epsilon/2=\epsilon.\end{aligned}$$

Hence the result.

For the other part, for $a\neq 0$, consider the function $f(x)=\begin{cases}|x| & \text{if } x\neq 0\\ a & \text{if } x=0.\end{cases}$

Now, $|f(0+h)-f(0-h)|=0$ but $|f(0+h)-f(0)|=|h|-a$ does not tend to 0 as $h\to 0$. $\square$

5.11.158 Problem. Give an alternative proof of the covergence of the sequence (v_n) given by $v_1=\sqrt{2}, v_{n+1}=\sqrt{v_n+2}$ using the inequality (to be established) $2-v_{n+1}=\frac{1}{3}(2-v_n)$.

5.11.158.1 Solution. $v_{n+1}=\sqrt{v_n+2}\implies 2-v_{n+1}=2-\sqrt{v_n+2}=\frac{2-v_n}{2+\sqrt{v_n+2}}<\frac{1}{3}(2-v_n)<\frac{1}{3^n}(2-v_1)$ as required. $\square$

5.11.159 Problem. For what real values of x do the sequences $(\cos nx), (\sin nx)$ converge?

5.11.159.1 Solution. Let $f_n(x)=\cos(nx)$. Suppose for some $x\in[0,2\pi]$, that $f_n(x)\to l$ as $n\to\infty$.Then $f_{n+1}(x)+f_{n-1}(x)=2\cos nx\sin x\implies 2l=2l\cos x$ as $n\to\infty$, so either $l=0$ or $\cos x=1$. But $f_{2n}(x)=\cos 2nx=2\cos^2 nx-1$ and so l must satisfy $l=2l^2-1$ and in particular $l\neq 0$. Hence for convergence we must have $\cos x=1\Rightarrow x=2k\pi$ for $k\in\mathbb{Z}$ then $f_n(x)=1$ $\forall$ $n\in\mathbb{N}$ and (f_n) converges. Hence $\cos nx$ has a limit iff $x=2k\pi$.
If $g_n(x)=\sin nx\to m$ as $n\to\infty$, then $g_{n+1}-g_{n-1}=2\cos nx\sin x\to 0$,so either $\sin x=0$, or $\cos nx\to 0$ which is impossible. Hence $\sin nx$ has a limit iff $x=k\pi$. $\square$

5.11.160 Problem. A sequence (x_n) defined by $x_{n+1}=\sqrt{ax_n^2+b}$ with $x_1=c$. Show that x_n converges whenever $0<a<1$ and $b>0$, hence find its limit.

5.11.160.1 Solution. Let $f(x)=\sqrt{ax^2+b}$, then

$$\begin{aligned}|f(x)-f(y)| &= \left|\sqrt{ax^2+b}-\sqrt{ay^2+b}\right|\\ &= \left|\frac{a(x+y)(x-y)}{\sqrt{ax^2+b}+\sqrt{ay^2+b}}\right|.\end{aligned}$$

Since $\left|\sqrt{ax^2+b}+\sqrt{ay^2+b}\right|\ge\sqrt{a}|x|+\sqrt{a}|y|\ge\sqrt{a}|x+y|$,
then $|f(x)-f(y)|\le\sqrt{a}|x-y|$ where $\sqrt{a}<1$ i.e., f is a contraction mapping, hence the limit of the sequence (x_n) is simply the fixed point of f. So solving $x=\sqrt{ax^2+b}$ we get

$$\lim_{n\to\infty}x_n=\sqrt{\frac{b}{1-a}}.$$

5.11.160.2 Solution. We see that

$$\begin{aligned}x_{n+1} &= \sqrt{a^nc^2+b(1+a+a^2+..+a^{n-1})}\\ &= \sqrt{a^nc^2+b\frac{1-a^n}{1-a}}\end{aligned}$$

Since $0 < a < 1, a^n \to 0$ as $n \to \infty$, thus

$$\lim_{n\to\infty} x_{n+1} = \sqrt{\frac{b}{1-a}}. \quad \square$$

5.11.161 Problem. Show that the following three are equivalent:

A. Let (x_n) be a bounded sequence in $\mathbb{R}$. A real number U is said to be the **limit superior or limsup**x_n if U satisfies the following conditions:

   (a) Given $\epsilon > 0 \ \exists \ m \in \mathbb{N}$ such that $n \geq m \Rightarrow x_n < U + \epsilon$.

   (b) Given $\epsilon > 0$ and $\forall \ p \in \mathbb{N} \ \exists \ n > p$ such that $x_n > U - \epsilon$.

B. Let (x_n) be a bounded sequence in $\mathbb{R}$. A real number V is said to be the **limit superior** if

$$V = \inf_{k\geq 1} \sup_{n\geq k} x_n.$$

C. If S be the set of all subsequential limits of (x_n), then **limit superior** of (x_n) is defined by $W = \sup S$.

5.11.161.1 Solution.

1. (A) implies (B)
   For each $k \in \mathbb{N}$. Let
   $$H_k = \{x_k, x_{k+1}, ...\} \text{ and } \sup H_k = h_k,$$
   then
   $$H_1 \supseteq H_2 \supseteq H_3 \supseteq ... \Rightarrow h_1 \geq h_2 \geq h_3 \geq ...$$
   Claim 1: U is a lower bound of $\{h_n; n \in \mathbb{N}\}$. If possible, let $h_p < U$ for some $p \in \mathbb{N}$. Then, let $\epsilon < U - h_p$, in this case by (b) $\exists \ q \in \mathbb{N}$ such that $q > p \Rightarrow x_q > U - \epsilon > h_p$ that is, $h_q > x_q > h_p$ contradicts that h_n is a decreasing sequence. Hence $h_n > U \ \forall \ n \in \mathbb{N}$. Thus U is a lower bound.
   Claim 2: U is the greatest lower bound of $\{h_n; n \in \mathbb{N}\}$. Let $\epsilon > 0$, then $\exists \ m \in \mathbb{N}$ such that $n \geq m \Rightarrow x_n < U + \epsilon$. i.e. $\sup H_m = h_m < U + \epsilon$. Thus $h_n < U + \epsilon \ \forall \ n \geq m$. Hence $\inf_{k\geq 1} h_k = U$ i.e., $U = \inf_{k\geq 1} h_k = \inf_{k\geq 1} \sup_{n\geq k} x_n = V$.

2. (B) implies (C)
   Let S be the set of all subsequential limits of (x_n). Suppose $s \in S$, then $\exists$ a subsequence (x_{n_k}) of (x_n) such that $x_{n_k} \to s$. Since $H_{n_k} \subseteq H_k$ implies $h_{n_k} \leq h_k$ and $x_{n_k} \in H_{n_k} \ \forall \ k \in \mathbb{N}$, hence $x_{n_k} \leq \sup H_{n_k} = h_{n_k}$ i.e., $\lim_{k\to\infty} x_{n_k} \leq \lim_{k\to\infty} h_{n_k} \Rightarrow s \leq V$. Which means that V is an upper bound of S. Again, since $V = \inf_n h_n$, so for $k \in \mathbb{N} \ \exists \ h_k < V + \frac{1}{k}$ and $x_{n_k} > h_k - \frac{1}{k} > V - \frac{1}{k}$. Then
   $$V - \frac{1}{k} < h_k - \frac{1}{k} < x_{n_k} < h_{n_k} < h_k < V + \frac{1}{k}.$$
   Hence $\lim_{k\to\infty} x_{n_k} = V \Rightarrow V \in S$. Thus $W = \sup S = V$.

3. (C) implies (A)
   Suppose $W = \sup S$, then $\forall\, \epsilon > 0\ \exists\, s_1 \in S$ such that $s_1 > W - \frac{\epsilon}{2}$ and there exists a subsequence (x_{n_k}) of (x_n) such that $x_{n_k} \to s_1$, i.e., $\exists\, m \in \mathbb{N}$ such that $k \geq m \Rightarrow x_{n_k} > s_1 - \frac{\epsilon}{2} > W - \epsilon$. We claim $T = \{x_n; x_n > W + \epsilon\}$ is finite. Suppose that T is infinite. Since T is bounded and infinite, by B-W theorem there exists a subsequence that converges to $l \in T$, i.e., $l > W + \epsilon > W$, which contradicts that $W = \sup S$. Now, consider the set $P = \{n \in \mathbb{N}; x_n \in T\}$ and let $\max P = m$. So for $n \geq m, x_n < W + \epsilon$. Thus W is the limit superior of (x_n). Hence $W = U$. □

5.11.162 Problem. If (u_n) is a bounded sequence of real numbers, then for each $\epsilon > 0$ the inequalities

$$u_k \geq \limsup u_n + \epsilon \text{ and } u_m \leq \liminf u_n - \epsilon$$

hold for finitely many k and finitely many m.

5.11.162.1 Solution. Assume that $u_k > \limsup u_n + \epsilon$ holds true for infinitely many k. Then there exists a subsequence (v_n) of (u_n) satisfying $v_n > \limsup u_n$ for each n. Since (u_n) is a bounded sequence, there exists a subsequence (w_n) of (v_n) (and hence of (u_n)) satisfying $w_n \to w \in \mathbb{R}$. We know that $w \geq \limsup u_n + \epsilon$, i.e., w is a cluster point of (u_n) which is greater than the largest cluster point ($\limsup u_n$) of (u_n), a contradiction. □

5.11.163 Problem. Let (u_n) be a bounded sequence of real numbers.

1. If $m \leq u_n \leq M\ \forall\, n \geq N$, then $\liminf u_n \geq m$ and $\limsup u_n \leq M$.
2. If $\beta > \limsup u_n$, then $\exists\, N \in \mathbb{N}$ such that $u_n < \beta\ \forall\, n \geq N$.
3. If $\alpha < \liminf u_n$, then $\exists\, N \in \mathbb{N}$ such that $u_n > \alpha\ \forall\, n \geq N$.
4. The sequence (u_n) converges iff $\liminf u_n = \limsup u_n$.
5. Both $\limsup(c + u_n) = c + \limsup u_n$ and $\liminf(c + u_n) = c + \liminf u_n$ are valid for any real number c.
6. If $c > 0$, then $\limsup(cu_n) = c \limsup u_n$ and $\liminf(cu_n) = c \liminf u_n$.
7. If $c < 0$, then $\limsup(cu_n) = c \liminf u_n$ and $\liminf(cu_n) = c \limsup u_n$.
8. For each $\epsilon > 0$, both sets $\{n \in \mathbb{N}; u_n < \liminf u_n + \epsilon\}$ and $\{n \in \mathbb{N}; u_n > \limsup u_n - \epsilon\}$ are infinite.

5.11.163.1 Solution. Left to the reader. □

5.11.164 Problem. Let $\mathbb{N} = \{1, 2, 3, ...\}$. Show that there exists an uncountable family $\mathcal{A}$ of infinite subsets of $\mathbb{N}$, such that for $A, B \in \mathcal{A}; A \cap B$ is finite.

5.11.164.1 Solution. Let $S = \{r_1, r_2, .., r_n, ...\}$ be the fixed enumeration of rationals in $[0, 1]$ and $t \in [0, 1]$ and consider a fixed sequence of rationals converging to t. Now we define $\mathbb{N}_t = \{i \in \mathbb{N}; r_i \to t\}$. Clearly $\mathbb{N}_t$ is infinite for each $t \in [0, 1]$. Again, if $t_1 \neq t_2$, then $\mathbb{N}_{t_1} \cap \mathbb{N}_{t_2}$ is a finite set, since the corresponding sequences are converging to distinct real numbers. □

5.11.165 Problem. Suppose that (a_n) is a sequence converging to a and assume b is an accumulation point of $\{a_n : n \in \mathbf{N}\}$. Prove that $a = b$.

5.11.165.1 Solution. Assume that $a \neq b$. Let $\epsilon = |b-a|/2$. Hence $N(a;\epsilon) \cap N(b;\epsilon) = \emptyset$. As (a_n) converges to a, so $\exists\ m \in \mathbb{N}$ such that $n \geq m$ implies $a_n \in N(a;\epsilon)$. i.e. $\exists$ finite number of points a_n are outside $N(a;\epsilon)$ Thus $N(b;\epsilon)$ contains atmost finite number of points of a_n. Suppose a_p,...a_q are in $N(b;\epsilon)$. Let $\delta = \min\{|b-a_p|, ..|b-a_q|\}$ so no $a_n \in (b-\delta, b+\delta)$.Thus this nbhd of b contains no points of a_n. Hence b cannot be limit point of $\{a_n\}$, consequently $a = b$. □

5.11.166 Problem. Let $x_n = 1 + \frac{1}{2!} + \frac{1}{3!} + .. + \frac{1}{n!}$ and show that $x_n \leqq 3 - \frac{1}{n!}, \forall\ n \in \mathbb{N}$.

5.11.166.1 Solution. Hint: Use induction. □

5.11.167 Problem. Show that $f(x) = O(|x-x_0|^2)$ as $x \to x_o$ implies $f(x) = o(|x-x_0|)$ as $x \to x_o$, but give an example to show that the converse is not true.

5.11.167.1 Solution. By the definition of "big"O, $\exists M > 0, n \in \mathbb{N}$ such that

$$\frac{|f(x)|}{|x-x_0|^2} \leq M \text{ as } |x-x_0| < \frac{1}{n},$$

for some $n > 0$, which implies

$$\frac{|f(x)|}{|x-x_0|} \leq \frac{|x-x_0||f(x)|}{|x-x_0|^2} \leq M|x-x_0|\ ,$$

for all x such that $|x-x_0| < \dfrac{1}{n}$, which implies $f(x) = o(|x-x_0|)$ as $x \to x_o$.
Consider $f(x) = |x-x_0|^{2/3}$. □

5.11.168 Problem. Define s_n by $s_n = 1+1/\sqrt{2}+..+1/\sqrt{n}$ and prove that $\forall\ n \in \mathbb{N}$, $\sqrt{n} \leq s_n \leq 2\sqrt{n}$.

5.11.168.1 Solution. Hint: Use induction. □

5.11.169 Problem. Let (a_n) be a sequence of positive numbers. If there exists a sequence (b_n) of positive numbers and a constant $A > 0$ such that

$$b_n \frac{a_n}{a_{n+1}} - b_{n+1} > A$$

show that the series $\sum_{n=1}^{\infty} a_n$ is convergent.

5.11.169.1 Solution. We have $a_n b_n - a_{n+1} b_{n+1} \geq A a_{n+1}$ for $n \geq 1$. Hence

$$\begin{aligned} a_1 b_1 &\geq a_1 b_1 - a_{N+1} b_{N+1} \\ &= \sum_{n=1}^{N} (a_n b_n - a_{n+1} b_{n+1}) \\ &\geq A \sum_{n=1}^{N} a_n + 1. \quad \square \end{aligned}$$

5.11.170 Problem. If

$$S_n = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}$$

show that

$$S_{2n} = \sum_{k=1}^{n} \frac{1}{n+k}.$$

5.11.170.1 Solution. We have

$$\begin{aligned} S_{2n} &= \sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k} \\ &= 1 + \left(\frac{1}{2} - \frac{2}{2}\right) + \frac{1}{3} + \left(\frac{1}{4} - \frac{2}{4}\right) + .. + \left(\frac{1}{2n} - \frac{2}{2n}\right) \\ &= \sum_{k=1}^{2n} \frac{1}{k} - \sum_{k=1}^{n} \frac{2}{2k} = \sum_{k=1}^{n} \frac{1}{k} + \sum_{k=1}^{n} \frac{1}{n+k} - \sum_{k=1}^{n} \frac{1}{k} \\ &= \sum_{k=1}^{n} \frac{1}{n+k}. \quad \square \end{aligned}$$

5.11.171 Problem. Find $\lim_{n\to\infty} \sqrt[n]{n!}$.

5.11.171.1 Solution.

$$\lim_{n\to\infty} \sqrt[n]{n!} > \lim_{n\to\infty} \sqrt[n]{(n/2)^{n/2}} = \lim_{n\to\infty} \sqrt{n/2} = \infty.$$

$\square$

5.11.172 Problem. Let $a_1 = 0$ and $a_2 = 1$, and for $n > 3$,

$$a_n = (n-1)(a_{n-1} + a_{n-2}).$$

Find

1. a formula for a_n and
2. $\lim_{n\to\infty} \frac{a_n}{n!}$.

5.11.172.1 Solution.

1. From $a_n = (n-1)(a_{n-1} + a_{n-2})$, we get

$$\begin{aligned} a_n - na_{n-1} &= -[a_{n-1} - (n-1)a_{n-2}] \\ &= (-1)^2[a_{n-2} - (n-2)a_{n-3}] \\ &= (-1)^3[a_{n-3} - (n-3)a_{n-4}] \\ &= .... \\ \text{after } (n-3) \text{ iterations} &= (-1)^{n-2}(a_2 - 2a_1) \\ &= (-1)^{n-2} = (-1)^n. \end{aligned}$$

Hence

$$\frac{a_n}{n!} - \frac{a_{n-1}}{(n-1)!} = \frac{(-1)^n}{n!}.$$

and

$$\sum_{r=2}^{n} \left(\frac{a_r}{r!} - \frac{a_{r-1}}{(r-1)!}\right) = \sum_{r=2}^{n} \frac{(-1)^r}{r!} = \sum_{r=0}^{n} \frac{(-1)^r}{r!}.$$

Since $a_1 = 0$, so we get $a_n = n!\left(\sum_{r=0}^{n} \frac{(-1)^r}{r!}\right)$.

2. Hence $\lim_n \dfrac{a_n}{n!} = \lim_n \left(\sum_{r=0}^n \dfrac{(-1)^r}{r!}\right) = e^{-1}$. $\square$

5.11.173 Problem. Let $a_1 = 1$ and $a_{i+1} = \sqrt{a_1 + a_2 + .. + a_i}$, for $i > 0$. Determine $\lim_{n\to\infty} \frac{a_n}{n}$.

5.11.173.1 Solution. We can write $a_{n+1}^2 = a_1 + a_2 + .. + a_{n-1} + a_n = a_n^2 + a_n$, thus

$$\begin{aligned}
a_{n+1} - a_n &= \sqrt{a_n^2 + a_n} - a_n \\
&= \frac{a_n^2 + a_n - a_n^2}{\sqrt{a_n^2 + a_n} + a_n} \\
&= \frac{a_n}{\sqrt{a_n^2 + a_n} + a_n} \\
&= \frac{1}{\sqrt{1 + \dfrac{1}{a_n}} + 1}.
\end{aligned}$$

Since $a_1 = 1$ and a_n is increasing, so $a_n \to \infty$, hence $a_{n+1} - a_n \to 1/2$. Let $b_n = a_{n+1} - a_n$, now by Cauchy's limit theorem $\dfrac{\sum_{r=1}^n b_r}{n} = \dfrac{a_{n+1} - a_1}{n} \to 1/2$.
Hence, $\lim_n \dfrac{a_{n+1} - 1}{n+1} . \dfrac{n+1}{n} \to 1/2$ implies $\lim_n \dfrac{a_{n+1}}{n+1} \to 1/2$.

5.11.173.2 Solution. Since $a_{n+1}^2 = a_n^2 + a_n$, show that

$$n/2 - \sqrt{n} < a_n < n/2.$$

Hence $\dfrac{a_n}{n} \to 1/2$. $\square$

5.11.174 Problem. Prove that $3 = \sqrt{5 + \sqrt{6 + 2\sqrt{7 + 3\sqrt{8 + 4\sqrt{9 + ...}}}}}$

5.11.174.1 Solution.

$$\begin{aligned}
3 = \sqrt{9} &= \sqrt{5 + \sqrt{16}} \\
&= \sqrt{5 + \sqrt{6 + 2\sqrt{25}}} = \sqrt{5 + \sqrt{6 + 2\sqrt{7 + 3\sqrt{36}}}} \\
&= \sqrt{5 + \sqrt{6 + 2\sqrt{7 + 3\sqrt{8 + 4\sqrt{49}}}}} \\
&= \sqrt{5 + \sqrt{6 + 2\sqrt{7 + 3\sqrt{8 + 4\sqrt{9 + 5\sqrt{64}}}}}} \\
&= .... \quad \square
\end{aligned}$$

Next, we present a famous identity of S.A. Ramanujan.

5.11.175 Problem. Prove that

$$\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+...}}}}=3.$$

5.11.175.1 Solution. Consider the function $f:[1,\infty)\to\mathbb{R}$ defined by

$$f(x)=\sqrt{1+x\sqrt{1+(x+1)\sqrt{1+(x+2)\sqrt{1+....}}}}$$

All we need to show is that this sequence is bounded above.

$$\begin{aligned}
f(x) &\leq \sqrt{(x+1)\sqrt{(x+2)\sqrt{(x+3)...}}}\\
&\leq \sqrt{2x\sqrt{3x\sqrt{4x...}}} \leq \sqrt{2x\sqrt{4x\sqrt{8x...}}} =\\
&= 2^{\sum\frac{k}{2^k}}x^{\sum\frac{1}{2^k}} \leq 2^{\frac{1}{2}+\frac{1}{2}+\frac{1}{4}+\frac{1}{4}+\frac{1}{8}+\frac{1}{8}+\cdots}x = 2x.
\end{aligned}$$

This shows that $f(x)\leq 2x$, for $x\geq 1$. Note also that

$$f(x)\geq\sqrt{x\sqrt{x\sqrt{x..}}}=x.$$

For reasons that will become apparent, we weaken this inequality to $f(x)\geq\frac{1}{2}(x+1)$. We then square the defining relation and obtain the functional equation

$$(f(x))^2=xf(x+1)+1.$$

Combining this with

$$\frac{1}{2}(x+1)\leq f(x+1)\leq 2(x+1),$$

we obtain $x.\dfrac{(x+1)}{2}+1\leq(f(x))^2\leq 2x(x+1)+1$, which yields the sharper double inequality

$$\frac{(x+1)}{\sqrt{2}}\leq f(x)\leq\sqrt{2}(x+1).$$

Repeating successively the argument, we find that

$$\frac{(x+1)}{2^{\frac{1}{2^n}}}\leq f(x)\leq 2^{\frac{1}{2^n}}(x+1)\text{ for }n\geq 1.$$

If in this double inequality we let $n\to\infty$, we obtain $x+1\leq f(x)\leq x+1$, and hence $f(x)=x+1$. The particular case $x=2$ yields Ramanujan's result

$$\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+...}}}}=3.\quad\square$$

5.11.176 Problem. Determine whether

$$x_n = \sqrt{1+\sqrt{2+\sqrt{3+\sqrt{4+\sqrt{5+..+\sqrt{n}}}}}}$$

converges or diverges.

5.11.176.1 Solution. Since $n+\sqrt{n+1} > n, x_{n+1} > x_n$

$$\begin{aligned}
x_n^2 &= 1+\sqrt{2+\sqrt{3+\sqrt{4+\sqrt{5+..+\sqrt{n}}}}}\\
&= 1+2\sqrt{\frac{2}{2^2}+\sqrt{\frac{3}{2^4}+\sqrt{\frac{4}{2^8}+\sqrt{\frac{5}{2^{16}}+..+\sqrt{\frac{n}{2^{2^{n-1}}}}}}}}\\
&\le 1+2\sqrt{1+\sqrt{2+\sqrt{3+\sqrt{4+..+\sqrt{n-1}}}}}\\
&\le 1+2x_{n-1} \le 1+2x_n.
\end{aligned}$$

Thus $x_n^2 - 2x_n + 1 \le 2 \Rightarrow x_n \le \sqrt{2}+1$, hence (x_n) is bounded and increasing. Therefore (x_n) is convergent. □

5.11.176.2 Solution. (Matematika v škole, 1971, solution from R. Honsberger, More Mathematical Morsels, Mathematical Association of America, 1991)
The sequence is increasing, so all we need to show is that it is bounded. The main trick is to factor a $\sqrt{2}$. The general term of the sequence becomes

$$\begin{aligned}
x_n &= \sqrt{1+\sqrt{2+\sqrt{3+\sqrt{4+\sqrt{5+..+\sqrt{n}}}}}}\\
&= \sqrt{2}\sqrt{\frac{1}{2}+\sqrt{\frac{2}{4}+\sqrt{\frac{3}{8}+\sqrt{\frac{4}{16}+..+\sqrt{\frac{n}{2^n}}}}}}\\
&< \sqrt{2}\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+..+\sqrt{1}}}}}
\end{aligned}$$

Let $b_n = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+..+\sqrt{1}}}}}$, where there are n radicals. Then $b_{n+1} = \sqrt{1+b_n}$. We see that $b_1 = 1 < 2$, and if $b_n < 2$, then $b_{n+1} < \sqrt{1+2} < 2$. Inductively we prove that $b_n < 2\ \forall\ n$. Therefore, $a_n < 2\ \forall\ n$. Being monotonic and bounded, the sequence (a_n) is convergent. □

5.11.177 Problem. Let

$$x_n = \sqrt[2]{2 + \sqrt[3]{3 + ... \sqrt[n]{n}}}$$

prove that $x_{n+1} - x_n < \frac{1}{n!}$ for $n = 2, 3, .$

5.11.177.1 Solution. For fixed $n > 2$, let

$$a_i = \sqrt[i]{i + \sqrt[i+1]{(i+1) + ... \sqrt[n]{n + \sqrt[n+1]{n+1}}}}$$
$$b_i = \sqrt[i]{i + \sqrt[i+1]{(i+1) + ... \sqrt[n]{n}}}$$
$$c_i = a_i^{i-1} + a_i^{i-2} b_i + .. + a_i b_i^{i-2} + b_i^{i-1}.$$

Then

$$a_k^k - b_k^k = a_{k+1} - b_{k+1} = \frac{a_{k+1}^{k+1} - b_{k+1}^{k+1}}{c_{k+1}};\ k = 2, 3, .., n-1,$$

and a multiplication of these $n - 2$ equations gives

$$a_2^2 - b_2^2 = \frac{a_n^2 - b_n^2}{c_3 c_4 ... c_n}.$$

So

$$x_{n+1} - x_n = a_2 - b_2 = \frac{a_2^2 - b_2^2}{c_2} = \frac{a_n^2 - b_n^2}{c_2 c_3 ... c_n} = \frac{(n+1)^{1/(n+1)}}{c_2 c_3 ... c_n}.$$

Since

$$k^{1/(n+1)} < k^{1/k} \le b_k < a_k,\ c_k > k^{1+(k-1)/(n+1)},$$
$$x_{n+1} - x_n \le \frac{(n+1)^{1/(n+1)}}{n! n^{(n-1)/(n+1)}}.$$

Since $(n+1)/n^{n-1} \le 2n/n^2$ for $n > 2$, the inequality is true in this case. In the case for $n = 2$ can be checked easily. Thus $x_{n+1} - x_n < \frac{1}{n!}$. □

5.11.178 Problem. Compute

$$\sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + ...}}}}$$

5.11.178.1 Solution. Left to the reader.

5.11.179 Problem. For an arbitrary number $x_0 \in (0, \pi)$, define recursively the sequence (x_n) by $x_{n+1} = \sin x_n,\ n \ge 0$. Compute $\lim_{n\to\infty} \sqrt{n} x_n$.

5.11.179.1 Solution. We compute the square of the reciprocal of the limit, namely $\lim_{n\to\infty} \dfrac{1}{nx_n^2}$. To this end, we apply the Cesáro–Stolz theorem to the sequences $a_n = \dfrac{1}{x_n^2}$ and $b_n = n$. First, note that $\lim_{n\to\infty} x_n = 0$. Indeed, in view of the inequality $0 < \sin x < x$ on $(0, \pi)$, the sequence is bounded and decreasing, and the limit L satisfies $L = \sin L$, so $L = 0$. We then have

$$\begin{aligned}
\lim_{n\to\infty}\left(\frac{1}{x_{n+1}^2}-\frac{1}{x_n^2}\right) &= \lim_{n\to\infty}\left(\frac{1}{\sin^2 x_n}-\frac{1}{x_n^2}\right)\\
&= \lim_{n\to\infty}\left(\frac{x_n^2-\sin^2 x_n}{x_n^2\sin^2 x_n}\right)\\
&= \lim_{x_n\to 0}\left(\frac{x_n^2-\frac{1}{2}(1-\cos 2x_n)}{x_n^2\frac{1}{2}(1-\cos 2x_n)}\right)\\
&= \lim_{x_n\to 0}\left(\frac{2x_n^2-\left[\frac{(2x_n)^2}{2!}-\frac{(2x_n)^4}{4!}+..\right]}{x_n^2\left[\frac{(2x_n)^2}{2!}-\frac{(2x_n)^4}{4!}+..\right]}\right)\\
&= \frac{1}{3}.
\end{aligned}$$

We conclude that the original limit is $\sqrt{3}$. □

(J. Dieudonné, Infinitesimal Calculus, Hermann, 1962, solution by Ch. Radoux)

5.11.180 Problem. Determine all subsequential limit points of the sequence $x_n = \cos n$.

5.11.180.1 Solution. Clearly, no number larger than 1 or less than -1 could be such a limit. Show that in fact the interval [-1,1] is the set of all such limit points. If $x \in [-1, 1]$ there must be a number y so that $\cos y = x$ (by Intermediate Value Property as $f(x) = \cos x$ is continuous). Now consider the set of numbers

$$G = \{n + 2m\pi; n, m \in \mathbb{Z}\}.$$

and this is dense. Hence for every $\epsilon > 0$ there are pairs of integers n, m so that

$$|y - n + 2m\pi| < \epsilon.$$

From this deduce that

$$|\cos y - \cos(n + 2m\pi)| < \epsilon$$

and so

$$|x - \cos n| < \epsilon.$$

□

5.11.181 Problem. Prove that the lim sup and lim inf preserve inequalities. That is, show that if two bounded sequences (x_n) and (y_n) of real numbers satisfy $x_n \leq y_n$ for all $n \geq n_0$, then

$$\limsup x_n \leq \limsup y_n \text{ and } \liminf x_n \leq \liminf y_n$$

that is, $\overline{\lim}\, x_n \leq \overline{\lim}\, y_n$ and $\underline{\lim}\, x_n \leq \underline{\lim}\, y_n$.

5.11.181.1 Solution. First, we shall show that if two sequences of real numbers (x_n) and (y_n) converge to x and y respectively in $\mathbb{R}$ and $x_n \leq y_n$ for all $n \geq n_0$, then $x \leq y$. If $x > y$, then let $\epsilon = \frac{x-y}{2} > 0$, then there exist $m_1, m_2 \in \mathbb{N}$ such that

$$\begin{aligned} n \geq m_1 \Rightarrow & x_n \in (x-\epsilon, x+\epsilon) = \left(\frac{x+y}{2}, \frac{3x-y}{2}\right) \text{ and} \\ n \geq m_2 \Rightarrow & y_n \in (y-\epsilon, y+\epsilon) = \left(\frac{3y-x}{2}, \frac{x+y}{2}\right). \end{aligned}$$

That is, $n \geq m = \max\{m_1, m_2\} \Rightarrow y_n < \frac{x+y}{2} < x_n$, which is impossible. Hence, $x \leq y$.
Now, assume that two bounded sequences of real numbers (x_n) and (y_n) satisfy $x_n \leq y_n$ for all $n \geq n_0$. Put

$$s_n = \inf_{k \geq n} x_k \text{ and } t_n = \inf_{k \geq n} y_k.$$

If $n \geq n_0$, then notice that for each $r \geq n$ we have $s_n = \inf_{k \geq n} x_k < x_r < y_r$ and so $s_n \leq \inf_{r \geq n} y_r = t_n$ for each $n \geq n_0$. By the discussion of the first part, we infer that

$$\liminf x_n = \lim s_n \leq \lim t_n = \liminf y_n.$$

The lim sup case can be established in a similar manner, or by $\limsup x_n = -\liminf(-x_n)$. $\square$

5.11.182 Problem. If (x_n) and (y_n) are two bounded sequences of real numbers, then the following inequalities hold. Give examples in each case to show that the equality need not occur.

1.
$$\begin{aligned} \underline{\lim}\, x_n + \underline{\lim}\, y_n \leq \underline{\lim}\, (x_n + y_n) \leq & \begin{cases} \underline{\lim}\, x_n + \overline{\lim}\, y_n \\ \overline{\lim}\, x_n + \underline{\lim}\, y_n \end{cases} \\ & \leq \overline{\lim}\, (x_n + y_n) \leq \overline{\lim}\, x_n + \overline{\lim}\, y_n. \end{aligned}$$

2. Assume that $x_n, y_n > 0\ \forall\, n \in \mathbb{N}$, then
$$\begin{aligned} \underline{\lim}\, x_n \cdot \underline{\lim}\, y_n \leq \underline{\lim}\, (x_n \cdot y_n) \leq & \begin{cases} \underline{\lim}\, x_n \cdot \overline{\lim}\, y_n \\ \overline{\lim}\, x_n \cdot \underline{\lim}\, y_n \end{cases} \\ & \leq \overline{\lim}\, (x_n \cdot y_n) \leq \overline{\lim}\, x_n \cdot \overline{\lim}\, y_n. \end{aligned}$$

3. Assume that $\lim y_n = y$. Then
$$\begin{aligned} \overline{\lim}\, (x_n + y_n) &= \overline{\lim}\, x_n + y. \\ \underline{\lim}\, (x_n + y_n) &= \underline{\lim}\, x_n + y. \end{aligned}$$

5.11.182.1 Solution. Let

$$\begin{aligned} A = \underline{\lim}\, x_n,\ B = \underline{\lim}\, y_n \text{ and } C = \underline{\lim}\, (x_n + y_n), \\ P = \overline{\lim}\, x_n,\ Q = \overline{\lim}\, y_n \text{ and } R = \overline{\lim}\, (x_n + y_n). \end{aligned}$$

1. Since (x_n) is a bounded sequence, there exists a subsequence (x_{n_k}) of (x_n) that converges to x and (y_{n_k}) of (y_n) that converges to y. So $A \leq x$, and $B \leq y$, and since C is the least of all subsequential limits of $(x_n + y_n)$ and there exists a strictly increasing sequence (m_k) of natural numbers such that $x = \lim x_{m_k}$ and $y = \lim y_{m_k}$. Hence,

$$\begin{aligned} \underline{\lim}\,(x_n + y_n) &= \lim_k (x_{m_k} + y_{m_k}) \\ &= \lim_k x_{m_k} + \lim_k y_{m_k} = x + y \geq A + B \\ \Rightarrow \underline{\lim}\,(x_n + y_n) &\geq \underline{\lim}\,(x_n) + \underline{\lim}\,(y_n) \end{aligned}$$

   and similarly for other parts.

2. Left to the reader.

3. Since $\overline{\lim}\, x_n = P$, so, there exists a subsequence (x_{n_k}) of (x_n) such that $\lim_{k\to\infty} x_{n_k} = P$ and it is the greatest of all subsequential limits of (x_n). Again, $\lim y_n = y$ implies every subsequence (y_{n_k}) of (y_n) converges to y and thus

$$\begin{aligned} &\lim_k (x_{n_k} + y_{n_k}) \leq \overline{\lim}(x_n + y_n) \\ \Rightarrow &P + y \leq \overline{\lim}(x_n + y_n) \leq \overline{\lim}\, x_n + \overline{\lim}\, y_n. \text{ by (1)} \\ \Rightarrow &P + y \leq \overline{\lim}\,(x_n + y_n) \leq P + y. \ (\overline{\lim}\, y_n = \underline{\lim}\, y_n = y) \end{aligned}$$

   Hence $\overline{\lim}\,(x_n + y_n) = P + y = \overline{\lim}\, x_n + \lim y_n$. □

5.11.183 Problem. Let $a > 0$ and $a_{n+1} = a^{a_n},\ n = 1, 2, ..., a_1 = a$. Show that

1. $a_n \to \infty$ when $a > e^{1/e}$.
2. a_n converges iff $e^{-e} \leq a \leq e^{1/e}$.
3. a_n does not tend to limit when $a < e^{-e}$.

Moreover, in (3), $x_{2n} \to A, x_{2n+1} \to B$ where A, B satisfy $a^A = B, a^B = A$.

5.11.183.1 Solution. We have

$$\frac{a_{n+1}}{a_n} = \frac{a^{a_n}}{a^{a_{n-1}}} = a^{a_n - a_{n-1}}.$$

Suppose $a > 1$, then $a_{n+1} > a_n$ if $a_n > a_{n-1}$. Now $a_2 = a^{a_1} = a^a > a = a_1$ so that a_n increases with n and must tend to a limit L or to infinity. Suppose k exists such that $a^k = k$. Clearly $k > a$ and

$$\frac{k}{a_{n+1}} = a^{k - a_n}.$$

Thus $k > a_{n+1}$ if $k > a_n$. But $k > a = a_1$. Hence if $\exists$ any root k of $a^x = x$ it follows $a_n \to L \leq k$. But

$$L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a^{a_n} = a^L.$$

Thus if there is any real root of the equation $a^x = x, x_n$ tends to the least root; if there is no root, $x_n \to \infty$. The equation $a^x = x$ $(x > 0, a > 0)$, putting $y = \frac{a^x}{x}$ we have

$$\frac{1}{y}\frac{dy}{dx} = \log a = \frac{1}{x}.$$

Thus if $a > 1$, $\frac{dy}{dx} \lesseqqgtr 0$ according as $x \lesseqqgtr \frac{1}{\log a}$ and $x = \frac{1}{\log a}$ gives a minimum value $e \log a$ for y. If $a > 1$, then for large x, and for small positive x, $y > 1$. Hence $a^x = x$ has 2,1,0 roots according as $a \lesseqqgtr e^{\frac{1}{e}}$. If $a < 1$, $\frac{dy}{dx}$ is always negative, and the equation $a^x = x$ has one and only one root.
The case $a = 1$ gives $x_n \to 1$. We now turn to the case $a < 1$. Our original equation shows that if $a_n > a_{n-1}$ then $a_{n+1} < a_n$ and so $a_{n+2} > a_{n+1}$. We are lead to consider seperately the odd and even sequences $a_1, a_3, a_5, ...$ and $a_2, a_4, a_6, ...$ The equation

$$\frac{a_{n+2}}{a_n} = a^{a_{n+1} - a_{n-1}} = a^{a^{a_n} - a^{a_{n-2}}},$$

shows (by induction) that these are respectively increasing and decreasing. Since a_n is essentially positive it follows that $x_{2n-1} \to B$ and hence that $x_{2n} \to A \geq B$. We have

$$A = \lim_{n\to\infty} a_{2n+2} = \lim_{n\to\infty} a^{a^{a_{2n}}} = a^{a^A} \text{ and similarly } B = a^{a^B}.$$

If k is any root of $a^{a^x} = x$ we have clearly $0 < k < 1$. Also

$$\frac{k}{a_n} = a^{a^k - a^{a_{n-2}}},$$

so that $a_n \lesseqqgtr k$ according as $a_{n-2} \lesseqqgtr k$. Thus A, B are respectively the greatest and least roots of $a^{a^x} = x$.
The equation $a^{a^x} = x$ $(a < 1, x > 0)$.
There is always one root of this equation - the root k of $a^x = x$ which have been known to exist. For convenience we put $a = e^{-b}$. Taking logarithms twice we have

$$-bx + \log b = \log\left(\log\frac{1}{x}\right).$$

Put $y = bx - \log b + \log\left(\log\frac{1}{x}\right)$, so that

$$\frac{dy}{dx} = b - \frac{1}{x\log(1/x)}.$$

Now

$$\frac{d}{dx}\left(x\log\frac{1}{x}\right) = \log\frac{1}{x} - 1,$$

so that $x\log\frac{1}{x}$ has a maximum value $1/e$ when $\log\frac{1}{x} = 1$ i.e. $x = \frac{1}{e}$. Hence $\frac{dy}{dx}$ has a minimum value $b - e$. If $b \leq e$, $\frac{dy}{dx}$ is always negative and there is one root only of the equation $y = 0$. Thus $A = B$ if

$a \geq e^{-e}$. If $b > e$ consider $\dfrac{dy}{dx}$ at the point $x = k$, where k is the root $a^x = x$ (and therefore $a^{a^x} = x$) already referred to. Then $x \log a = \log k$ i.e. $-kb = \log k$, and so,

$$\frac{dy}{dx} = b - \frac{1}{x \log \dfrac{1}{x}} = b - \frac{1}{k^2 b}.$$

To decide the sign of the expression we note that the increasing function $bx + \log x$ is negative when $x = 1/b$ and $b > e$. Thus $1/b < k$, the only valueof x for which $bx + \log x = 0$. Thus, when $x = k$, $\dfrac{dy}{dx} > 0$. It follows that $y > 0$ for $x = k + \delta$ and $y < 0$ for $x = k - \delta$, δ being small. But $y > 0$ for small values of x and $y < 0$ when x approaches 1. Thus there is a real root of the equation $y = 0$ in each of the intervals $(0, k), (k, 1)$, and it follows that $A > B$. Thus x_n does not tend to a limit. the properties (1), (2) and (3) are therefore proved.
From these results and the fact that

$$a_{n+1} = a^{a_n},$$

it follows at once that $A = a^B$ and $B = a^A$. □

5.11.184 Problem.

1. If a sequence (a_n) has the property that $\lim_{n\to\infty}(a_{n+p} - a_n) = 0$ ($p \in \mathbb{N}$ is fixed), then (a_n) converges. True or false?

2. Prove that the condition $\lim_{n\to\infty}(a_{n+p} - a_n) = 0$ for every positive integer p is necessary but not sufficient for the convergence of the sequence (a_n).

5.11.184.1 Solution.

1. Consider the sequence (a_n) defined by $a_n = 1 + \frac{1}{2} + ... + \frac{1}{n}$. Now,

$$\begin{aligned} a_{2^k} &= 1 + \frac{1}{2} + .. + \frac{1}{2^k} \\ &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + ... \\ &+ \left(\frac{1}{2^{k-1}+1} + \frac{1}{2^{k-1}+2} + .. + \frac{1}{2^k}\right) \\ &> 1 + \frac{1}{2} + 2\frac{1}{2^2} + 4\frac{1}{2^3} + .. + 2^{k-1}\frac{1}{2^k} = 1 + k\frac{1}{2} \to \infty. \end{aligned}$$

   Therefore, the chosen subsequence is divergent, that implies the divergence of the original sequence. At the same time, the condition of the statement is satisfied:

$$\lim_{n\to\infty}(a_{n+p} - a_n) = \lim_{n\to\infty}\left(\frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{n+p}\right) = 0. \quad \square$$

   Remark: This is a wrong reformulation of Cauchy's criterion: a sequence a_n converges if, and only if, for any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\forall\, n > N$ and all $p \in \mathbb{N}$, then $|a_{n+p} - a_n| < \epsilon$.

2. Left to the reader.

5.11.185 Problem. Show that the sequence (x_n) defined by $x_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}$ does not converge.

5.11.185.1 Solution. Suppose that (x_n) converges to x, then $x_{2n} \to x$, but we observe that

$$\begin{aligned} x_{2n} - x_n &= \frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{n+n} \\ &> \frac{1}{2n} + \frac{1}{2n} + ... + \frac{1}{2n} = \frac{1}{2}. \end{aligned}$$

So we have $0 = \lim_{n\to\infty}(x_{2n} - x_n) \geq \frac{1}{2}$, a contradiction. $\square$

5.11.185.2 Solution. We claim that $x_{2^n} \geq 1 + \frac{n}{2}$, and prove it by induction. For $n = 1$, we have $x_{2^1} \geq 1 + \frac{1}{2}$. Now, if we assume the inequality is true for some n, then

$$\begin{aligned} x_{2^{n+1}} &= x_{2^n} + \frac{1}{2^n+1} + \frac{1}{2^n+2} + .. + \frac{1}{2^n+2^n} \\ &\geq 1 + \frac{n}{2} + 2^n.\frac{1}{2.2^n} = 1 + \frac{n+1}{2} \end{aligned}$$

and, we get $x_{2^{n+1}} - x_{2^n} \geq \frac{1}{2}$.

So we have $0 = \lim_{n\to\infty}(x_{2^{n+1}} - x_{2^n}) \geq \frac{1}{2}$, a contradiction. $\square$

5.11.185.3 Solution. Note that

$$\begin{aligned} x_{10^n-1} = \sum_{k=1}^{10^n-1}\frac{1}{k} &= \left[1 + \frac{1}{2} + .. + \frac{1}{9}\right] + \left[\frac{1}{10} + .. + \frac{1}{99}\right] + .. + \left[\frac{1}{10^{n-1}} + .. + \frac{1}{10^n-1}\right] \\ &\geq 9.\frac{1}{10} + 90.\frac{1}{100} + .. + 9.10^{n-1}.\frac{1}{10^n} \\ &= \frac{9}{10} + \frac{9}{10} + .. + \frac{9}{10} = \frac{9n}{10} \to \infty. \quad \square \end{aligned}$$

5.11.186 Problem. Determine the limits of the following sequences:

1. $x_n = \prod_{k=2}^n \left(1 - \frac{1}{k^2}\right)$
2. $x_n = \prod_{k=2}^n \left(1 - \frac{1}{\frac{k(k+1)}{2}}\right)$
3. $x_n = \prod_{k=2}^n \left(\frac{k^3+1}{k^3-1}\right)$
4. $x_n = \prod_{k=0}^n \left(1 + a^{2^k}\right); a \in (0,1)$
5. $x_n = \left(1 - \frac{1}{n}\right)^n$.
6. $x_n = \sum_{k=1}^n \frac{a_k}{S_{k-1}S_k}$, where (a_n) is an increasing sequence and $S_n = \sum_{k=0}^n a_k,\ S_0 = a_0$.
7. $x_n = \frac{a^n}{(1+a)(1+a^2)...(1+a^n)};\ a > 0.$

5.11.186.1 Solution.

1.

$$\begin{aligned}\lim_{n\to\infty} x_n &= \lim_{n\to\infty}\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)...\left(1-\frac{1}{n^2}\right)\\ &= \lim_{n\to\infty}\left(\frac{1.3}{2^2}.\frac{2.4}{3^2}...\frac{(k-1)(k+1)}{n^2}\right) = \lim_{n\to\infty}\frac{1}{2}.\frac{n+1}{n} = \frac{1}{2}.\end{aligned}$$

2.

$$\begin{aligned}\lim_{n\to\infty} x_n &= \lim_{n\to\infty}\left(1-\frac{1}{3}\right)\left(1-\frac{1}{6}\right)...\left(1-\frac{2}{n^2+n}\right)\\ &= \lim_{n\to\infty}\left(\frac{1.4}{2.3}.\frac{2.5}{3.4}...\frac{(n-1)(n+2)}{n(n+1)}\right) = \lim_{n\to\infty}\frac{1}{3}.\frac{n+2}{n} = \frac{1}{3}.\end{aligned}$$

3.

$$\begin{aligned}\lim_{n\to\infty} x_n &= \lim_{n\to\infty}\left(\frac{2^3-1}{2^3+1}\right)\left(\frac{3^3-1}{3^3+1}\right)...\left(\frac{n^3-1}{n^3+1}\right)\\ &= \lim_{n\to\infty}\left(\frac{1.7}{3.3}.\frac{2.13}{4.7}.\frac{3.21}{5.13}.\frac{4.34}{6.21}...\frac{(n-1)(n^2+n+1)}{(n+1)((n-1)^2+(n-1)+1)}\right)\\ &= \lim_{n\to\infty}\frac{2}{3}.\frac{(n^2+n+1)}{n(n+1)} = \frac{2}{3}.\end{aligned}$$

4. Here

$$\begin{aligned}x_n &= \prod_{k=0}^{n}\left(1+a^{2^k}\right) = (1+a)\left(1+a^2\right)...\left(1+a^{2^k}\right)\\ &= \frac{1-a}{1-a}(1+a)\left(1+a^2\right)...\left(1+a^{2^k}\right) = \frac{1}{1-a}\left(1-a^2\right)\left(1+a^2\right)...\left(1+a^{2^k}\right)\\ &= .... = \frac{1-a^{2^{n+1}}}{1-a}.\end{aligned}$$

Hence $\lim_{n\to\infty} x_n = \dfrac{1}{1-a}$ as $a\in(0,1)$.

5. Now,

$$x_n = \left(1-\frac{1}{n}\right)^n = \frac{1}{\left(\frac{n}{n-1}\right)^n} = \frac{1}{\left(1+\frac{1}{n-1}\right)^n} = \left(\left(1+\frac{1}{n-1}\right)^{n-1}\right)^{-1}\left(1+\frac{1}{n-1}\right)^{-1}.$$

Hence $\lim_{n\to\infty} x_n = e^{-1}$.

6. Use the equalities $\dfrac{a_k}{S_{k-1}S_k} = \dfrac{1}{S_{k-1}} - \dfrac{1}{S_k}; k=1,2,..,n.$

7. Case $0 < a < 1, a = 1, a > 1$
If $0 < a < 1$, we have

$$0 < \frac{a^n}{(1+a)(1+a^2)...(1+a^n)} < a^n \Rightarrow \lim_{n\to\infty} x_n = 0.$$

If $a > 1$, we have

$$0 < \frac{a^n}{(1+a)(1+a^2)...(1+a^n)} < \frac{a^n}{a.a^2...a^n} = \frac{a^n}{a^{n(n+1)/2}} \Rightarrow \lim_{n\to\infty} x_n = 0.$$

If $a = 1$, we have

$$x_n = \frac{a^n}{(1+a)(1+a^2)...(1+a^n)} = \frac{1}{2^n} \Rightarrow \lim_{n\to\infty} x_n = 0. \quad \square$$

5.11.187 Problem. Find the largest term in the sequence (x_n) defined by

$$x_n = \frac{1000^n}{n!}.$$

5.11.187.1 Solution. Since $x_{n+1} = \dfrac{1000^{n+1}}{(n+1)!}$, so $\dfrac{x_{n+1}}{x_n} = \dfrac{1000}{n+1}$, which shows that $x_{999} = x_{1000}$. Thus (x_n) is increasing when $1 \le n \le 999$ and (x_n) is decreasing when $n \ge 1000$. Hence $\max_{n\ge 1} x_n = x_{1000} = \dfrac{1000^{1000}}{1000!}$. $\square$

5.11.188 Problem. Suppose that (x_n) and (y_n) are Cauchy. Prove that $(|x_n - y_n|)$ is also Cauchy.

5.11.188.1 Solution. Let $\epsilon > 0$. Since (x_n) and (y_n) are Cauchy, $\exists\, N \in \mathbb{N}$ such that $|x_m - x_n| < \epsilon/2$ and $|y_m - y_n| < \epsilon/2$ whenever $m, n \ge N$. Hence, by using the inequality $||a| - |b|| \le |a - b|$, we get

$$\begin{aligned} |\,|x_m - y_m| - |x_n - y_n|\,| &\le |x_m - y_m - x_n + y_n| \\ &\le |x_m - x_n| + |y_n - y_m| \\ &< \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

Hence $(|x_n - y_n|)$ is Cauchy. $\square$

5.11.189 Problem. A Cauchy sequence that contains a convergent subsequence is convergent.

5.11.189.1 Solution. Let (a_n) be a Cauchy sequence with $a_{n_k} \to \ell$. We show that $a_n \to \ell$. Let $\epsilon > 0$. Since (a_n) is a Cauchy sequence, there exists an $N_1 \in \mathbb{N}$ such that $m, n \ge N_1 \Rightarrow |a_m - a_n| < \epsilon/2$. Since $a_{n_k} \to \ell$ there exists an $N_2 \in \mathbb{N}$ such that $k \ge N_2 \Rightarrow |a_{n_k} - \ell| < \epsilon/2$. Let $N = \max\{N_1, N_2\}$, then as $n_k > k$ we get

$$n, k \ge N \Rightarrow |a_n - \ell| \le |a_n - a_{n_k}| + |a_{n_k} - l| < \epsilon,$$

which shows that $a_n \to \ell$. $\square$

5.11.190 Problem. Let x_1 be a real number, $0 < x_1 < 1$, and define a sequence by $x_{n+1} = x_n - x_n^{n+1}$. Show that $\liminf x_n > 0$.

5.11.190.1 Solution. It is clear that $x_{n+1} = x_n(1 - x_n^n) \le x_n \le ... \le x_1$. Thus

$$x_{n+1} = x_n(1 - x_n^n) \ge x_n(1 - x_1^n),$$

and therefore

$$x_n \ge x_1 \prod_{k=1}^{n} (1 - x_1^k) = x_1 \exp\left(\sum_{k=1}^{n} \log\left(1 - x_1^k\right)\right).$$

Since $\log(1 - x_1^k) = O(x_1^k)$ as $k \to \infty$, the sum converges to a finite value L as $n \to \infty$ and we get $\liminf x_n \ge x_1 \exp L > 0$.

5.11.191 Problem. Let x_n be a sequence of real numbers so that

$$\lim_{n\to\infty} (2x_{n+1} - x_n) = x.$$

Show that $\lim_{n\to\infty} x_n = x$.

5.11.191.1 Solution. First we show, by induction, that the sequence (x_n) is bounded. For that, choose M large so that $\max\{|x_1|, |2x_{n+1} - x_n|\} < M$, $\forall\, n \in \mathbb{N}$. Now

$$|x_{n+1}| = \left|\frac{x_n - (2x_{n+1} - x_n)}{2}\right| \le \frac{1}{2}(|x_n| + |2x_{n+1} - x_n|) \le M$$

showing that (x_n) is bounded. Now to compute the limit we write

$$x_{n+1} = \frac{x_n - (2x_{n+1} - x_n)}{2}$$

and taking the lim sup, we have

$$\limsup x_n \le \frac{\limsup x_n + x}{2}$$

showing that $\limsup x_n \le x$. In the same way we obtain $\limsup x_n \ge x$ showing that $\lim_{n\to\infty} x_n = x$.

5.11.192 Problem. For which values of the real number a does the series

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \sin\frac{1}{n}\right)^a$$

converge?

5.11.192.1 Solution. If $a \le 0$, the general term does not go to zero, so the series diverges. If $a > 0$, we have, using the Maclaurin's series for $\sin x$, and, therefore,

$$\begin{aligned} \frac{1}{n} - \sin\frac{1}{n} &= \frac{1}{6n^3} - o(n^{-3})\ (n \to \infty) \\ \Rightarrow \left(\frac{1}{n} - \sin\frac{1}{n}\right)^a &= \frac{1}{6^a n^{3a}} - o(n^{-3a})\ (n \to \infty). \end{aligned}$$

Thus, the series converges if and only if $3a > 1$.

5.11.193 Problem. Let $\alpha \in \mathbb{R}$ such that $|\alpha| < 1$. Show that $\lim_{n\to\infty} \alpha^n = 0$.

5.11.193.1 Solution. If $\alpha = 0$, then the conclusion is obvious. Assume first that $0 < \alpha < 1$. Then the sequence (α^n) is decreasing and bounded below by 0. So it has a limit L. Let us prove that $L = 0$. We have $\alpha^{n+1} = \alpha\alpha^n$ so taking limits we get $\lim_{k\to\infty} \alpha^{n+1} = \alpha \lim_{k\to\infty} \alpha^n \Rightarrow L = \alpha L \Rightarrow L = 0$. If $-1 < \alpha < 0$ then $0 < -\alpha < 1 \Rightarrow (-\alpha)^n = (-1)^n\alpha^n$, and then we use the fact that the product of a bounded sequence with a sequence which converges to 0 also converges to 0 to get $\lim_{n\to\infty}(-\alpha)^n = 0$. Therefore, for any α, with $|\alpha| < 1$ we have $\lim_{n\to\infty} \alpha^n = 0$. □

5.11.194 Problem. Let (x_n) be a sequence such that there exist $A > 0$ and $\beta \in (0, 1)$ for which $|x_{n+1} - x_n| \le A\beta^n$ for any $n \ge 1$. Show that (x_n) is Cauchy. Is this conclusion still valid if we assume only $\lim_{n\to\infty} |x_{n+1} - x_n| = 0$?

5.11.194.1 Solution. Let $n \ge 1$ and $p \ge 1$, then we have

$$|x_{n+p} - x_n| = \left|\sum_{k=0}^{p-1} x_{n+k+1} - x_{n+k}\right| \le \sum_{k=0}^{p-1} |x_{n+k+1} - x_{n+k}|.$$

By the condition, we get

$$|x_{n+p} - x_n| \le \sum_{k=0}^{p-1} A\beta^{n+k} = A\beta^n \frac{1-\beta^p}{1-\beta} < A\frac{\beta^n}{1-\beta}.$$

By the previous problem, $\beta^n \to 0 \Rightarrow |x_{n+p} - x_n|$ can be made to less than arbitrary ϵ, showing that (x_n) is Cauchy.
No. As for example, consider the sequence (x_n) defined by $x_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}$ is divergent, but $|x_{n+1} - x_n| = \frac{1}{n} \to 0$. □

5.11.195 Problem. Discuss the convergence or divergence of a sequence x_n defined by

$$x_n = \frac{[\alpha] + [2\alpha] + ... + [n\alpha]}{n^2},$$

where $[x]$ denotes the greatest integer less than or equal to the real number x, and α is an arbitrary real number.

5.11.195.1 Solution. By definition of the greatest integer function $[\cdot]$, we have $[x] < x < [x] + 1$ for any real number x which implies $x - 1 < [x] < x$. So

$$\begin{aligned}
&\frac{(\alpha-1) + (2\alpha-1) + ... + (n\alpha-1)}{n^2} < x_n \le \frac{\alpha + 2\alpha + ... + n\alpha}{n^2}\\
\Rightarrow &\frac{(1+2+..+n)\alpha - n}{n^2} < x_n \le \frac{(1+2+..+n)\alpha}{n^2}\\
\Rightarrow &\frac{\frac{n(n+1)}{2}\alpha - n}{n^2} < x_n \le \frac{\frac{n(n+1)}{2}\alpha}{n^2}\\
\Rightarrow &\frac{(n+1)\alpha}{2n} - \frac{1}{n} < x_n \le \frac{(n+1)\alpha}{2n}.
\end{aligned}$$

Since $\frac{(n+1)\alpha}{2n} - \frac{1}{n} \to \frac{\alpha}{2}$ and $\frac{(n+1)\alpha}{2n} \to \frac{\alpha}{2}$, so by Sandwich theorem, we get $x_n \to \frac{\alpha}{2}$. □

5.11.196 Problem. The real line $\mathbb{R}$ is complete, i.e. every Cauchy sequence is convergent.

5.11.196.1 Solution. Let (x_n) be a Cauchy sequence of real numbers. Define the sequence (n_k) of integers by induction in the following way: $n_0 = 1$ and n_{k+1} is the smallest integer greater than n_k a such that, for $p \geq n_{k+1}$ and $q \geq n_{k+1}$, $|x_p - x_q| < 1/2^{k+2}$; the possibility of the definition follows from the fact that (x_n) is a Cauchy sequence. Let $I_k = [x_{n_k} - 2^{-k}, x_{n_k} + 2^{-k}]$ be the closed interval; we have $I_{k+1} \subseteq I_k$ for ; on the other hand, $|x_{n_k} - x_{n_{k+1}}| < 2^{-k-1}]$ for $m > n_k, x_m \in I_k$, by definition. Now from Cantor's theorem it follows that the nested interval theorem, I_k have a nonempty intersection; let $a \in I_k \ \forall\, k \in \mathbb{N}$. Then it is clear that $|a - x_m| < 2^{-k+1}\ \forall\, m \geq n_k$, hence $a = \lim_{n\to\infty} x_n$. □

5.11.197 Problem. Does there exist a nonconstant function $f : (1, \infty) \to \mathbb{R}$ satisfying the relation $f(x) = f\left(\frac{x^2+1}{2}\right)\ \forall\, x > 1$ and such that $\lim_{x\to\infty} f(x)$ exists?

5.11.197.1 Solution. For $x > 1$ define the sequence (x_n) by $x_0 = x$ and $x_{n+1} = \frac{x_n^2+1}{2}$. The sequence is increasing because of the AM-GM inequality. Hence it has a limit L, finite or infinite. Passing to the limit in the recurrence relation, we obtain $L = \frac{L^2+1}{2}$; hence either $L = 1$ or $L = \infty$. Since the sequence is increasing, $L \geq x_0 > 1$, so $L = \infty$. We therefore have

$$f(x) = f(x_0) = f(x_1) = f(x_2) = .. = \lim_{n\to\infty} f(x_n) = \lim_{x\to\infty} f(x).$$

This implies that f is constant, which is ruled out by the hypothesis. So the answer to the question is no.

5.11.198 Problem. Examine whether the following implications are true.

1. $\lim_{n\to\infty} x_n^2 = x^2 \Rightarrow \lim_{n\to\infty} x_n = x$.
2. $\lim_{n\to\infty} x_n^3 = x^3 \Rightarrow \lim_{n\to\infty} x_n = x$.

5.11.198.1 Solution.

1. False. Consider $x_n = (-1)^n$.
2. True. □

5.11.199 Problem. Show that there exist sequences (a_n) and (b_n) such that $\lim_{n\to\infty} a_n = \infty$ and $\lim_{n\to\infty} b_n = -\infty$, but

1. $\lim_{n\to\infty}(a_n + b_n)$ converges.
2. $\lim_{n\to\infty}(a_n + b_n) = \infty$
3. $\lim_{n\to\infty}(a_n + b_n) = -\infty$
4. the sequence $(a_n + b_n)$ neither converges nor diverges to ∞ or to $-\infty$.

5.11.199.1 Solution.

1. $a_n = \frac{1}{n} + n,\ b_n = \frac{1}{n} - n.$
2. $a_n = -\frac{1}{n} + 2n,\ b_n = \frac{1}{n} - n.$

3. $a_n = -\frac{1}{n} + 2n,\ b_n = \frac{1}{n} - 3n.$

4. $a_n = (-1)^n + n,\ b_n = -n.$ □

5.11.200 Problem. Examine whether the following are Cauchy sequences.

1. $a_n = 1 + \dfrac{1}{4} + \dfrac{1}{9} + ... + \dfrac{1}{n^2};$

2. $b_n = \dfrac{\sin 1}{2} + \dfrac{\sin 2}{2^2} + ... + \dfrac{\sin n}{2^n};$

3. $c_n = \dfrac{\cos 1!}{1.2} + \dfrac{\cos 2!}{2.3} + ... + \dfrac{\cos n!}{n(n+1)};$

4. $d_n = 1 + \dfrac{1}{2} + \dfrac{1}{3} + ... + \dfrac{1}{n};$

5. $e_n = 1 + \dfrac{1}{\ln 2} + \dfrac{1}{\ln 3} + ... + \dfrac{1}{\ln n};$

5.11.200.1 Solution.

1. Hint: For any $n \in \mathbb{N}$, we get $\dfrac{1}{n^2} < \dfrac{1}{n(n-1)} = \dfrac{1}{n-1} - \dfrac{1}{n}$, hence

$$\begin{aligned} a_{n+p} - a_n &= \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + .. + \frac{1}{(n+p)^2} \\ &< \frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+1} - \frac{1}{n+2} + ... + \frac{1}{n+p-1} - \frac{1}{n+p} \\ &< \frac{1}{n} - \frac{1}{n+p} < \frac{1}{n} \end{aligned}$$

2. Left to the reader.

3. Left to the reader.

4. Left to the reader.

5. Since $\ln x < x$ for $x > 1$, we have

$$\begin{aligned} |e_{n+p} - e_n| &= \left| \frac{1}{\ln(n+1)} + \frac{1}{\ln(n+2)} + ... + \frac{1}{\ln(n+p)} \right| \\ &> \frac{p}{\ln(n+p)} > \frac{p}{n+p}. \end{aligned}$$

For $p = n$, we obtain

$$|e_{n+p} - e_n| > \frac{1}{2}.$$

Thus it cannot be made less than $\epsilon < \frac{1}{2}$. Hence (x_n) is not a Cauchy sequence. □

5.11.201 Problem. Consider the sequence (u_n) defined by $u_1 = u_2 = u_3 = 1$, and

$$\det \begin{pmatrix} u_{n+3} & u_{n+2} \\ u_{n+1} & u_n \end{pmatrix} = n!,\ n \geq 1$$

Prove that u_n is an integer for all n.

5.11.201.1 Solution. The condition gives us

$$\begin{aligned} u_{n+3}u_n - u_{n+2}u_{n+1} &= n! \\ \Rightarrow u_{n+3} &= \frac{u_{n+2}u_{n+1} + n!}{u_n}. \end{aligned}$$

Substituting $n = 1, 2, 3, ...n$, we get

$$\begin{aligned} u_4 &= \frac{u_3u_2 + 1!}{u_1} = 2 \\ u_5 &= \frac{u_4u_3 + 2!}{u_2} = 3 \\ u_6 &= \frac{u_5u_4 + 3!}{u_3} = 4.2 \\ u_7 &= \frac{u_6u_5 + 4!}{u_4} = 5.3 \\ u_8 &= \frac{u_7u_6 + 5!}{u_5} = 6.4.2 \\ u_9 &= \frac{u_8u_7 + 6!}{u_6} = 7.5.3 \\ &\dots\dots\dots\dots\dots\dots \end{aligned}$$

We claim that $u_{n+1} = (n-1)(n-3)(n-5)(n-7)......$ and our claim can be established easily by induction and left to the reader. □

5.11.202 Problem. Find a formula for the general term of the sequence
1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5,....

5.11.202.1 Solution.

$$\begin{aligned} x_k = k,\ \text{if } \frac{(n-1)n}{2} &< k \leq \frac{n(n+1)}{2} \\ &\Rightarrow n^2 - n + \frac{1}{4} < 2k < n^2 + n + \frac{1}{4} \\ &\Rightarrow \left(n - \frac{1}{2}\right)^2 < 2k < \left(n + \frac{1}{2}\right)^2 \\ &\Rightarrow n - \frac{1}{2} < \sqrt{2k} < n + \frac{1}{2} \\ &\Rightarrow n < \sqrt{2k} + \frac{1}{2} < n + 1. \end{aligned}$$

Now this happens if and only if $n = \left[\sqrt{2k} + \frac{1}{2}\right]$, which then gives the formula for the general term of the sequence i.e.

$$a_n = \left[\sqrt{2n} + \frac{1}{2}\right]. \quad \square$$

5.11.203 Problem. (Russian Mathematical Olympiad, 1995) The sequence $a_0, a_1, a_2, ...$ satisfies

$$a_{m+n} + a_{m-n} = \frac{1}{2}\left(a_{2m} + a_{2n}\right)$$

for all nonnegative integers m and n with $m \geq n$. If $a_1 = 1$, determine a_n.

5.11.203.1 Solution. The given relations produces the following by putting $n = 0, n = m$

$$a_m + a_m = \frac{1}{2}\left(a_{2m} + a_0\right) \text{ and } a_{2m} + a_0 = \frac{1}{2}\left(a_{2m} + a_{2m}\right)$$

imply $a_{2m} = 4a_m$, as well as $a_0 = 0$. We compute $a_2 = 4, a_4 = 16$. Also, $a_1 + a_3 = (a_2 + a_4)/2 = 10$, so $a_3 = 9$. At this point we claim that $a_k = k^2 \ \forall \ k \geq 1$. We prove our claim by induction on k. Suppose that $a_j = j^2 \ \forall \ j < k$. The given equation with $m = k - 1$ and $n = 1$ gives

$$\begin{aligned}
a_n &= \frac{1}{2}(a_{2n-2} + a_2) - a_{n-2} \\
&= 2a_{n-1} + 2a_1 - a_{n-2} \\
&= 2(n^2 - 2n + 1) + 2 - (n^2 - 4n + 4) = n^2.
\end{aligned}$$

This completes the proof. $\square$

5.11.204 Problem. Let (x_n) be a sequence with $x_n > 0 \ \forall \ n \in \mathbb{N}$, show that

$$\liminf_{n\to\infty} \frac{x_{n+1}}{x_n} \leq \liminf_{n\to\infty} \sqrt[n]{x_n} \leq \limsup_{n\to\infty} \sqrt[n]{x_n} \leq \limsup_{n\to\infty} \frac{x_{n+1}}{x_n}.$$

Deduce that if $\lim_{n\to\infty} \dfrac{x_{n+1}}{x_n}$ exists, then $\lim_{n\to\infty} \sqrt[n]{x_n}$ exists. What happens to the converse?

5.11.204.1 Solution. Assume first that $\lim_{n\to\infty} \dfrac{x_{n+1}}{x_n} = l \in \mathbb{R}$. So for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for any $n \geq N$, we have $l - \epsilon \leq \inf_{n\geq N} \dfrac{x_{n+1}}{x_n}$, which implies $(l-\epsilon)x_n \leq x_{n+1}$ for any $n \geq N$. This clearly implies $(l-\epsilon)^{n-N} x_N \leq x_n \to l - \epsilon$, for any $n \geq N$. Hence

$$(l-\epsilon)^{(n-N)/n} x_N^{1/n} \leq x_n^{1/n}.$$

Since $(l-\epsilon)^{(n-N)/n} x_N^{1/n} \to (l-\epsilon)$ as $n \to \infty$, we get

$$l - \epsilon \leq \liminf_{n\to\infty} x_n^{1/n}.$$

Since $\epsilon > 0$ is arbitrary, we get

$$\liminf_{n\to\infty} \frac{x_{n+1}}{x_n} \leq \liminf_{n\to\infty} \sqrt[n]{x_n}.$$

A similar proof will lead to

$$\limsup_{n\to\infty} \sqrt[n]{x_n} \le \limsup_{n\to\infty} \frac{x_{n+1}}{x_n}.$$

If (x_{n+1}/x_n) is convergent, then we have

$$\liminf_{n\to\infty} \frac{x_{n+1}}{x_n} = \limsup_{n\to\infty} \frac{x_{n+1}}{x_n},$$

which obviously implies

$$\liminf_{n\to\infty} \sqrt[n]{x_n} = \limsup_{n\to\infty} \sqrt[n]{x_n} = \lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \sqrt[n]{x_n}.$$

The converse is not true. Indeed, take $x_n = 2+(-1)^n, n \in \mathbb{N}$. It is easy to check that $\lim_{n\to\infty} \sqrt[n]{x_n} = 1$. But

$$\liminf_{n\to\infty} \frac{x_{n+1}}{x_n} = \frac{1}{3} \text{ and } \limsup_{n\to\infty} \frac{x_{n+1}}{x_n} = 3. \quad \square$$

5.11.205 Problem. Given two natural numbers k and m, let $a_1, a_2, .., a_k, b_1, b_2, .., b_m$ be positive numbers greater than 1 such that

$$\sqrt[n]{a_1} + \sqrt[n]{a_2} + ... + \sqrt[n]{a_k} = \sqrt[n]{b_1} + \sqrt[n]{b_2} + ... + \sqrt[n]{b_m},$$

for all positive integers n. Prove that $k = m$ and $a_1 a_2 .... a_k = b_1 b_2 .... b_m$.

5.11.205.1 Solution. Using the fact that $\lim_{n\to\infty} \sqrt[n]{a} = 1$, taking the limits in the relation from the statement to obtain

$$\underbrace{1 + 1 + ... + 1}\{k \ times\} = \underbrace{1 + 1 + ... + 1}\{m \ times\}$$

Hence $k = m$. Using L'Hôpitals theorem, one can prove that $\lim_{x\to 0} \frac{a^x - 1}{x} = \ln a$, and hence $\lim_{n\to\infty} n(\sqrt[n]{a} - 1) = \ln a$. Transform the relation from the hypothesis into

$$n(\sqrt[n]{a_1} - 1) + ... + n(\sqrt[n]{a_k} - 1) = n(\sqrt[n]{b_1} - 1) + ... + n(\sqrt[n]{b_k} - 1).$$

Taking the limit with $n \to \infty$, we obtain

$$\ln a_1 + \ln a_2 + ... + \ln a_k = \ln b_1 + \ln b_2 + ... + \ln b_k.$$

This implies $a_1 a_2 .... a_k = b_1 b_2 .... b_k$. $\square$

5.11.206 Problem. Prove that for any sequence $a_1, a_2, ..., a_n$ of positive real numbers,

$$\frac{1}{\frac{1}{a_1}} + \frac{2}{\frac{1}{a_1} + \frac{1}{a_2}} + \frac{3}{\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3}} + ... + \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + .. + \frac{1}{a_n}} < 2(a_1 + a_2 + ... + a_n).$$

(KöMaL N. 189., November 1998)

5.11.206.1 Solution. Applying the weighted AM-HM inequality,

$$\begin{aligned}\sum_{k=1}^{n}\frac{k}{\frac{1}{a_1}+\frac{1}{a_2}+..+\frac{1}{a_k}} &= \sum_{k=1}^{n}\frac{2}{k+1}\cdot\frac{1+2+..+k}{\frac{1}{a_1}+\frac{2}{2a_2}+..+\frac{k}{ka_k}} \leq \\ &\leq \sum_{k=1}^{n}\frac{2}{k+1}\cdot\frac{1.a_1+2.2a_2+..+k.ka_k}{1+2+..+k} = \\ &= \sum_{k=1}^{n}\frac{4}{k(k+1)^2}\sum_{i=1}^{k}i^2a_i \\ &\leq .... \\ &< \sum_{i=1}^{k}i^2a_i\cdot\frac{2}{i^2} = 2(a_1+a_2+...+a_n). \quad \square\end{aligned}$$

5.11.207 Problem. Calculate $\lim_{n\to\infty}\sqrt[n]{2^n-n}$.

5.11.207.1 Solution. Hint: $2=\sqrt[n]{2^n}>\sqrt[n]{2^n-n}>\sqrt[n]{2^n-2^{n-1}}=2\sqrt[n]{\frac{1}{2}}$, for large n. $\square$

5.11.208 Problem. Is $\sqrt[n]{n^2+\cos n}$ convergent?

5.11.208.1 Solution. Hint:$1<\sqrt[n]{n^2+\cos n}<\sqrt[n]{n^3}=(\sqrt[n]{n})^3$. $\square$

5.11.209 Problem. Prove that

$$\left(1+\frac{1}{n}\right)^{n+1} > \left(1+\frac{1}{n+1}\right)^{n+2}$$

in other words the sequence (a_n) given by

$$a_n = \left(1+\frac{1}{n}\right)^{n+1}$$

is strictly monotone decreasing.

5.11.209.1 Solution. We observe that the above inequality is equivalent to

$$\sqrt[n+2]{\left(\frac{n}{n+1}\right)^{n+1}} < \frac{n+1}{n+2} = \frac{(n+1)\frac{n}{n+1}+1.1}{(n+1)+1}.$$

Thus applying weighted AM-GM inequality to the the numbers $\frac{n}{n+1}, 1$ with respective weights $n+1, 1$, required result follows. $\square$

5.11.210 Problem. Suppose that f is a real-valued function of a real variable, and that $f(x+y)=f(x)f(y)$ for all x and y, $f(1)\neq 0$, and $\lim_{x\to 0}f(x)$ exists. Prove that $\lim_{x\to 0}f(x)=1$.

5.11.210.1 Solution. Since for all $x\in\mathbb{R}, x=2y$ for some y, we have $f(x)=f(2y)=f^2(y)>0$. Further, $f(1)=f(0+1)=f(0)f(1)$, and since $f(1)\neq 0, f(0)=1$. Now let $L=\lim_{x\to 0}f(x)$. Then

$$1=f(0)=f(x-x)=f(x)f(-x)\to L^2.$$

Since L cannot be equal to -1 since this would imply $f(x)<0$ for some x, we have $L=1$. $\square$

5.11.211 Problem. Given $a > 0$ and $x > 0$, show that there exists one and only one sequence of positive numbers $\{x_0, x_1, x_2, ..., x_n, ...\}$ such that

$$x_n = \sum_{i=n+1}^{\infty} x_i^a$$

for $n = 0, 1, 2, ...$

5.11.211.1 Solution. First note that for $a > 0, y > 0$, the equation $x + x^a = y$ has a unique solution with $x > 0$. This is because $x + x^a$ is a continuous, monotonic increasing function of x with $\lim_{x\to\infty}(x+x^a) = \infty$ and is equal to zero at $x = 0$. Thus it assumes every positive real value exactly once for $x \in (0, \infty)$. Now, if the sequence (x_n) satisfies the conditions given in the problem, then

$$\begin{aligned} x_n &= \sum_{i=n+1}^{\infty} x_i^a = x_{n+1}^a + \sum_{i=n+2}^{\infty} x_i^a \\ &= x_{n+1}^a + x_{n+1} \end{aligned}$$

So x_{n+1} must be the unique solution of this equation. If follows that the sequence (x_n) is unique. Further, if we define x_n inductively by taking x_{n+1} as a solution of $x + x^a = x_n$, then it is clear that

$$\begin{aligned} x_n &= x_{n+1}^a + x_{n+1} \\ &= x_{n+1}^a + x_{n+2}^a + x_{n+2} = ... \\ &= \sum_{i=n+1}^{\infty} x_i^a. \quad \square \end{aligned}$$

5.11.212 Problem. Find two decreasing sequences (a_n) and (b_n) of positive numbers such that

$$\sum_{n=1}^{\infty} a_n = \infty \text{ and } \sum_{n=1}^{\infty} b_n = \infty \text{ but } \sum_{n=1}^{\infty} c_n < \infty,$$

where $c_n = \min\{a_n, b_n\}$.

5.11.212.1 Solution. Consider any two sequences (c_n) and (d_n) each tending monotonically to zero, with $d_n > c_n$, and $\sum c_n$ converging while $\sum d_n$ diverges. Since $\sum d_n$ diverges, for each n there exists an integer $\phi(n)$ such that $\sum_{i=n}^{\phi(n)} d_i \geq p$ where p is any fixed positive number. Define the sequence n_k recursively by $n_1 = 1$ and $d_{n_{k+1}} < c_{\phi(n_k)}$. It is clear that $n_k < \phi(n_k) < n_{k+1}$. Now consider the sequences

$$\begin{aligned} (a_n) &\equiv (c_{n_1}, c_{n_1+1}, .., c_{\phi(n_1)}, d_{n_2}, d_{n_2+1}, .., d_{\phi(n_2)}, c_{n_3} .., c_{\phi(n_3)}, d_{n_4}, ...) \\ (b_n) &\equiv (d_{n_1}, d_{n_1+1}, .., d_{\phi(n_1)}, c_{n_2}, c_{n_2+1}, .., c_{\phi(n_2)}, d_{n_3} .., d_{\phi(n_3)}, c_{n_4}, ...). \end{aligned}$$

Clearly, each of the series $\sum a_n$ and $\sum b_n$ diverge since they contain infinitely many stretches of terms adding up to more than p, and $\sum \min\{a_n, b_n\}$ converges by comparison with $\sum c_n$. $\square$

5.11.213 Problem. Determine the limits of the following sequences x_n:

1. $x_n = \dfrac{\sqrt[n]{n!}}{n}$, Ans:$1/e$.

2. $x_n = \dfrac{\sqrt[n]{(kn)!}}{n^k}$, $k \in \mathbb{N}$ Ans: $\frac{k^k}{e^k}$.

3. $x_n = \dfrac{1}{n}\sqrt[n]{(n+1)(n+2)...(n+n)}$,

5.11.213.1 Solution.

1. Denote $y_n = \dfrac{n!}{n^n}$ then find $\dfrac{y_n}{y_{n-1}}$

2. In this case put $y_n = \dfrac{(kn)!}{n^{kn}}$ and proceed.

3. $\frac{4}{e}$. $\square$

5.11.214 Problem. If $U_n = \frac{1}{1.n} + \frac{1}{2.(n-1)} + \frac{1}{3.(n-2)} + ... + \frac{1}{n.1}$, prove that $\lim U_n = 0$.

5.11.214.1 Solution. Hint:

$$(n+1)U_n = \left(1 + \frac{1}{n}\right) + \left(\frac{1}{2} + \frac{1}{n-1}\right) + ... + \left(\frac{1}{n} + 1\right). \quad \square$$

5.11.215 Problem. Suppose $\lim_{x\to c} f(x) = L$ and $\lim_{y\to L} g(y) = M$ and suppose that the $\lim_{x\to c}(f \circ g)(x)$ exists (it might not!). Show that the limit of $g \circ f$ at c must equal either M or $g(L)$ (or both).[1]

5.11.215.1 Solution. Hint: Choose a sequence (x_n) such that $(g \circ f)(x_n)$ is defined, $x_n \neq c$, and $x_n \to c$; either $f(x_n) \neq L$ ultimately, or $f(x_n) = L$ frequently.

5.11.216 Problem. Study the convergence of the sequence:

$$\sqrt{2}, \sqrt{1+\sqrt{2}}, \sqrt{1+\sqrt{1+\sqrt{2}}}, \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{2}}}}, ...$$

5.11.216.1 Solution. Let $f(x) = \sqrt{1+x}$. Then the above sequence can be represented as the recurrence $x_{n+1} = f(x_n), x_0 = \sqrt{2}$. We claim that the sequence (x_n) is convergent. Define $g(x) = x - f(x)$. Then $g(x) = 0$ has the unique solution $\alpha = \dfrac{1+\sqrt{5}}{2}$, and g is increasing on $[0, \infty)$ since $g'(x) = 1 - \dfrac{1}{2\sqrt{1+x}} > 0$ for $x \geq 0$. Thus $g(x) < 0$ on $[0, \alpha)$ and $g(x) \geq 0$ on $[\alpha, \infty)$. Now for $t \in [0, \alpha), g(t) < 0$ means $t \leq f(t)$, so if $x_n \in [0, \alpha)$ then $x_n < f(x_n) = x_{n+1}$. On the other hand, as f is increasing and $x_n < \alpha$, we get $x_{n+1} < \alpha$. Combining these observations, we conclude that if $x_n \in [0, \alpha)$, then $x_n < x_{n+1} < \alpha$. Since $x_0 = \sqrt{2}$ is in this range, (xn) is an increasing sequence which is bounded above by α.To find the limit, let $\beta \geq 0$ be this limit and solve the equation $\beta = \sqrt{1+\beta}$, which would yield $\beta = \alpha$. This proves the claim. $\square$

5.11.217 Problem. Let $[a_1, b_1], [a_2, b_2], ..., [a_n, b_n]$ be subintervals of $[a, b]$. Assume that each point $x \in [a, b]$ lies in at least q of these subsets. Prove that there exists $k \in \{1, 2, .., n\}$ such that $(b_k - a_k) \geq (b - a)\frac{q}{n}$.

[1] P. Ramankutty and M. K. Vamanamurthy [American Mathematical Monthly 82 (1975), 63].

5.11.217.1 Solution. The $\sum_{i=1}^{n}(b_i - a_i)$ must be at least $q(b-a)$ since the intervals cover the whole interval $[a, b]$ at least q many times. Thus $\sum_{i=1}^{n}(b_i - a_i) \geq q(b-a)$ from which it follows that there is a k so that $(b_k - a_k) \geq (b-a)\frac{q}{n}$.

5.11.218 Problem. Show that the set $S = \{x_n; n \in \mathbb{N}\} \cup \{x\}$ is closed in $\mathbb{R}$ if (x_n) is a sequence in $\mathbb{R}$ that converges to $x \in \mathbb{R}$.

5.11.218.1 Solution. Fix $y \in \mathbb{R}, y \notin S$. Let $r = |y - x|(> 0)$ and $r_n = |y - x_n|(> 0)$ for $n \in \mathbb{N}$. Since $x_n \to x$, there exists $n_0 \in \mathbb{N}$ such that $r_n < r/2 \ \forall\ n \geq n_0$. Put $\delta = \min\{r_1, r_2, .., r_{n_0}, r/2\}$. Clearly, $(y - \delta, y + \delta) \cap S = \emptyset$, hence $\mathbb{R} \setminus S$ is open. □

5.11.219 Problem. Let $a_n > 0 \ \forall\ n \in \mathbb{N}$, and $\lim a_n = 0$. Show that there is an unbounded sequence (b_n) of positive numbers such that $\lim_n a_n b_n = 0$. This shows that there is no "slowest" convergence to zero.

5.11.219.1 Solution. Hint. Consider $b_n = 1/\sqrt{a_n}$. □

5.11.220 Problem. Assume that $a_n > 0 \ \forall\ n \in \mathbb{N}$, that $a_1 = 1$, and that $a_2 < 1$. Assume also that the sequence (a_n) is decreasing, and that

$$a_{nk} \leq a_n a_k \ \forall\ n, k \in \mathbb{N}.$$

Show that there is $\epsilon > 0$ such that

$$a_n \leq \frac{1}{n^\epsilon} \ \forall\ n \in \mathbb{N}.$$

5.11.220.1 Solution. Given $n \in \mathbb{N}, n \geq 2$, choose $k \in \mathbb{N}$ so that $2^k \leq n < 2^{k+1}$. We have $a_n \leq a_{2^k} \leq a_2^k$. Therefore, due to the monotonicity of the function $\ln x$, we get $\ln a_n \leq k \ln a_2$. Thus, since $k \ln 2 \leq \ln n$, we have

$$\frac{\ln a_n}{\ln n} \leq \frac{\ln a_2}{\ln 2}.$$

Put

$$\epsilon = -\frac{\ln a_2}{\ln 2}.$$

Then, for $n \in \mathbb{N}$ we have $a_n \leq n^{-\epsilon}$. □

5.11.221 Problem. Show that

$$\lim_{n \to \infty} \sin(\sin(\sin ...(\sin x))) = 0$$

for every $x \in \mathbb{R}$, where $\sin(\sin(\sin ...(\sin x)))$ denotes the function sin applied n times to x.

5.11.221.1 Solution. Let $x \geq 0$, and $a_n = \sin(\sin(\sin ...(\sin x)))$ then $0 \leq a_{n+1}(x) \leq a_n(x) \leq 1$ for each n. Thus $\lim_{n\to\infty} a_n(x)$ exists, denote it by $\alpha \in [0, 1]$. We have $\sin(a_n(x)) = a_{n+1}(x)$ for each $n \in \mathbb{N}$. Take the limit for $n \to \infty$ in this last equality, and use the continuity of the function $\sin x$. We get $\sin \alpha = \alpha$. Thus, we get $\alpha = 0$. □

5.11.222 Problem. Define $x_0 = 0, x_1 = 1$, and

$$x_{n+1} = \frac{n}{n+1}x_n + \frac{1}{n+1}x_{n-1}$$

Prove that (x_n) converges and determine its limit.

5.11.222.1 Solution. Note that

$$\begin{aligned} x_{n+1} - x_n &= \frac{n-(n+1)}{n+1}x_n + \frac{1}{n+1}x_{n-1} \\ &= \frac{-1}{n+1}x_n + \frac{1}{n+1}x_{n-1} \\ &= \frac{-1}{n+1}(x_n - x_{n-1}) \\ &= \frac{(-1)^2}{(n+1)n}(x_{n-1} - x_{n-2}) \\ &\cdots\cdots\cdots\cdots \\ &= \frac{(-1)^n}{(n+1)!}(x_1 - x_0) = \frac{(-1)^n}{(n+1)!} \end{aligned}$$

and thus

$$x_n = \sum_{k=0}^{n-1} \frac{(-1)^k}{(k+1)!} \to e^{-1}. \quad \square$$

5.11.223 Problem. Prove that $(n/e)^n < n! \;\forall\, n \in \mathbb{N}$.

5.11.223.1 Solution. We prove by induction on n. It is clearly true for $n = 1$. Assuming for $n = m$, $(m/e)^m < m!$, we have

$$\left(\frac{m+1}{e}\right)^{m+1} = \frac{m+1}{e}\left(1 + \frac{1}{m}\right)^m \left(\frac{m}{e}\right)^m < \frac{m+1}{e}em! = (m+1)!.$$

Hence it is true for $n = m + 1$. The result follows. $\square$

5.11.224 Problem. For $n \in \mathbb{N}$, write $n = 2^{p-1}(2q - 1)$ where $p, q \in \mathbb{N}$ and write

$$s_n = \frac{1}{p} + \frac{1}{q}$$

Find all limit points of the sequence (s_n). Evaluate $\overline{\lim}_n s_n$ and $\underline{\lim}_n s_n$.

5.11.224.1 Solution. We show that the function $f(p, q) = 2^{p-1}(2q-1)$ is a bijection from $\mathbb{N}\times\mathbb{N} \to \mathbb{N}$. Let

$$\begin{aligned} &f(p, q) = f(h, k) \\ &\Rightarrow 2^{p-1}(2q - 1) = 2^{h-1}(2k - 1) \\ &\Rightarrow 2^{p-h}(2q - 1) = (2k - 1) \\ &\Rightarrow p = h \text{ and } q = k \\ &\Rightarrow (p, q) = (h, k) \Rightarrow f \text{ is one-one.} \end{aligned}$$

We show that f is onto. Let $n \in \mathbb{N}$ be odd, then $n = 2k - 1$ for some $k \in \mathbb{N}$, hence $f(1, k) = 2k - 1$ and if n is even, we can write $n = 2^p(2k - 1)$ some $p, k \in \mathbb{N}$, so $n = f(p + 1, k)$ implies f is onto. Therefore,

$$S = \{s_n; n \in \mathbb{N}\} = \left\{\frac{1}{p} + \frac{1}{q}; p, q \in \mathbb{N}\right\}.$$

Thus $S' = \left\{0, 1, \frac{1}{2}, .., \frac{1}{n}, ...\right\}$, and every member of S' is the subsequential limits of S greatest and least of them are 0 and 1. Hence $\overline{\lim}_n s_n = 1$ and $\underline{\lim}_n s_n = 0$. $\square$

5.11.225 Problem. Let $a \in \mathbb{R}, a \notin \{0, 1, 2\}$ and define $x_1 = a, x_{n+1} = 2 - 2/x_n$ for $n \in \mathbb{N}$. Find the limit points of the sequence (x_n).

5.11.225.1 Solution. Note that $x_{n+4} = x_n \ \forall \ n \in \mathbb{N}$. Therefore the sequence takes on the values $\{x_1, x_2, x_3, x_4\}$ only. $\square$

5.11.226 Problem. Find a divergent sequence (x_n) in $\mathbb{R}$ such that $\lim_{n\to\infty} |x_{n^2} - x_n| = 0$.

5.11.226.1 Solution. For $n \geq 4$, let $k(n)$ be the unique integer such that $2^{2^{k(n)}} \leq n < 2^{2^{k(n)+1}}$ and define $x_n = \sum_{r=1}^{k(n)} 1/r$. Note that $k(n) \to \infty$ and $k(n^2) = k(n) + 1$. Therefore $x_n \to \infty$ and $|x_{n^2} - x_n| = 1/(k(n) + 1) \to 0$. $\square$

5.11.227 Problem. Let S be the set of all sequences $\left\{X = (x_1, x_2, ...) \in \{0, 1\}^{\mathbb{N}}\right\}$, that is, for all n, $x_n \in \{0, 1\}$. Prove that there does not exist a one-to-one mapping from the set $\mathbb{N}$ onto the set S.

5.11.227.1 Solution. Suppose that there exists a bijection $f : \mathbb{N} \to \{0, 1\}^{\mathbb{N}}$, i.e., $f(n) = Z$ and $Z : \mathbb{N} \to \{0, 1\}$ Define the sequence $Y = (y_n)$ so that for all natural numbers n,

$$Y(n) = y_n = \begin{cases} 0, & \text{if } Z(n) = (f(n))\,(n) = 1 \\ 1, & \text{if } Z(n) = (f(n))\,(n) = 0. \end{cases}$$

Then $Y = (y_n) \in S$. Hence there exists some natural number m such that $f(m) = Y$. But this means

$$Y(m) = y_m = (f(m))\,(m)$$

which contradicts the construction of $Y(m)$. Thus no such f exists. $\square$

5.11.228 Problem. Let S_n be the sum of first n terms of the sequence (x_n) where

$$x_n = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n-1}{2} & \text{if } n \text{ is odd} \end{cases}$$

Show that if x and y are positive integers and $x > y$ then $xy = f(x + y) - f(x - y)$.

5.11.228.1 Solution. It is easily verified by induction that

$$S_n = \begin{cases} \frac{n^2}{4} & \text{if } n \text{ is even} \\ \frac{n^2-1}{4} & \text{if } n \text{ is even} \end{cases}$$

Therefore, since $x + y$ and $x - y$ always have the same parity, in any case we must have

$$f(x + y) - f(x - y) = \frac{(x + y)^2 - (x - y)^2}{4} = xy.$$

$\square$

5.11.229 Problem. Let $0 < x_1 < 1$ and $x_{n+1} = x_n(1 - x_n), n \in \mathbb{N}$. Show that $\lim_{n\to\infty} nx_n = 1$.

5.11.229.1 Solution. Multiplying the defining relation by $(n+1)$ we get

$$\begin{aligned}(n+1)x_{n+1} &= nx_n + x_n - (n+1)x_n^2 \\ &= nx_n + x_n[1 - (n+1)x_n]\end{aligned}$$

To prove that nx_n is increasing, we need to show that $1-(n+1)x_n > 0$. From the graph of $x(1-x)$, we note that $x_2 \le 1/4$ and that $x_n \le a \le 1/2$ implies $x_{n+1} \le a(1-a)$. So by induction,

$$(n+1)x_n \le (n+1)\frac{1}{n}\left(1 - \frac{1}{n}\right) = 1 - \frac{1}{n^2} < 1.$$

Furthermore, $nx_n < (n+1)x_n < 1$ and so nx_n is bounded above by 1. Thus nx_n converges to a limit α, with $0 < nx_n < \alpha < 1$. Now, summing (1) from 2 to n we obtain

$$\begin{aligned}1 &\ge (n+1)x_{n+1} \\ &= 2x_2 + x_2(1-3x_2) + x_3(1-4x_3) + ... + x_n[1-(n+1)x_n]\end{aligned}$$

If $\alpha \ne 1$ then $[1-(n+1)x_n] > (1-\alpha)/2$ for all large n and thus (2) would show that $\sum x_n$ is convergent. However $nx_n \ge x_1$ and so $\sum x_n \ge x_1 \sum \left(\frac{1}{n}\right)$, a contradiction. □

5.11.230 Problem. Let a sequence (x_n) be given, and let $y_n = x_{n-1} + 2x_n, n = 2, 3, 4, ...$ Suppose that the sequence (y_n) converges. Prove that the sequence (x_n) also converges.

5.11.230.1 Solution. Let $\lim_{n\to\infty} y_n = y$, and let $x = y/3$. We will show that $\lim_{n\to\infty} x_n = x$. Now, for any $\epsilon > 0$ there is an N such that for all $n > N, |y_n - y| < \epsilon/2$. Again,

$$\begin{aligned}\epsilon/2 > |y_n - y| &= |x_{n-1} + 2x_n - 3x| = |2(x_n - x) + (x_{n-1} - x)| \\ &\ge 2|x_n - x| - |x_{n-1} - x| \\ \Rightarrow &|x_n - x| < \epsilon/4 + \frac{1}{2}|x_{n-1} - x|.\end{aligned}$$

$$\begin{aligned}\text{Thus, } |x_{n+m} - x| &< \epsilon/4 + \frac{1}{2}|x_{n+m-1} - x| \\ &< \epsilon/4 + \frac{1}{2}\left(\epsilon/4 + \frac{1}{2}|x_{n+m-2} - x|\right) = \\ &< \epsilon/4\left(1 + \frac{1}{2}\right) + \frac{1}{2^2}|x_{n+m-2} - x| \\ &< \epsilon/4\left(1 + \frac{1}{2} + \frac{1}{2^2}\right) + \frac{1}{2^3}|x_{n+m-3} - x| \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &< \epsilon/4\sum_{i=0}^{m} 2^{-i} + \frac{1}{2^{m+1}}|x_{n-1} - x| \\ &< \epsilon/2 + 2^{-(m+1)}|x_{n-1} - x|\end{aligned}$$

By taking m large enough, $2^{-(m+1)}|x_{n-1} - x| < \epsilon/2$. Thus for all sufficiently large $k, |x_n - x| < \epsilon$. Hence the result. □

5.11.231 Problem. The terms of a sequence (x_n) satisfy $x_n x_{n+1} = n, n \in \mathbb{N}$ and $\lim_{n\to\infty} \frac{x_n}{x_{n+1}} = 1$. Show that $\pi x_1^2 = 2$.

5.11.231.1 Solution. The first relation shows that

$$x_n = \begin{cases} \dfrac{(n-1)(n-3)...3}{(n-2)(n-4)...2} \dfrac{1}{x_1} & \text{if } n \text{ is even} \\ \dfrac{(n-1)(n-3)...2}{(n-2)(n-4)...1} \dfrac{1}{x_1} & \text{if } n \text{ is odd.} \end{cases}$$

If n is odd,

$$\frac{x_n}{x_{n+1}} = x_1^2 . \frac{2}{1} . \frac{2}{3} . \frac{4}{3} . \frac{4}{5} . \frac{6}{5} ... \frac{n-1}{n-2} . \frac{n-1}{n}.$$

The Wallis product

$$\frac{\pi}{2} = \frac{2}{1} . \frac{2}{3} . \frac{4}{3} . \frac{4}{5} . \frac{6}{5} ......$$

After an even number of factors the partial product is less than $\frac{\pi}{2}$ and after an odd number of factors the partial product is greater than $\frac{\pi}{2}$. Thus for the case when n is odd, $\frac{x_n}{x_{n+1}} < \frac{1}{2}\pi x_1^2$. A similar calculation shows that, when n is even, $\frac{x_n}{x_{n+1}} < \frac{2}{\pi x_1^2}$. Since $\lim_{n\to\infty} \frac{x_n}{x_{n+1}} = 1$, 1 is less than or equal to both $\frac{1}{2}\pi x_1^2$ and its reciprocal. This implies that $\pi x_1^2 = 2$. □

5.11.232 Problem. The sequence (x_n) defined by $x_n = \sum_{k=1}^{n} \sin k;\ n = 1, 2, ...$ is bounded. Find an upper and lower bound.

5.11.232.1 Solution. We observe that

$$\left| \sum_{k=1}^{n} \sin k \right| = \left| Im \sum_{k=1}^{n} e^{ik} \right| \le \frac{|e^{in} - 1|}{|e^i - 1|} \le \frac{2}{|e^i - 1|}.$$

Hence the result follows. □

5.11.233 Problem. If a sequence a_n is bounded, positive and $(a_{n+1} - a_n) \to 0$, then a_n is convergent. True or false?

5.11.233.1 Solution. Consider the sequence (a_n) defined by $a_n = 2 + \sin \sqrt{n}\frac{\pi}{2}$ bounded (located between 1 and 3) and

$$\begin{aligned} a_{n+1} - a_n &= \sin \sqrt{n+1}\frac{\pi}{2} - \sin \sqrt{n}\frac{\pi}{2} \\ &= 2 \cos \frac{\pi}{4} \left(\sqrt{n+1} + \sqrt{n}\right) \sin \frac{\pi}{4} \left(\sqrt{n+1} - \sqrt{n}\right). \end{aligned}$$

Since $\sqrt{n+1} - \sqrt{n} \to 0$, so $a_{n+1} - a_n \to 0$.
At the same time, this sequence is divergent, because for $n = (2k)^2, k \in \mathbb{N}$ one has $a_n = 2 + \sin 2k\frac{\pi}{2} = 2$, while for $n = (4m+1)^2, m \in \mathbb{N}$ one gets $a_n = 2 + \sin(4m+1)\frac{\pi}{2} = 3$.
The converse is true: if (a_n) is convergent, then a_n is bounded and $a_{n+1} - a_n \to 0$. Another example for unbounded sequence is $a_n = \sum_{r=1}^{n} \frac{1}{r}$. □

5.11.234 Problem. If a sequence (a_n) is bounded, positive and $\frac{a_{n+1}}{a_n} \to 1$, then (a_n) is convergent. True or false?

5.11.234.1 Solution. Consider the sequence (a_n) defined by $a_n = 2 + \sin\sqrt{n}\frac{\pi}{2}$ bounded (located between 1 and 3) and we see that

$$\left|\frac{a_{n+1}}{a_n} - 1\right| = \frac{\sin\sqrt{n+1}\frac{\pi}{2} - \sin\sqrt{n}\frac{\pi}{2}}{2 + \sin\sqrt{n}\frac{\pi}{2}} \to 0.$$

And by the previous problem, we that the sequence is divergent. □

5.11.235 Problem.

1. If $\lim_{n\to\infty} x_n = x$, then x is the only cluster point of (x_n).
2. Give an example that the converse is not true.

5.11.235.1 Solution.

1. Left to the reader.
2. For the sequence (x_n) where $x_n = n^{(-1)^{n+1}}$. Here 0 is the cluster point of (x_n), but (x_n) does not tend to 0.

5.11.236 Problem. Let $a_1 = 1$ and $a_{n+1} = a_n + \frac{1}{2a_n}$ for $n \geq 1$. Prove that

1. $n \leq a_n^2 < n + \sqrt[3]{n}$.
2. $\lim_{n\to\infty}(a_n - \sqrt{n}) = 0$.

5.11.236.1 Solution.

1. We have that

$$a_{n+1}^2 = a_n^2 + 1 + \frac{1}{4a_n^2}.$$

In particular, $a_{n+1}^2 > a_n^2 + 1$ and it follows by induction that $a_n^2 \geq n$. We shall prove the other inequality by induction again. It is obvious for $n = 1$. Suppose that it is true for $n = m$ that is, $a_m^2 < m + \sqrt[3]{m}$. Then

$$a_{m+1}^2 = a_m^2 + 1 + \frac{1}{4a_m^2} < m + \sqrt[3]{m} + 1 + \frac{1}{4m}$$

and it is enough to check that

$$\begin{aligned}
&\sqrt[3]{m} + \frac{1}{4m} < \sqrt[3]{m+1} \\
\Leftrightarrow &\frac{1}{4m} < \sqrt[3]{m+1} - \sqrt[3]{m} = \frac{1}{\sqrt[3]{(m+1)^2} + \sqrt[3]{m(m+1)} + \sqrt[3]{m^2}} \\
\Leftrightarrow &\sqrt[3]{(m+1)^2} + \sqrt[3]{m(m+1)} + \sqrt[3]{m^2} < 4m \\
\Leftrightarrow &\sqrt[3]{\left(1 + \frac{1}{m}\right)^2} + \sqrt[3]{1 + \frac{1}{m}} + 1 < 4\sqrt[3]{m}.
\end{aligned}$$

This inequality follows by the inequalities $1 + \frac{1}{n} \leq 2$ and $1 + \sqrt[3]{2} + \sqrt[3]{4} = \frac{1}{\sqrt[3]{2}-1} < 4$.

2. The statement is a consequence of the inequalities

$$0 \leq a_n - \sqrt{n} = \sqrt{n + \sqrt[3]{n}} - \sqrt{n} = \frac{\sqrt[3]{n}}{\sqrt{n + \sqrt[3]{n}} + \sqrt{n}} < \frac{\sqrt[3]{n}}{\sqrt{n}} = \frac{1}{\sqrt[6]{n}}. \quad \square$$

5.11.237 Problem. Let $a \geq 2$ be a real number. Denote x_1 and x_2 the roots of the equation $x^2 - ax + 1 = 0$ and let $S_n = x_1^n + x_2^n, n = 1, 2, ...$

1. Prove that the sequence $\left(\frac{S_n}{S_{n+1}}\right)$ is decreasing.
2. Find all a such that

$$\frac{S_1}{S_2} + \frac{S_2}{S_3} + ... + \frac{S_n}{S_{n+1}} > n - 1$$

for any $n = 1, 2, ...$

5.11.237.1 Solution. If $a \geq 2$ then the roots x_1 and x_2 of the equation $x^2 - ax + 1 = 0$ are positive and $x_1 x_2 = 1$. In particular, $S_n > 0$ for $n = 1, 2, ...$

1. We have

$$\begin{aligned} \frac{S_{n-1}}{S_n} \geq \frac{S_n}{S_{n+1}} &\Leftrightarrow (x_1^{n-1} + x_2^{n-1})(x_1^{n+1} + x_2^{n+1}) \geq (x_1^n + x_2^n)^2 \\ &\Leftrightarrow x_1^{n-1}x_2^{n+1} + x_2^{n-1}x_1^{n+1} \geq 2x_1^n x_2^n \\ &\Leftrightarrow (x_1 x_2)^{n-1}(x_1 - x_2)^2 \geq 0 \end{aligned}$$

holds.

2. Let $a \geq 2$ have the desired property. Then (1) implies

$$n\frac{S_1}{S_2} \geq \frac{S_1}{S_2} + \frac{S_2}{S_3} + ... + \frac{S_n}{S_{n+1}} > n - 1,$$

i.e., $\frac{S_1}{S_2} > 1 - \frac{1}{n} > 1$, and we get $S_1 = a, S_2 = a^2 - 2$ and therefore

$$\frac{a}{a^2 - 2} \geq 1 \Leftrightarrow \frac{(a+1)(a-2)}{a^2 - 2} \leq 0.$$

Since $a \geq 2$ we get $a = 2$.
Conversely, if $a = 2$, then $x_1 = x_2 = 1$ and $S_n = 2$ for any $n = 1, 2, ..$ Hence

$$\frac{S_1}{S_2} + \frac{S_2}{S_3} + ... + \frac{S_n}{S_{n+1}} = n > n - 1. \quad \square$$

5.11.238 Problem. Consider the sequence

$$x_1 = x_2 = 1, x_{n+2} = (4k - 5)x_{n+1} - x_n + 4 - 2k, n \geq 1.$$

Find all integers k such that any term of the sequence is a perfect square.

5.11.238.1 Solution. Let k have the given property. We have $x_3 = 2k - 2 = 4a^2, a \geq 0$ i.e., $k = 2a^2 + 1$. Further $x_4 = 8k^2 - 20k + 13$ and $x_5 = 32k^3 - 120k^2 + 148k - 59 = 256a^6 - 96a^4 + 8a^2 + 1$, If $a = 0$, we get $k = 1$ and the given sequence is 1,1,0,1,1,0,.... Hence $k = 1$ is the solution of the problem.
Let $a > 0$. It is easy to check that

$$(16a^3 - 3a)^2 \geq x_5 = 256a^6 - 96a^4 + 8a^2 + 1 > (16a^3 - 3a - 1)^2.$$

Since x_5 is a perfect square, the first inequality must be equality. i.e., $a = 1$ and then $k = 3$.
We shall prove that $k = 3$ is a solution of the problem. In this case the sequence is defined by $x_1 = x_2 = 1, x_{n+2} = 7x_{n+1} - x_n - 2, n \geq 1$. Since $x_3 = 2^2, x_4 = 5^2, x_5 = 13^2$, it is natural to conjecture $x_n = u_{2n-3}^2$ for $n \geq 2$ where (u_n) is the Fibonacci sequence: $u_1 = u_2 = 1$ and $u_{n+2} = u_{n+1} + u_n$ for $n \geq 1$. To prove this, we first note that $u_{n+2} = 3u_n - u_{n-2}$ and $u_{n+2}u_{n-2} - u_n^2 = 1$ for any odd $n \geq 3$. It follows that $(u_{n+2} + u_{n-2})^2 = 9u_n^2$ and thus

$$u_{n+2}^2 = 9u_n^2 - u_{n-2}^2 - 2u_{n+2}u_{n-2} = 7u_n^2 - u_{n-2}^2 - 2.$$

Hence $x_n = u_{2n-3}^2$ for $n \geq 2$. □

5.11.239 Problem. Let $x_1 > 0$. The sequence (x_n) is defined by $x_{n+1} = x_n + \frac{n}{x_n}$ for $n \geq 1$. Prove that

1. $x_n \geq n$ for $n \geq 2$;
2. the sequence $\left(\frac{x_n}{n}\right)$ converges to a limit and find its limit.

5.11.239.1 Solution.

1. We have, $x_2 = x_1 + \frac{1}{x_1} \geq 2$ and if $x_n \geq n$, then

$$x_{n+1} - n - 1 = x_n + \frac{n}{x_n} - n - 1 = \frac{(x_n - 1)(x_n - n)}{x_n} \geq 0$$

   and the assertion follows by induction.

2. Let $n \geq 2$. It follows from (1) that $x_{n+1} \leq x_n + 1$ then $x_n \leq x_2 + n - 1$ whence $1 \leq \frac{x_n}{n} \leq 1 + \frac{x_2 - 2}{n}$. Therefore the sequence $\left(\frac{x_n}{n}\right)$ is convergent and its limit is 1. (One can prove a stronger statement $\lim_{n\to\infty}(x_n - n) = 0$.)

5.11.240 Problem. For any positive integer n the sum $1 + \frac{1}{2} + .. + \frac{1}{n}$ is written in the form $\frac{p_n}{q_n}$ where p_n and q_n are coprime numbers.

1. Prove that 3 does not divide p_{67}.
2. Find all n, for which 3 divides p_n.

5.11.240.1 Solution.

1. Let $S_n = 1 + \frac{1}{2} + .. + \frac{1}{n}$ and we have $S_2 = \frac{3}{2}, S_7 = \frac{3.120}{140}$,

$$\begin{aligned} S_{22} - S_7 =& \frac{1}{8} + \frac{1}{22} + \frac{1}{10} + \frac{1}{20} + \frac{1}{11} + \frac{1}{19} + \frac{1}{14} + \frac{1}{16} + \frac{1}{13} + \frac{1}{17} + \\ & \frac{1}{9} + \frac{1}{12} + \frac{1}{15} + \frac{1}{18} + \frac{1}{21} = \frac{30a}{b} + \frac{51}{140} = \frac{3c}{d} \end{aligned}$$

where $(a,b)=(c,d)=1$. It is easy to see that $a\equiv b \pmod 3$. Hence $3\nmid c,d, c\not\equiv d \pmod 3), p_{22}=3p'_{22}$ and $3\nmid p'_{22}$. Similarly, we have $S_{67}-S_{22}=\frac{90e}{f}+\frac{c}{d}$ where $3\nmid f$. It follows that $3\nmid p_{67},q_{67}$.

2. Let $S_n=\frac{k_n}{3^{m_n}l_n}$ where $3\nmid k_n,l_n$. Then

$$\begin{aligned}S_{3n}&=\frac{S_n}{3}+1+\frac{1}{2}+\frac{1}{4}+\frac{1}{5}+\frac{1}{7}+....+\frac{1}{3n-1}+\frac{1}{3n+1}\\&=\frac{k_n}{3^{m_n+1}l_n}+3\frac{a_n}{b_n}=\frac{k_nb_n+3^{m_n+2}l_na_n}{3^{m_n+1}l_nb_n}\end{aligned}$$

where $3\nmid b_n$. Therefore if $m_n\geq -1$, then $m_{3n}=m_n+1$. Analogously, we have $m_{3n+2}=m_n+1$ for $m_n\geq -1$ and $m_{3n+1}=m_n+1$ for $m_n\geq 0$. Since $m_1=0, m_2=m_7=m_{22}=-1$ and $m_{67}=0$, It is easy to see that the answer is $n=2,7,22$. □

5.11.241 Problem. Define $f:[0,\infty)\to\mathbb{R}$ by

$$f(x)=\begin{cases}a_n+\sin\pi x & \text{if } x\in[2n,2n+1]\\ b_n+\cos\pi x & \text{if } x\in(2n-1,2n), \forall\ n\in\mathbb{N}\cup\{0\}.\end{cases}$$

Find all possible sequences (a_n) and (b_n) such that f is continuous on $[0,\infty)$.

5.11.241.1 Solution. We write the function explicitly in the form

$$f(x)=\begin{cases}a_0+\sin\pi x & \text{if } x\in[0,1]\\ b_1+\cos\pi x & \text{if } x\in(1,2)\\ a_1+\sin\pi x & \text{if } x\in[2,3]\\ b_2+\cos\pi x & \text{if } x\in(3,4)\\ \dots\dots\dots \\ b_n+\cos\pi x & \text{if } x\in(2n-1,2n),\\ a_n+\sin\pi x & \text{if } x\in[2n,2n+1]\\ b_{n+1}+\cos\pi x & \text{if } x\in(2n+1,2n+2),\end{cases}$$

Since the function is continuous at $0,1,2,...,2n-1,2n,2n+1,...$ we get

$$\begin{aligned}&f(0)=\lim_{x\to 0}f(x)=\lim_{x\to 0+}f(x)=a_0+\sin\pi 0=a_0\\&f(1)=\lim_{x\to 1-}f(x)=\lim_{x\to 1+}f(x)\\&\Rightarrow a_0=b_1+\cos\pi=b_1-1 && (1)\\&\text{and } f(2)=\lim_{x\to 2-}f(x)=\lim_{x\to 2+}f(x)\\&\Rightarrow a_1=b_1+\cos 2\pi=b_1+1 && (2)\\&\text{and } \lim_{x\to 3-}f(x)=\lim_{x\to 3+}f(x)=f(3)\\&\Rightarrow a_1+\sin 3\pi=b_2+\cos 3\pi\\&\Rightarrow a_1=b_2-1 && (3)\end{aligned}$$

$$\begin{aligned}
&\text{and} \lim_{x\to(2n)-} f(x) = \lim_{x\to(2n)+} f(x) = f(2n)\\
&\Rightarrow b_n + \cos 2n\pi = a_n + \sin 2n\pi\\
&\Rightarrow b_n + 1 = a_n\\
&\text{and} \lim_{x\to(2n+1)-} f(x) = \lim_{x\to(2n+1)+} f(x) = f(2n+1)\\
&\Rightarrow a_n + \sin(2n+1)\pi = b_{n+1} + \cos(2n+1)\pi\\
&\Rightarrow a_n = b_{n+1} - 1,
\end{aligned}$$

Thus, we get by (1) and (2) $a_0 + 1 = a_1 - 1$ implies $a_1 = a_0 + 2$ and $b_1 = a_0 + 1$ $b_2 = a_1 + 1 = a_0 + 3, a_2 = b_2 + 1 = a_0 + 4$ and hence $a_n = a_0 + 2n, b_n = a_0 + 2n - 1$ where $a_0 = f(0)$. □

5.11.242 Problem. The sequences $(a_n), (b_n)$ are such that $a_{n+1} = 2b_n - a_n$ and $b_{n+1} = 2a_n - b_n$. Prove that

1. $a_{n+1} = 2(a_1 + b_1) - 3a_n$.
2. if $a_n > 0$ for every n, then $a_1 = b_1$.

5.11.242.1 Solution.

1. Since $a_{n+1} + b_{n+1} = 2b_n - a_n + 2a_n - b_n = a_n + b_n$, we have

$$a_{n+1} = 2(a_n + b_n) - 3a_n = a_{n+1} = 2(a_1 + b_1) - 3a_n.$$

2. Using (1), we obtain

$$a_{n+1} - \frac{a_1 + b_1}{2} = -3\left(a_n - \frac{a_1 + b_1}{2}\right),$$

$$\text{which produces } a_{n+1} - \frac{a_1 + b_1}{2} = (-3)^n\left(a_1 - \frac{a_1 + b_1}{2}\right).$$

Since $\lim_{n\to\infty} 3^n = \infty$, it follows that if $a_1 > b_1$, then $\lim_{n\to\infty} a_{2n} = -\infty$, a contradiction. Analogously it is not possible to have $a_1 < b_1$. Thus $a_1 = b_1$. □

5.11.243 Problem. The sequence (a_n) is defined by $a_1 = 0$ and $a_{n+1} = a_n + 4n + 3, n \geq 1$.

1. Express a_n as a function of n.
2. Find the limit

$$\lim_{n\to\infty} \frac{\sqrt{a_n} + \sqrt{a_{4n}} + \sqrt{a_{4^2n}} + .. + \sqrt{a_{4^{10}n}}}{\sqrt{a_n} + \sqrt{a_{2n}} + \sqrt{a_{2^2n}} + .. + \sqrt{a_{2^{10}n}}}.$$

5.11.243.1 Solution.

1. Using recurrence relation, we get

$$\begin{aligned}
a_k &= a_{k-1} + 4(k-1) + 3 = a_{k-2} + 4(k-2) + 4(k-1) + 2.3 = ...\\
&= a_1 + 4(1 + 2 + ... + (k-1)) + (k-1)3 = 2k(k-1) + 3(k-1)\\
&= (2k+3)(k-1).
\end{aligned}$$

2. We have $\lim_{n\to\infty} \frac{\sqrt{a_{kn}}}{n} = \lim_{n\to\infty} \sqrt{\left(2k+\frac{3}{n}\right)\left(k-\frac{1}{n}\right)} = \sqrt{2}k$. Therefore required limit is

$$\frac{1+4+4^2+...+4^{10}}{1+2+2^2+...+2^{10}} = 683. \quad \square$$

5.11.244 Problem. Define the sequence $x_1, x_2, ...$ inductively by $x_1 = \sqrt{5}$ and $x_{n+1} = x_n^2 - 2$ for each $n \geq 1$. Show that

$$\lim_{n\to\infty} \frac{x_1 x_2 ... x_n}{x_{n+1}} = 1.$$

5.11.244.1 Solution. Let $y_n = x_n^2$. Then

$$\begin{aligned} y_{n+1} &= x_{n+1}^2 = (x_n^2 - 2)^2 = (y_n - 2)^2 \\ \Rightarrow y_{n+1} - 4 &= y_n(y_n - 4). \end{aligned}$$

Since $y_2 = 9 > 5$, we have $y_3 = (y_2 - 2)^2 > 5$ and inductively $y_n > 5$ for $n \geq 2$. Hence $y_{n+1} - y_n = y_n^2 - 5y_n - 4 > 4$ for all $n \geq 2$ so $y_n \to \infty$. Now by $y_{n+1} - 4 = y_n(y_n - 4)$.

$$\begin{aligned} \left(\frac{x_1 x_2 ... x_n}{x_{n+1}}\right)^2 &= \frac{y_1 y_2 ... y_n}{y_{n+1}} \\ &= \frac{y_{n+1} - 4}{y_{n+1}} . \frac{y_1 y_2 ... y_n}{y_{n+1} - 4} = \frac{y_{n+1} - 4}{y_{n+1}} . \frac{y_1 y_2 ... y_{n-1}}{y_n - 4} = ... \\ &= \frac{y_{n+1} - 4}{y_{n+1}} . \frac{1}{y_1 - 4} = \frac{y_{n+1} - 4}{y_{n+1}} \to 1. \end{aligned}$$

Hence

$$\lim_{n\to\infty} \frac{x_1 x_2 ... x_n}{x_{n+1}} = 1. \quad \square$$

5.11.245 Problem.

1. A sequence (x_n) of real numbers satisfies $x_{n+1} = x_n \cos x_n \ \forall \ n \geq 1$. Does it follow that this sequence converges for all initial values x_1?

2. A sequence (y_n) of real numbers satisfies $y_{n+1} = y_n \sin y_n \ \forall \ n \geq 1$. Does it follow that this sequence converges for all initial values y_1?

5.11.245.1 Solution.

1. No. For example, for $x_1 = \pi$ we have $x_n = (-1)^{n-1}\pi$, and the sequence is divergent.

2. Similar to the first solution, $|y_n| \to a$ for some real number a. Notice that $t. \sin t = (-t)\sin(-t) = |t| \sin |t| \ \forall \ t \in \mathbb{R}$, hence $y_{n+1} = |y_n| \sin |y_n| \ \forall \ n \geq 2$. Since the function $t \to t \sin t$ is continuous, $y_{n+1} = |y_n| \sin |y_n| \to |a|| \sin a|$. (For continuity see chapter 6) $\square$

5.11.246 Problem. Let (a_n) be a sequence with $1/2 < a_n < 1$ for all $n \geq 1$. Define the sequence (x_n) by

$$x_1 = a_1, \quad x_{n+1} = \frac{a_{n+1} + x_n}{1 + a_{n+1} x_n}, (n \geq 1).$$

What are the possible values of $\lim_{n\to\infty} x_n$? Can such a sequence diverge?

5.11.246.1 Solution. We prove by induction that

$$0 < 1 - x_n < \frac{1}{2^{n+1}}.$$

Then we will have $(1-x_n) \to 0$ and therefore $x_n \to 1$. The case $n = 1$ is true since $1/2 < x_1 = a_1 < 1$. Suppose that the induction hypothesis holds for n, from the recurrence relation we get

$$1 - x_{n+1} = 1 - \frac{a_{n+1} + x_n}{1 + a_{n+1}x_n} = \frac{1 - a_{n+1}}{1 + a_{n+1}x_n}(1 - x_n)$$

By the relation $1/2 < a_n < 1$, we get $1 - a_n < 1/2$ and

$$0 < \frac{1 - a_{n+1}}{1 + a_{n+1}x_n} < 1 - \frac{1}{2},$$

we obtain

$$0 < 1 - x_{n+1} < \frac{1}{2}(1 - x_n) < \frac{1}{2} \cdot \frac{1}{2^{n+1}} = \frac{1}{2^{n+2}}.$$

Hence, the sequence converges in all cases and $x_n \to 1$. □

5.11.247 Problem. Find $\lim_{n\to\infty} n^{\frac{1}{\sqrt{n}}}$.

5.11.247.1 Solution. Hint: $x_n = n^{\frac{1}{\sqrt{n}}} = \left((\sqrt{n})^{\frac{1}{\sqrt{n}}}\right)^2$.

5.12 Additional Exercises on Chapter 5.

5.12.1 Exercise. For each $x \in \mathbb{R}$, let $\langle x\rangle$ denote the distance from x to the integer nearest x and, recall that $[x]$ denotes the greatest integer $\leq x$. Find each limit if it exists. If it doesn't, explain why!

1. $\lim_{x\to\infty}[x]$
2. $\lim_{x\to\infty}(x - [x])$
3. $\lim_{x\to\infty} \frac{[x]}{x}$
4. $\lim_{x\to\infty}\langle x\rangle$
5. $\lim_{x\to\infty}(x - \langle x\rangle)$
6. $\lim_{x\to\infty} \frac{\langle x\rangle}{x}$

5.12.2 Exercise. Find all limit points of the following sets:

1. $\{\sqrt{m} - \sqrt{n}; m, n \in \mathbb{N}\}$.
2. $\left\{\frac{\sqrt{m}-\sqrt{n}}{\sqrt{m}+\sqrt{n}}; m, n \in \mathbb{N}\right\}$.
3. $\left\{m + n\sqrt{2}; m, n \in \mathbb{Z}\right\}$.

4. $\{\sin n; n \in \mathbb{N}\}$

5. $\{\tan n; n \in \mathbb{N}\}$

6. $\{\cot n; n \in \mathbb{N}\}$

5.12.3 Exercise. Give a counterexample to the statement: if sequences (a_n) is convergent and (b_n) is divergent, then the sequence (a_n/b_n) is divergent.

5.12.4 Exercise. If $(x_n), (y_n) \in (0, \infty)^{\mathbb{N}}$ and (x_n/y_n) is monotone, then the sequence (z_n) defined by

$$z_n = \frac{x_1 + ... + x_n}{y_1 + ... + y_n}$$

is also monotone. Hint. $a/b \leq c/d \Rightarrow a/b \leq (a+c)/(b+d) \leq c/d$.

5.12.5 Exercise. Let $A = \{a_1, a_2, .., a_r\}$ be a set of positive reals and $a = \max\{a_1, a_2, .., a_r\}$. Prove that for any positive integer n,

$$a^n \leq (a_1^n + a_2^n + .. + a_r^n) \leq ra^n.$$

and deduce that

$$a \leq (a_1^n + a_2^n + .. + a_r^n)^{\frac{1}{n}} \leq r^{\frac{1}{n}} a.$$

and hence show that

$$\lim_{n\to\infty} (a_1^n + a_2^n + .. + a_r^n)^{\frac{1}{n}} = a.$$

5.12.6 Exercise. Give an example where $\lim_{x\to a} f(x) = b$ and $\lim_{x\to b} g(x) = c$ but $\lim_{x\to a} g(f(x)) \neq c$.

5.12.7 Exercise. Show that

1. $\sin x = o(\sqrt{x})$ as $x \to 0$, and that $\sin x \neq o(x)$ as $x \to 0$.

2. $\sin x = O(x)$ as $x \to 0$.

5.12.8 Exercise. Suppose that every subsequence of (x_n) has a subsequence that converges to 0. Show that $\lim_n x_n = 0$.

5.12.9 Exercise. Let I be an interval, and let $x_0 \in I$. Suppose that f, g and h are functions defined on I (except possibly at x_0). Then, as $x \to x_0$ the following are true.

1. If $f = o(h)$ and $g = o(h)$, then $f \pm g = o(h)$ and $fg = o(h^2)$.

2. If $f = o(h)$ and $c \in \mathbb{R}$; then $cf = o(h)$ and (if $c \neq 0$) $cf = o(ch)$.

3. If $f = o(h)$ and g is bounded away from zero $(x \to x_0)$, then $f/g = o(h)$.

4. If $f = o(g)$ and if $g = O(h)$ (e.g., if $g = o(h)$, then $f = o(h)$.

5. If $f = o(1)$ then $1/(1 + f) = 1 - f + o(h)$.

6. If $f = O(h)$ and $g = O(h)$, then $f \pm g = O(h)$ and $fg = O(h^2)$.

7. If $f = O(h)$ and $c \in \mathbb{R}$; then $cf = O(h)$ and (if $c \neq 0$) $cf = O(ch)$.
8. If $f = O(h)$ and g is bounded away from zero ($x \to x_0$), then $f/g = O(h)$.
9. If $f = O(g)$ and if $g = O(h)$, then $f = O(h)$.
10. If $f = O(1)$ and if $g = O(h)$, then $fg = O(h)$.
11. If $f = o(1)$ and if $g = O(h)$, then $fg = O(h)$.

5.12.10 Exercise. Show that every point of a closed set S is either an accumulation point or an isolated point.

5.12.11 Exercise. (Bolzano-Weierstrass theorem) Every bounded infinite subset of real numbers has at least one accumulation point.

5.12.12 Exercise. A real-valued function f of a real variable is said to satisfy a **Holder condition** with exponent a if there is a constant c such that $|f(x) - f(y)| < c|x - y|^a$ for all x, y. Wherever these functions are used, a is restricted to be < 1. Can you explain why?

5.12.13 Exercise. Let (x_n) be a Cauchy sequence such that x_n is an integer for every $n \in \mathbb{N}$. Show that (x_n) is ultimately constant.

5.12.14 Exercise. Let (x_n) and (y_n) be sequences of positive numbers such that $\lim(x_n/y_n) = \infty$.

1. Show that if $\lim y_n = \infty$, then $\lim x_n = \infty$.
2. Show that if (x_n) is bounded, then $\lim y_n = 0$.

5.12.15 Exercise. Show that if $\lim(a_n/n) = L$, where $L > 0$, then $\lim a_n = \infty$.

5.12.16 Exercise. Let (a_k) be given by the following recurrence formula:

$$\begin{cases} a_1 = a_2 = 1, \\ a_{k+1} = \dfrac{1}{a_{k-1} + \dfrac{1}{a_k}} \quad \forall\, k \geq 2. \end{cases}$$

Show that this sequence is convergent and find its limit.

5.12.17 Exercise. Let (x_n) be properly divergent and let (x_n) be such that $\lim(x_n y_n)$ belongs to $\mathbb{R}$. Show that (y_n) converges to 0.

5.12.18 Exercise. Is the sequence $(n \sin n)$ properly divergent?

5.12.19 Exercise. Give an example of a function $f : [0, 1] \to \mathbb{R}$ for which $f([0, 1])$ is a countably infinite union of disjoint intervals.

5.12.20 Exercise. Let $f : [a, b] \to \mathbb{R}$ be monotone. Let $c \in (a, b)$ and assume that $\lim_{x\to c} f(x)$ exists. Show that $\lim_{x\to c} f(x) = f(c)$.

5.12.21 Exercise. Consider the definition of $a_n \to b$ as $n \to \infty$ as

$$(\forall\, \epsilon > 0)\, (\exists\, n_0)\, (\forall\, n \geq n_0)\, (|a_n - b| < \epsilon)$$

Changing the quantifiers and their order we can produce the following statements:

1. $(\forall\ \epsilon > 0)\,(\exists\ n_0)\,(\exists\ n \geq n_0)\,(|a_n - b| < \epsilon)\,;$
2. $(\forall\ \epsilon > 0)\,(\forall\ n_0)\,(\forall\ n \geq n_0)\,(|a_n - b| < \epsilon)\,;$
3. $(\exists\ \epsilon > 0)\,(\exists\ n_0)\,(\exists\ n \geq n_0)\,(|a_n - b| < \epsilon)\,;$
4. $(\exists\ n_0)\,(\forall\ \epsilon > 0)\,(\forall\ n \geq n_0)\,(|a_n - b| < \epsilon)\,;$
5. $(\forall\ n_0)\,(\exists\ \epsilon > 0)\,(\exists\ n \geq n_0)\,(|a_n - b| < \epsilon)\,.$

Which properties of the sequence (a_n) are expressed by these statements? Give examples of sequences (if they exist) satisfying these properties.

5.12.22 Exercise. Consider the definition of $a_n \to \infty$ as $n \to \infty$ as

$$(\forall\ P)\,(\exists\ n_0)\,(\forall\ n \geq n_0)\,(a_n > P)\,.$$

Changing the quantifiers and the orders we can produce the following statements:

1. $(\forall P)(\exists n_0)(\exists n \geq n_0)(a_n > P);$
2. $(\forall P)(\forall n_0)(\forall n \geq n_0)(a_n > P);$
3. $(\exists P)(\exists n_0)(\forall n \geq n_0)(a_n > P);$
4. $(\exists P)(\exists n_0)(\exists n \geq n_0)(a_n > P);$
5. $(\exists n_0)(\forall P)(\forall n \geq n_0)(a_n > P);$
6. $(\forall n_0)(\exists P)(\exists n \geq n_0)(a_n > P).$

Which properties of the sequence (a_n) are expressed by these statements? Give examples of sequences (if they exist) satisfying these properties.

5.12.23 Exercise. Let r be a rational number, say, $r = p/q$ where p and q are integers with no common divisors. Define $x_n = nr - [nr]$, where $[x]$ denotes the greatest integer in x. Determine the cluster points of (x_n).

5.12.24 Exercise. If $a_0, a_1 > 0$ and $a_{n+1} = \sqrt{a_n} + \sqrt{a_{n-1}}$, then (a_n) is convergent.

5.12.25 Exercise. Let (x_n) be a sequence of real numbers defined as follows:

$$x_1 = 1, x_{n+1} = \frac{3 + 2x_n}{3 + x_n}\ \forall\ n \in \mathbb{N}.$$

1. Show that $\exists\ \lambda \in (0, 1)$ such that for all $n \geq 2$,

$$|x_{n+1} - x_n| \leq \lambda |x_n - x_{n-1}|.$$

2. Prove that $\lim_{n\to\infty} x_n$ exists and find its value.

5.12.26 Exercise. Show that if (a_n) is a sequence that does not converge to L, then there exists an $\epsilon > 0$ and there exists a subsequence (a_{n_k}) of (a_n) such that for all $k \in \mathbb{N}, |a_{n_k} - L| \geq \epsilon$.

5.12.27 Exercise. Let $a_1 = 0, a_2 = 1$, and

$$a_{n+2} = \frac{(n+2)a_{n+1} - a_n}{n-1}.$$

Prove that $\lim_n a_n = e$.

5.12.28 Exercise. If $a_1 = a_2 = 1$ and $a_n = \frac{1}{a_{n-1}} + \frac{1}{a_{n-2}}$, show that (a_n) converges.

5.12.29 Exercise. Show that the sequence (a_n) converges and find its limit.

1. $a_1 = 2,\ a_{n+1} = 2 + \dfrac{1}{3 + \dfrac{1}{a_n}}$.
2. $a_1 = 0, a_2 = \frac{1}{2}, a_{n+2} = \frac{1}{3}(1 + a_{n+1} + a_n^3)$.

5.12.30 Exercise. A sequence (x_n) is said to be **eventually or ultimately bounded** if $\exists$ a positive number M and a positive integer N such that $|x_n| < M$ for all $n \geq N$. Prove that a sequence is bounded iff it is eventually bounded.

5.12.31 Exercise. A sequence (a_n) is a **decimal sequence** in case a_1 is an integer (positive or negative), and

$$a_{n+1} = a_n + \frac{b_n}{10^n},$$

where b_n is one of the numbers 0,1,2,...,9. A decimal sequence is **normal** in case $\forall\ k \in \mathbb{N}\ \exists\ n > k$ such that $b_n \neq 9$.

1. Show that every decimal sequence is bounded and nondecreasing.
2. It has a limit in the real number system.
3. Using the Archimedean property, prove that every real number is the limit of exactly one normal decimal sequence.

5.12.32 Exercise. A sequence (x_n) is said to be **eventually or ultimately monotone** if $\exists$ a positive integer N such that $x_{n+1} \geq x_n$ for all $n \geq N$.
Give an example of a sequence that is not eventually monotone and give an example of a sequence that is eventually increasing but not increasing.

5.12.33 Exercise. Let (x_n) be a bounded sequence and suppose that every convergent subsequence of (x_n) converges to L. (This statement does not guarantee that every subsequence converges!) Prove that (x_n) converges to L.

5.12.34 Exercise. Let (x_n) be a bounded sequence that does not converge. Prove that (x_n) has at least two subsequences that converge to different limits.

5.12.35 Exercise. Prove the following statement about Cauchy sequences without using the fact that a Cauchy sequence of real numbers converges: if a subsequence of a Cauchy sequence converges, then the Cauchy sequence converges.

5.12.36 Exercise. Let r_n be any enumeration of rational numbers in the interval (a, b). Find with proof, $\lim_{n\to\infty} \inf r_n$ and $\lim_{n\to\infty} \sup r_n$.

5.12.37 Exercise. If a is limit point of a set $S = \{a_n; n \in \mathbb{N}\}$, where (a_n) is a sequence in $\mathbb{R}$, then

1. there is a subsequence (a_{n_k}) that converges to a.
2. if $a \leq a_n \ \forall n \in \mathbb{N}$, then there is a decreasing subsequence (a_{n_k}) that converges to a.

5.12.38 Exercise. Let (p_n/q_n) be a sequence of rationals, each being expressed in its lowest terms and let p_n/q_n tend to a finite limit ξ as $n \to \infty$. Prove that if ξ is irrational, then $q_n \to +\infty$ as $n \to \infty$, if $\xi > 0$, then $p_n \to +\infty$ as $n \to \infty$.

5.12.39 Exercise. Let (a_n) be a sequence of real numbers such that $2a_n \leq a_{n-1} + a_{n+1}$ for $n \geq 1$, show that a_n converges.

5.12.40 Exercise. Let $X \subseteq \mathbb{R}$ and $a \in X$, let $\phi : X \to X$ and a sequence a_n be given by

$$a_0 = a, a_{n+1} = \phi(a_n) \ \forall \ n \geq 1.$$

Prove that

1. if ϕ is increasing, then (a_n) is increasing or decreasing according as $\phi(a) \geq a$ or $\phi(a) \leq a$,
2. if ϕ is decreasing, then the sequences a_{2n} and a_{2n+1} are monotonic in opposite senses,
3. if ϕ is continuous on X and if $a_n \to l \in X$, then $\phi(l) = l$.
4. Use the results of (i) and (ii) to discuss the behaviour of the sequence (a_n) for different values of a when $X = \{x \in \mathbb{R}; x \geq 1\}$ and $\phi(x) = 3 - \frac{2}{x}$.

5.12.41 Exercise. Suppose that the sequences $(x_n), (y_n)$ converge to x, y respectively. Then

1. $(x_n + y_n)$ converges to $x + y$,
2. $(x_n y_n)$ converge to xy,
3. $(1/y_n)$ converge to $1/y$ provided $y \neq 0$.

5.12.42 Exercise. Let (x_n) be a sequence and let (x_{n_k}) be any of its subsequences. Show that

$$\liminf_{n\to\infty} x_n \leq \liminf_{n_k\to\infty} x_{n_k} \leq \limsup_{n_k\to\infty} x_{n_k} \leq \limsup_{n\to\infty} x_n.$$

In particular, if (x_{n_k}) is convergent, then

$$\liminf_{n\to\infty} x_n \leq \lim_{n_k\to\infty} x_{n_k} \leq \limsup_{n\to\infty} x_n.$$

Is the converse true? That is, for any l between $\liminf_{n\to\infty} x_n$ and $\limsup_{n\to\infty} x_n$, there exists a subsequence (x_{n_k}) which converges to l.

5.12.43 Exercise. Let $x_1 \in \mathbb{R}$ and $x_{n+1} = \left(2 + \frac{1}{n}\right) x_n - 1$. For what values of x_1 does the sequence diverge? For what values of x_1 does the sequence converge? What is $\lim x_n$?

5.12.44 Exercise. Suppose $b \in \mathbb{R}, a_n \to a \in \mathbb{R}$ and $x_{n+1} = a_n + bx_n$. Prove that:

1. $x_n \to \dfrac{a}{1-b}$, if $|b| < 1$.

2. $x_n \to \dfrac{a}{1-b}$, if $|b| > 1$ and $x_1 + \sum \dfrac{a_k}{b^k} = 0$.

5.12.45 Exercise. Let (a_n) be a sequence of real numbers such that $a_{n+1} - a_n \to 0$. Prove that the set of limit points of its convergent subsequences is the interval with endpoints $\underline{\lim} a_n, \overline{\lim} a_n$.

5.12.46 Exercise. Let (a_n) be a sequence of real numbers such that $a_{n+1} + a_n \to 0$. Prove that the set of limits of its convergent subsequences is either infinite or contains at most two points.

5.12.47 Exercise. The sequence (x_n) is defined by

$$x_1 = 2, x_{n+1} = \frac{2 + x_n}{1 - 2x_n}, n = 1, 2, 3..$$

Prove that $x_n \neq \frac{1}{2}$ or 0 for all n and the terms of the sequence are all distinct.

5.12.48 Exercise. If $a_{n+2} = \frac{a_n + a_{n+1}}{2}$ for all $n \geq 1$, show that $a_n \to \frac{a_1 + 2a_2}{3}$.

5.12.49 Exercise. If $x_1 \in (0, 1)$ and $x_{n+1} = 1 - \sqrt{1 - x_n}$ for all $n \geq 1$, show that (x_n) is decreasing with limit 0. Show also that $\frac{x_{n+1}}{x_n} \to \frac{1}{2}$.

5.12.50 Exercise. If $|a_n| < 2$ and $|a_{n+1} - a_n| \leq \frac{1}{8} \left| a_{n+1}^2 - a_n^2 \right|$ for all $n \geq 1$, show that (a_n) converges.

5.12.51 Exercise. Establish which of the following statements are true.

1. A sequence is convergent if and only if all of its subsequences are convergent.
2. A sequence is bounded if and only if all of its subsequences are bounded.
3. A sequence is monotonic if and only if all of its subsequences are monotonic.
4. A sequence is divergent if and only if all of its subsequences are divergent.

5.12.52 Exercise. Establish which of the following statements are true for an arbitrary sequence (x_n).

1. If all monotone subsequences of a sequence (x_n) are convergent, then (x_n) is bounded.
2. If all monotone subsequences of a sequence (x_n) are convergent, then (x_n) is convergent.
3. If all convergent subsequences of a sequence (x_n) converge to 0, then (x_n) converges to 0.
4. If all convergent subsequences of a sequence (x_n) converge to 0 and (x_n) is bounded, then (x_n) converges to 0.

5.12.53 Exercise. Establish which of the following statements are true.

1. Every subsequence of a convergent real sequence is convergent.
2. Every subsequence of a divergent real sequence is divergent.
3. Every subsequence of a bounded real sequence is bounded.
4. Every subsequence of an unbounded real sequence is unbounded.

5. Every subsequence of a monotone real sequence is monotone.

6. Every subsequence of a nonmonotone real sequence is nonmonotone.

7. If every subsequence of a real sequence converges, the sequence itself converges.

8. If for a real sequence (a_n), the sequences (a_{2n}) and (a_{2n+1}) both converge, then (a_n) converges.

9. If for a real sequence (a_n), the sequences (a_{2n}) and (a_{2n+1}) both converge to the same limit, then (a_n) converges.

5.12.54 Exercise. Show that if a sequence (x_n) converges to a finite limit or diverges to $\pm\infty$, then every subsequence has precisely the same behavior.

5.12.55 Exercise. Suppose a sequence (x_n) has the property that every subsequence has a further subsequence convergent to L. Show that (x_n) converges to L.

5.12.56 Exercise. Let (x_n) be a bounded sequence, let

$$y = \inf\{x_n; n \in \mathbb{N}\},\ x = \sup\{x_n; n \in \mathbb{N}\}.$$

Suppose that, moreover, $y < x_n < x$ for all n. Prove that there is a pair of convergent subsequences (x_{n_k}) and (x_{m_k}) so that

$$\lim_{k\to\infty} |x_{n_k} - x_{m_k}| = x - y.$$

5.12.57 Exercise. Describe all sequences that have only finitely many different subsequences.

5.12.58 Exercise. Let (a_n) be a bounded sequence in $\mathbb{R}$.

1. Prove that if $b = \limsup a_n$ then b satisfies the following condition: $(\forall\ \epsilon > 0)\ a_n < b + \epsilon$ ultimately and $a_n > b - \epsilon$ frequently.

2. Show that the condition in (1) characterizes the limit superior, in the sense that if $b \in \mathbb{R}$ satisfies the condition then necessarily $b = \limsup a_n$.

5.12.59 Exercise. Let (a_n) be a bounded sequence in $\mathbb{R}$.

1. Prove that if $c = \liminf a_n$ then c satisfies the following condition: $(\forall\ \epsilon > 0)\ a_n > c - \epsilon$ ultimately and $a_n < c + \epsilon$ frequently.

2. Show that the condition in (1) characterizes the limit inferior.

5.12.60 Exercise. A subset A of real numbers is a closed subset of $\mathbb{R}$ if, whenever a convergent sequence has all of its terms in A, the limit of the sequence must also be in A. That is, if $x_n \to x$ and $x_n \in A$ for all n, then necessarily $x \in A$.

5.12.61 Exercise. Let I be an interval, (x_n) a sequence in I that converges to a point $x \in I$. Prove that there exists a closed interval $J \subseteq I$ that contains x and every x_n.

5.12.62 Exercise. Prove that if (a_n) is a monotone sequence in $\mathbb{R}$ that has a bounded subsequence, then (a_n) is convergent.

5.12.63 Exercise. If (a_n) is a sequence in $\mathbb{R}$ and $a \in \mathbb{R}$, the following conditions are equivalent:

1. $\forall \epsilon > 0, \quad |a_n - a| < \epsilon$ frequently;
2. there exists a subsequence of (a_n) converging to a.

If the word "frequently" in (1) is replaced by "ultimately", how must (2) be changed so as to remain equivalent to (1)?

5.12.64 Exercise. Prove: If the sequences (x_n) and (y_n) are null then so is $(x_n y_n)$.

5.12.65 Exercise. Prove that the sequence (x_n) is null if and only if (x_n^2) is null.

5.12.66 Exercise. True or false (explain): If (x_n) and (y_n) are sequences in $\mathbb{R}$ such that $(x_n y_n)$ is null, then either (x_n) or (y_n) is null.

5.12.67 Exercise. Let $x_n = \sqrt{n+1} - \sqrt{n}$. True or false (explain): (x_n) is null. Hint: Consider $y_n = \sqrt{n+1} + \sqrt{n}$ and calculate $(x_n y_n)$.

5.12.68 Exercise. Prove that $\lim_{n\to\infty} \left[a_1\sqrt{n+1} + a_2\sqrt{n+2} + a_3\sqrt{n+3}\right] = 0$ where $a_1, a_2, a_3 \in \mathbb{R}$ and $a_1 + a_2 + a_3 = 0$.

5.12.69 Exercise. Show that

$$\lim_{n\to\infty} \frac{1 - \sin\left\{\frac{\pi}{2}\sin\left[\frac{\pi}{2}\sin..\sin\left(\frac{\pi}{2}\cos\frac{\pi}{2}x\right)\right]\right\}}{x^N} = \left(\frac{\pi^2}{8}\right)^{N-1},$$

where 'sin' occurs n times in the functional symbol and $N = 2^{n+1}$.

5.12.70 Exercise. Obtain the limit

$$\lim_{n\to\infty} \frac{\log\{\log[..\log(x+1)]\}}{\log\{\log[..\log x]\}}$$

where 'log' occurs n times in the functional symbol.

5.12.71 Exercise. Prove that $\chi_{\mathbb{Q}}(x) = \lim_{m\to\infty}\left[\lim_{n\to\infty}\{\cos(m!\pi x)\}^{2n}\right] \ \forall\, x \in \mathbb{R}$.

5.12.72 Exercise. Prove that $sign(x) = \lim_{n\to\infty} \frac{2}{\pi}\tan^{-1}(nx) \ \forall\, x \in \mathbb{R}$.

5.12.73 Exercise. Prove that $1 - \chi_{\mathbb{Q}}(x) = \lim_{m\to\infty} sign\left\{\sin^2(m!\pi x)\right\} \ \forall\, x \in \mathbb{R}$.

5.12.74 Exercise. Let α and β be positive numbers. Show that

$$\lim_{n\to\infty} \sqrt[n]{\alpha^n + \beta^n} = \max\{\alpha, \beta\}.$$

5.12.75 Exercise. Let $x_1 = \theta$ and define a sequence recursively by

$$x_{n+1} = \frac{x_n}{1 + x_n/2}.$$

For what values of θ is it true that $x_n \to 0$?

5.12.76 Exercise. If

$$\lim_{n\to\infty} \frac{s_n - \alpha}{s_n + \alpha} = 0$$

what can you conclude about the sequence (s_n)?

5.12.77 Exercise. A sequence of real numbers (x_n) has the property that $(2 - x_n)x_{n+1} = 1$. Show that $\lim_{n\to\infty} x_n = 1$.

5.12.78 Exercise. Suppose that the sequence whose n-th term is $s_n + 2s_{n+1}$ is convergent. Show that (s_n) is also convergent.

5.12.79 Exercise. Show that the sequence

$$\sqrt{7},\ \sqrt{7-\sqrt{7}},\ \sqrt{7-\sqrt{7+\sqrt{7}}},\ \sqrt{7-\sqrt{7+\sqrt{7-\sqrt{7}}}}, ...$$

converges and find its limit.

5.12.80 Exercise. Recall that a real number a is a **cluster point** of a sequence (a_n) in $\mathbb{R}$ if there is a subsequence (a_{n_k}) of (a_n) such that $a_{n_k} \to a$.

1. Show that if $a_n \to a$, then a is the only cluster point of (a_n).
2. Show that the converse of (1) is not true. In other words, show that there is a divergent sequence that has a unique cluster point. Hint. $\left(1, 1, \frac{1}{2}, 2, \frac{1}{3}, 3, \frac{1}{4}, 4, ..., \frac{1}{n}, n, ...\right)$
3. Show that if $a_n \to \infty$ or if $a_n \to -\infty$, then (a_n) has no cluster point.
4. Show that the converse of (3) is not true. In other words, show that there is a sequence without a cluster point that neither tends to ∞ nor tends to $-\infty$. Hint. Any finite set.

5.12.81 Exercise. Let (x_n) be a sequence in $\mathbb{R}$ with no convergent subsequence. Prove that $\{x_n; n \in \mathbb{N}\}$ is a closed subset of $\mathbb{R}$.

5.12.82 Exercise. Prove that for $a > b^2$,

$$\sqrt{a - b\sqrt{a + b\sqrt{a - b\sqrt{a + ...}}}} = \sqrt{a - \frac{3}{4}b^2} - \frac{1}{2}b.$$

5.12.83 Exercise. Let A and B be two sets. Let (A_n) be the sequence of sets defined by

$$A_n = \begin{cases} A & \text{if } n \text{ is odd,} \\ B & \text{if } n \text{ is even.} \end{cases}$$

Then $\limsup A_n = A \cup B$ and $\liminf A_n = A \cap B$. Thus $\lim A_n$ exists if and only if $A = B$.

5.12.84 Exercise. Find the lim sup and lim inf of the sequences $\left(|\sin n|^{\sin n}\right)$ and $(|\cos n|^{\cos n})$.

5.12.85 Exercise. Suppose a sequence (x_n) in $\mathbb{R}$ satisfies the following condition: for each $p \in \mathbb{N}$ and each $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that if $n > N$, then $|x_{n+p} - x_n| < \epsilon$. Is (x_n) necessarily a Cauchy sequence?

5.12.86 Exercise. Prove that

1. $\lim_{n\to\infty} \frac{1}{n}\left\{(2n+1)(2n+2)...(2n+n)\right\}^{\frac{1}{n}} = \frac{27}{4e}$.

2. $\lim_{n\to\infty} \frac{1}{n} \{(a+1)(a+2)...(a+n)\}^{\frac{1}{n}} = \frac{1}{e},\ a > 0.$

5.12.87 Exercise. Prove that the sequence (a_n) is convergent by showing that the subsequences a_{2n} and a_{2n-1} converge to the same limit.

1. $0 < a_1 < a_2$ and $a_{n+2} = \frac{1}{3}(a_{n+1} + a_n)$. for $n \geq 1$.
2. $0 < a_1 < a_2$ and $a_{n+2} = \sqrt{a_{n+1}a_n}$. for $n \geq 1$.
3. $0 < a_1 < a_2$ and $\frac{2}{a_{n+2}} = \frac{1}{a_{n+1}} + \frac{1}{a_n}$. for $n \geq 1$.

5.12.88 Exercise. Prove that the sequence

1. $\left(2,\ \frac{2}{1+2},\ \frac{2}{1+\frac{2}{1+2}}, ...\right)$ converges to 1.
2. $\left(6,\ \frac{6}{1+6},\ \frac{6}{1+\frac{6}{1+6}}, ...\right)$ converges to 2.

5.12.89 Exercise. Prove that the sequence (x_n), where

$$x_n = \frac{\{\sqrt{3}\}}{1.2} + \frac{\{\sqrt{4}\}}{2.3} + ... + \frac{\{\sqrt{n+2}\}}{n(n+1)}$$

is a Cauchy sequence. Note that for any real number a, $a = [a] + \{a\}$ where $[a]$ is the integral part and $\{a\}$ is the fractional part.

5.12.90 Exercise. Assume that $a_n \to a$ and $a < a_n$ for all n. Prove that (a_n) can be rearranged to a monotone decreasing sequence. Hint: Study the sequence $b_n = \max\{a_k; k \geq n\}$.

5.12.91 Exercise. Let $x \in \mathbb{R}$ and consider the sequence $([nx])$, where, as usual, $[t]$ denotes the greatest integer $\leq t$. Show that

$$\lim_{n\to\infty} \frac{[nx]}{n} = x.$$

Deduce, in particular, that $\mathbb{Q}$ is dense in $\mathbb{R}$.

5.12.92 Exercise. Prove or disprove each of the following assertions:

1. Every monotonic sequence converges.
2. Every bounded sequence converges.
3. If (s_n) is bounded then so too is the sequence $(|s_n|)$.
4. If (s_n) is monotonic then so too is the sequence (s_n^2).
5. If (s_n) is bounded then so too is the sequence (s_n^2).
6. If (s_n) is monotonic, $s_n > 0$ then so too is the sequence $(1/s_n)$.
7. If (s_n) is bounded but not monotonic then it is divergent.

5.12.93 Exercise. Give a complete proof for the statement that

$$\lim \frac{1}{s_n} = \frac{1}{\lim s_n}$$

for a sequence (s_n) under some appropriate hypotheses. (Include an example to show that your hypotheses cannot be dropped.)

5.12.94 Exercise. A sequence (x_n) is defined by $x_1 = \mu$ and

$$x_{n+1} = \frac{x_n^2 - 2x_n + 3}{2}$$

Discuss the behaviour of convergence. (Hint: the cases $\mu < 1, 1 < \mu < 3$, etc. should be considered separately.)

5.12.95 Exercise. Give an example of a sequence (s_n) having the stated property or else explain briefly (using appropriate theorems) why no such example can exist.

1. Each $s_n < 0$ and (s_n) has no convergent subsequence.
2. Each $-2 < (s_n) < 0$ and (s_n) has no convergent subsequence.
3. The sequence (s_n) diverges but every subsequence converges.
4. The sequence (s_n) diverges and every subsequence diverges.
5. The sequence (s_n) converges but every subsequence diverges.
6. The sequence (s_n) diverges and every subsequence either diverges or else converges to 2.

5.12.96 Exercise. Prove (or disprove) directly from the definition of a Cauchy sequence:

1. Every bounded sequence is Cauchy.
2. If (s_n) is Cauchy then so too is $(|s_n|)$.
3. If $(|s_n|)$ is Cauchy then so too is (s_n).

5.12.97 Exercise. If $x_{2k} \to L$ and $x_{2k+1} \to L$ as $k \to \infty$ then $x_k \to L$.

5.12.98 Exercise. A sequence (x_n) is said to be contractive if there is some constant $\mu < 1$ so that $|x_{n+2} - x_{n+1}| \leq \mu |x_{n+1} - x_n|$ for all $n \in \mathbb{N}$. Show that every contractive sequence is convergent. Give an example of a contractive sequence. Give an example of a non-contractive sequence. Is every convergent sequence contractive?

5.12.99 Exercise. Does every divergent sequence contain a divergent monotonic subsequence?

5.12.100 Exercise. Does every divergent sequence contain a divergent bounded subsequence?

5.12.101 Exercise. What relation, if any, can you state for the lim sups and lim infs of a sequence (a_n) and one of its subsequences (a_{n_k})?

5.12.102 Exercise. If a sequence (a_n) has no convergent subsequences, what can you state about the lim sups and lim infs of the sequence?

5.12.103 Exercise. Let S denote the set of all real numbers t with the property that some subsequence of a given sequence (a_n) converges to t. What is the relation between the set S and the lim sups and lim infs of the sequence (a_n)?

5.12.104 Exercise. Let a_1 and a_2 be positive numbers and suppose that the sequence (a_n) is defined recursively by

$$a_{n+2} = \sqrt{a_{n+1}} + \sqrt{a_n}.$$

Show that this sequence converges and find its limit.
(Problem posed by A. Emerson in the Amer. Math. Monthly, 85 (1978), p. 496.)

5.12.105 Exercise. Let a sequence be defined by the phrase "consider the sequence of prime numbers 2, 3, 5, 7, 11, 13,.. ". Are you sure that this defines a sequence?

5.12.106 Exercise. One frequently encounters statements such as "what is the next term in the sequence 3, 1, 4, 1, 5,...?" In terms of our definition of a sequence is this correct usage? (By the way, what do you suppose the next term in the sequence might be?)

5.12.107 Exercise. Give two different formulas (for two different sequences) that generate a sequence whose first four terms are 2, 4, 6, 8.

5.12.108 Exercise. Give a formula that generates a sequence whose first five terms are 2, 4, 6, 8, π.

5.12.109 Exercise. The examples listed here are the first few terms of a sequence that is either an arithmetic progression or a geometric progression. What is the next term in the sequence? Give a general formula for the sequence.

1. 7, 4, 1, ...
2. .1, .01, .001, . . .
3. 2, $\sqrt{2}$, 1, . . .

5.12.110 Exercise. If $x_n \to x$ and $y_n \to x$, show that the sequence (z_n), where

$$z_n = \begin{cases} x_{\frac{n+1}{2}}, & \text{if } n \text{ is odd} \\ y_{\frac{n}{2}}, & \text{if } n \text{ is even} \end{cases}$$

also converges to x.

5.12.111 Exercise. Show that there is a sequence (n_k) of distinct integers such that $\lim_k \sin n_k$ exists.

5.12.112 Exercise.

1. Show that, if $0 \leq a \leq b \leq c$ and $b > 0$, then

$$2\sqrt{\frac{a}{c}} \leq \frac{a}{b} + \frac{b}{c} \leq 1 + \frac{a}{c}.$$

2. Let $m = \min\{a_1, a_2, .., a_n\}$ and $M = \max\{a_1, a_2, .., a_n\}$, where $a_k > 0 \ \forall \ k = 1, 2, .., n$. Show that we have
$$2\sqrt{\frac{m}{M}} \le \sum_{k=1}^{n} \frac{a_k}{M} + \sum_{k=1}^{n} \frac{m}{a_k} \le n\left(1 + \frac{m}{M}\right).$$

3. With notation as in (2), prove the inequalities
$$n^2 \le \left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} \frac{1}{a_k}\right) \le n^2 \frac{(m+M)^2}{4mM}.$$

5.12.113 Exercise. Let (x_n) be a sequence of real numbers ≥ 0 such that $\lim x_n = 0$. Show that there are infinitely many indices n such that $x_n \ge x_m$ for every $m \ge n$.

5.12.114 Exercise. Suppose that x_0 is a point of accumulation of both A and B and that $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$, and that $f(x) = g(x)$ if $x \in A \cap B$.

1. What conditions on A and B ensure that if $\lim_{x \to x_0} f(x)$ exists, then $\lim_{x \to x_0} g(x)$ exists?

2. What conditions on A and B ensure that if $\lim_{x \to x_0} f(x)$ and $\lim_{x \to x_0} g(x)$ both exist then they must be equal.

5.12.115 Exercise. Using your definition prove that if (s_n) is a convergent sequence of positive real numbers then the limit must be nonnegative.

5.12.116 Exercise. Let E be a closed set and let (s_n) be a sequence of real numbers converging to a number L. Suppose that $s_n \in E$ for every $n \in \mathbb{N}$. Prove that $L \in E$.

5.12.117 Exercise. Prove that if a sequence (s_n) of real numbers converges to a positive number L then there must exist an integer M so that $s_n > 0$ for every $n \ge M$.

5.12.118 Exercise. Let G be an open set, F a closed set and (x_n) a sequence of real numbers converging to a number x. Prove (or disprove):

1. If $x \in G$ then there exist an integer m so that $x_n \in G$ for all $n \ge m$.

2. If $x \in F$ then there exist an integer m so that $x_n \in F$ for all $n \ge m$.

3. If every $x_n \notin F$ then necessarily $x \notin F$.

4. If every $x_n \in G$ then necessarily $x \in G$.

5.12.119 Exercise. Let (x_n) be a convergent sequence and let E be the range of the sequence. What is the closure of E?

5.12.120 Exercise. Prove that the following definitions for “accumulation point” are equivalent:

1. A point x_0 is an *accumulation point* of a set A provided that every deleted neighbourhood of x_0 contains some points of A.

2. A point x_0 is an *accumulation point* of a set A provided that every neighbourhood of x_0 contains two points of A.

3. A point x_0 is an *accumulation point* of a set A provided that every deleted neighbourhood of x_0 contains infinitely many points of A.

4. A point x_0 is an *accumulation point* of a set A provided that there is a sequence (x_n) of distinct points of A such that $\lim_{n\to\infty} x_n = x_0$.

5.12.121 Exercise. Define a sequence of real numbers recursively by writing $x_0 = a$ and

$$x_{n+1} = \sqrt{\frac{ab^2 + x_n}{a+1}}.$$

Prove that if $0 < a < b$ then the sequence (x_n) is convergent and obtain its limit.

5.12.122 Exercise. Let $(x_n, y_n)_{(n,n)\in\mathbb{N}\times\mathbb{N}}$ be a sequence in $X \times X$ and A and B be two infinite subsets of N. Is the sequence $(x_k, y_p)_{(k,p)\in A\times B}$ a subsequence of (x_n, y_n)?

5.12.123 Exercise. Show that for $x > 1$,

$$\frac{x - x^{-1}}{1} < \frac{x^2 - x^{-2}}{2} < \frac{x^3 - x^{-3}}{3} < .. < \frac{x^n - x^{-n}}{n} < ..$$

5.12.124 Exercise. Evaluate $\lim_{n\to\infty}\{(\sqrt{2}+1)^{2n}\}$ where $\{a\}$ denotes the fractional part of a, i.e. $\{a\} = a - [a]$.

5.12.125 Exercise. If (x_n) is a sequence in $\mathbb{R}$ such that $|x_{n+1} - x_n| \leq 2^{-n}$, then (x_n) is a Cauchy sequence.

5.12.126 Exercise.

1. For any positive sequence (x_n), prove that $\overline{\lim}\left(\frac{x_1+x_{n+1}}{x_n}\right)^n \geq e$.

2. For any number $x \geq e$, determine a positive sequence (x_n) such that $\left(\frac{x_1+x_{n+1}}{x_n}\right)^n \to x$.

5.12.127 Exercise. If $a > 0$ and $x_n = \sqrt{a + \sqrt{a + ... + \sqrt{a}}}$ (n roots). Prove that $x_n \to l_a = \frac{1}{2}(1 + \sqrt{1+4a})$.

5.12.128 Exercise. Let E be any subset of $\mathbb{R}$. Prove that the following statements are equivalent:

1. Any bounded monotone sequence in E converges to a point of E.

2. For any $A \subseteq E$ ($A \neq \emptyset, A$ bounded), $\sup A$ and $\inf A$ both belong to E.

3. Every sequence (x_n) in E satisfying

$$\lim_{N\to\infty}\{\sup\{|x_n - x_m|; m, n \geq N\}\} = 0$$

converges to a point in E.

5.12.129 Exercise. Find a sequence (a_n) of real numbers such that $\lim_{n\to\infty} a_n = 0$, but na_n converge to a transcendental number.

5.12.130 Exercise. Let (x_n) be a sequence in [0,1]. Assume that $2x_n \leq x_{n-1} + x_{n+1} \; \forall \; n \geq 1$, then which of the following is true?

1. $\lim_{n\to\infty}(x_{n+1} - x_n)$ does not exist.
2. $\lim_{n\to\infty}(x_{n+1} - x_n) = 0$.
3. $\lim_{n\to\infty}(x_{n+1} - x_n) = \frac{1}{2}$.
4. $\lim_{n\to\infty}(x_{n+1} - x_n) = 1$.

5.12.131 Exercise. Prove that

$$\lim_{n\to\infty} x^{\frac{1}{2n-1}} = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x < 0. \end{cases}$$

5.12.132 Exercise. Suppose that $x_0 = 1, y_0 = 0$,

$$\begin{aligned} x_n &= x_{n-1} + 2y_{n-1} \\ \text{and } y_n &= x_{n-1} + y_{n-1} \end{aligned}$$

for $n \in \mathbb{N}$. Prove that $x_n^2 - 2y_n^2 = \pm 1$ for $n \in \mathbb{N}$, and

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \sqrt{2}.$$

5.12.133 Exercise. Let (x_n) be a sequence in $\mathbb{R}$. Prove that $\lim_{n\to\infty} x_n = 0$ iff

$$\lim_{n\to\infty} \limsup x_n = 0.$$

5.12.134 Exercise. Let $f : (0,1) \to \mathbb{R}$ be bounded but such that $\lim_{x\to 0} f(x)$ does not exist. Show that there are two sequences (x_n) and (y_n) in (0,1) with $\lim(x_n) = 0 = \lim(y_n)$, but such that $\lim(f(x_n))$ and $\lim(f(y_n))$ exist but are not equal.

5.12.135 Exercise. Let $I = (a, b)$ be a bounded open interval and $f : I \to \mathbb{R}$ be a monotone increasing function on I.

1. If f is bounded above on I then $\lim_{x\to b-} f(x) = \sup_{x\in(a,b)} f(x)$.
2. If f is bounded below on I then $\lim_{x\to a+} f(x) = \inf_{x\in(a,b)} f(x)$.
3. If f is unbounded above on I then $\lim_{x\to b-} f(x) = \infty$.
4. If f is unbounded below on I then $\lim_{x\to a+} f(x) = -\infty$.

5.12.136 Exercise. Let $I = (a, b)$ be a bounded open interval and $f : I \to \mathbb{R}$ be a monotone decreasing function on I.

1. If f is bounded above on I then $\lim_{x\to b-} f(x) = \inf_{x\in(a,b)} f(x)$.
2. If f is bounded below on I then $\lim_{x\to a+} f(x) = \sup_{x\in(a,b)} f(x)$.

3. If f is unbounded above on I then $\lim_{x\to a+} f(x) = \infty$.

4. If f is unbounded below on I then $\lim_{x\to b-} f(x) = -\infty$.

5.12.137 Exercise. Let $I = (a, b)$ be a bounded open interval and $c \in (a, b)$. If $f : I \to \mathbb{R}$ be a monotone function on I, then $\lim_{x\to c+} f(x)$ and $\lim_{x\to c-} f(x)$ both exist.

5.12.138 Exercise. Let $a \in \mathbb{R}$ and $I = (a, \infty)$. Let $f : I \to \mathbb{R}$ be a monotone increasing function on I.

1. If f is bounded above on I then $\lim_{x\to\infty} f(x) = \sup_{x\in(a,\infty)} f(x)$.

2. If f is bounded below on I then $\lim_{x\to a+} f(x) = \inf_{x\in(a,\infty)} f(x)$.

3. If f is unbounded above on I then $\lim_{x\to\infty} f(x) = \infty$.

4. If f is unbounded below on I then $\lim_{x\to a+} f(x) = -\infty$.

5.12.139 Exercise. Let $a \in \mathbb{R}$ and $I = (-\infty, a)$. Let $f : I \to \mathbb{R}$ be a monotone increasing function on I.

1. If f is bounded above on I then $\lim_{x\to a-} f(x) = \sup_{x\in(-\infty,a)} f(x)$.

2. If f is bounded below on I then $\lim_{x\to-\infty} f(x) = \inf_{x\in(-\infty,a)} f(x)$.

3. If f is unbounded above on I then $\lim_{x\to a-} f(x) = \infty$.

4. If f is unbounded below on I then $\lim_{x\to-\infty} f(x) = -\infty$.

5.12.140 Exercise. Let $a \in \mathbb{R}$ and $I = (a, \infty)$. Let $f : I \to \mathbb{R}$ be a monotone decreasing function on I.

1. If f is bounded above on I then $\lim_{x\to\infty} f(x) = \inf_{x\in(a,\infty)} f(x)$.

2. If f is bounded below on I then $\lim_{x\to a+} f(x) = \sup_{x\in(a,\infty)} f(x)$.

3. If f is unbounded above on I then $\lim_{x\to\infty} f(x) = \infty$.

4. If f is unbounded below on I then $\lim_{x\to a+} f(x) = -\infty$.

5.12.141 Exercise. Let $a \in \mathbb{R}$ and $I = (-\infty, a)$. Let $f : I \to \mathbb{R}$ be a monotone decreasing function on I.

1. If f is bounded above on I then $\lim_{x\to a-} f(x) = \sup_{x\in(-\infty,a)} f(x)$.

2. If f is bounded below on I then $\lim_{x\to-\infty} f(x) = \inf_{x\in(-\infty,a)} f(x)$.

3. If f is unbounded above on I then $\lim_{x\to a-} f(x) = \infty$.

4. If f is unbounded below on I then $\lim_{x\to-\infty} f(x) = -\infty$.

5.12.142 Exercise. The set of all rational numbers can be listed in a sequence $r_1, r_2, r_3, ...$ (in other words, there exists a surjection $\mathbb{N} \to \mathbb{Q}$. [Hint: First list the positive rational numbers in a sequence $x_1, x_2, x_3, ...$, for example,

$$\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{2}{2}, \frac{3}{1}, ...$$

(the fractions m/n with $m + n = 2$, then $m + n = 3$, then $m + n = 4$ and so on); the list $0, x_1, -x_1, x_2, -x_2, x_2, x_3, -x_3...,$. then the list contains every rational number.]

5.12.143 Exercise. There exists a nested sequence of closed intervals $[a_n, b_n]$ such that for every n, there exists a real number $r_n \notin [a_n, b_n]$.

5.12.144 Exercise. If $x, y \in \mathbb{R}$, and $0 < x < y$ then $\sqrt{y} - \sqrt{x} < \sqrt{y - x}$.

5.12.145 Exercise. If $x = .2847$ then $x \in [.284, 285] \subset [.28, .29] \subset [.2, .3]$. Interpret a 'decimal' $.d_1 d_2 d_3...$ as the intersection of a nested sequence of closed intervals.

5.12.146 Exercise. Let (a_n, b_n) be a nested sequence of open intervals, where $a_n < b_n$ for all n, and let $A = (a_n, b_n)$ be their intersection.

1. It can happen that $A = \emptyset$. (Example?)
2. If (a_n) is strictly increasing and (b_n) is strictly decreasirig, then A is a closed interval (in particular, $A \neq \emptyset$).

5.12.147 Exercise. Let $[a, b]$ be a closed interval in $\mathbb{R}, a < b$, and let (x_n) be any sequence in $\mathbb{R}$. Prove that $[a, b]$ contains a real number not equal to any term of the sequence.

5.12.148 Exercise. Let (a_n) be a sequence in $\mathbb{R}$ and let $b = \limsup a_n$. Then

1. $r > b \Rightarrow a_n < r$ ultimately.
2. $r < b \Rightarrow a_n > r$ frequently.
3. the condition $r = b$ is inconclusive. For example, if (a_n) is the sequence

$$-1, 0, -1/2, 0, -1/3, 0, ...$$

then $b = 0$ but neither of the conditions "$a_n < 0$ ultimately" or "$a_n > 0$ frequently" holds.

5.12.149 Exercise. Let (a_n) be a sequence in $\mathbb{R}$ and let $a = \liminf a_n$. Then

1. $r < a \Rightarrow a_n > r$ ultimately.
2. $r > b \Rightarrow a_n < r$ frequently.

5.12.150 Exercise. For a subset A of $\mathbb{R}$, the following conditions are equivalent:

1. A is open;
2. if $x_n \to x$ and $x \in A$, then $x_n \in A$ ultimately;
3. if $x_n \to x$ and $x \in A$, then $x_n \in A$ frequently.

5.12.151 Exercise. Let (x_n) be a sequence of distinct elements in $\mathbb{R}$, and suppose that $x_i \to x$. Let f be a one-to-one map of the set of x_i's into itself. Prove that $f(x_i) \to x$.

5.12.152 Exercise. Define $s_1 = \alpha > 0$ and $s_{n+1} = \sqrt{\frac{1+s_n^2}{1+\alpha}}; n \geq 1$. Which of the following is true?

1. If $s_n^2 < \frac{1}{\alpha}$, then (s_n) is monotonically increasing and $\lim_{n\to\infty} s_n = \frac{1}{\sqrt{\alpha}}$.
2. If $s_n^2 < \frac{1}{\alpha}$, then (s_n) is monotonically decreasing and $\lim_{n\to\infty} s_n = \frac{1}{\alpha}$.
3. If $s_n^2 > \frac{1}{\alpha}$, then (s_n) is monotonically increasing and $\lim_{n\to\infty} s_n = \frac{1}{\sqrt{\alpha}}$.
4. If $s_n^2 > \frac{1}{\alpha}$, then (s_n) is monotonically decreasing and $\lim_{n\to\infty} s_n = \frac{1}{\alpha}$.

5.12.153 Exercise. Let $a_n = n + \frac{1}{n}, n \in \mathbb{N}$. Then the sum of the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{a_{n+1}}{n!}$ is (1) $e^{-1} - 1$ (2) e^{-1} (3) $1 - e^{-1}$ (4) $1 + e^{-1}$.

5.12.154 Exercise. Let $a_n = \frac{b_{n+1}}{b_n}$, where $b_1 = 1, b_2 = 1$ and $b_{n+2} = b_n + b_{n+1}, n \in \mathbb{N}$. Then $\lim_{n\to\infty} a_n$ is (1) $\frac{1-\sqrt{5}}{2}$. (2) $\frac{1-\sqrt{3}}{2}$. (3) $\frac{1+\sqrt{3}}{2}$. (4) $\frac{1+\sqrt{5}}{2}$.

5.12.155 Exercise. Let $a_n = \begin{cases} 2 + \frac{(-1)^{\frac{n-1}{2}}}{n} & \text{if } n \text{ is odd} \\ 1 + \frac{1}{2^n} & \text{if } n \text{ is even,} \end{cases}$ for $n \in \mathbb{N}$. Then which one of the following is true?

1. $\sup\{a_n; n \in \mathbb{N}\} = 3$ and $\inf\{a_n; n \in \mathbb{N}\} = 1$.
2. $\sup\{a_n; n \in \mathbb{N}\} = 2$ and $\inf\{a_n; n \in \mathbb{N}\} = 1$.
3. $\liminf a_n = \limsup a_n = \frac{3}{2}$.
4. $\liminf a_n = 1$ and $\limsup a_n = 3$.

5.12.156 Exercise. Let S be the set of all limit points of the set $\left\{\frac{n}{\sqrt{2}} + \frac{\sqrt{2}}{n}\right\}$. Let $\mathbb{Q}^+$ be the set of all positive rational numbers. Then which of the following is true?

1. $\mathbb{Q}^+ \subseteq S$
2. $S \subseteq \mathbb{Q}^+$
3. $S \cap (\mathbb{R} \setminus \mathbb{Q}^+) \neq \emptyset$.
4. $S \cap \mathbb{Q}^+ \neq \emptyset$.

5.12.157 Exercise. Let (x_n) be a sequence in $\mathbb{R}$, defined by

$$a_n = \max\left\{\sin\frac{n\pi}{3}, \cos\frac{n\pi}{3}\right\} \; n \geq 1.$$

Then which of the following statements is/are true about the subsequences a_{6n-1} and a_{6n+4}?

1. Both the subsequences are convergent.
2. Only one of the subsequences is convergent.

3. a_{6n-1} converges to $-\frac{1}{2}$.
4. a_{6n+4} converges to $\frac{1}{2}$.

5.12.158 Exercise. Let (a_n) be a sequence of real numbers such that

$$a_1 = 1,\ a_{n+1} = a_n + a_n^2\ \forall\ n \geq 1.$$

Then which of the following statement is true?

1. $a_4 = a_1 = a_1(1+a_1)(1+a_2)(1+a_3)$.
2. $\lim_{n\to\infty} \frac{1}{a_n} = 0$.
3. $\lim_{n\to\infty} \frac{1}{a_n} = 1$.
4. $\lim_{n\to\infty} a_n = 0$.

5.12.159 Exercise. Define the sequence (s_n) by

$$s_n = \begin{cases} \frac{1}{2^n}\sum_{j=0}^{n-2} 2^{2j} & \text{if } n > 0 \text{ is even} \\ \frac{1}{2^n}\sum_{j=0}^{n-1} 2^{2j} & \text{if } n > 0 \text{ is odd.} \end{cases}$$

Define $\sigma_m = \sum_{n=1}^{m} s_n$, Find the number of limit points of the sequence σ_m.

5.12.160 Exercise.

a. Suppose that f and g have limits in $\mathbb{R}$ as $x \to \infty$ and that $f(x) \leq g(x)\ \forall\ x \in (a,\infty)$. Prove that $\lim_{x\to\infty} f(x) \leq \lim_{x\to\infty} g(x)$.

b. Let f be defined on $(0,\infty) \to \mathbb{R}$. Prove that $\lim_{x\to\infty} f(x) = L$ if and only if $\lim_{x\to\infty} f(1/x) = L$.

c. Show that if $f : (a,\infty) \to \mathbb{R}$ is such that $\lim_{x\to\infty} xf(x) = L$ where $L \in \mathbb{R}$, then $\lim_{x\to\infty} f(x) = 0$.

d. Suppose that $\lim_{x\to\infty} f(x) = L$ where $L > 0$, and that $\lim_{x\to\infty} g(x) = \infty$. Show that $\lim_{x\to\infty} f(x)g(x) = \infty$. If $L = 0$, show by example that this conclusion may fail.

e. Find functions f and g defined on $(0,\infty)$ such that $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty}$, and $\lim_{x\to\infty}$. Can you find such functions, with $g(x) > 0\ \forall\ x \in (0,\infty)$, such that $\lim_{x\to\infty}(f/g)(x) = 0$?

f. Let f and g be defined on (a,∞) and suppose $\lim_{x\to\infty} f(x) = L$ and $\lim_{x\to\infty} g(x) = \infty$. Prove that $\lim_{x\to\infty}(f \circ g)(x) = L$.

Chapter 6

Continuity

It isn't that they can't see the solution. It is that they can't see the problem.
- G.K.Chesterton, The scandal of Father Brown "The point of a Pin."

6.1 Introduction

Notion of continuity, informally, is that $f(x)$ is near to $f(a)$ provided $x \in X$ is sufficiently near to a. The degree of nearness to $f(a)$ (namely ϵ) is specified in advance (and arbitrary); the degree of nearness to a (namely δ) has to he found. If a smaller ϵ is specified, the chances are that δ will also have to be taken smaller (but not necessarily; for a constant function, whatever the given $\epsilon > 0$, any $\delta > 0$ will do). In other words, for $a \in S$, if $x \in X$ is δ-near a then $f(x)$ is ϵ-near to $f(a)$.

6.1.1 Definition. Let $X \subseteq \mathbb{R}$ and $a \in X$. Let $f : X \to \mathbb{R}$. Then a function f is said to be **continuous** at a point $a \in X$ iff $\forall\, \epsilon > 0,\ \exists\, \delta > 0$ such that $x \in X$ and $|x-a| < \delta \Rightarrow |f(x)-f(a)| < \epsilon$ and is **continuous** on X iff it is continuous at each point of X. Equivalently, f is said to be **continuous** at a point $a \in X$ iff $\forall\, \epsilon > 0,\ \exists\, \delta > 0$ such that $f(B(a;\delta) \cap X) \subseteq B(f(x);\epsilon)$.

6.1.2 Theorem. Let $X \subseteq \mathbb{R}$, and $f : X \to \mathbb{R}$. Then the following are equivalent:

1. f is continuous on X.
2. $\forall\, x \in X$, and $\forall\, V$ a nbhd. of $f(x)$ in $\mathbb{R} \Longrightarrow f^{-1}(V)$ is nbhd. of $x \in X$.
3. For $x \in X$ and for each nbhd. V of $f(x)$ in $\mathbb{R}$, there exists a nbhd. U of $x \in X$, such that $f(U) \subseteq V$.
4. V is open in $\mathbb{R} \Rightarrow f^{-1}(V)$ is open X.
5. (Heine's characterization of continuity) Each sequence (x_n) in X converging to x implies the sequence $(f(x_n))$ in $\mathbb{R}$ converges to$f(x)$.
6. $f^{-1}(P^\circ) \subseteq (f^{-1}(P))^\circ\ \forall\, P \subseteq \mathbb{R}$.
7. V is closed in $\mathbb{R} \Rightarrow f^{-1}(V)$ is closed in X.
8. $f(\bar{A}) \subseteq \overline{f(A)}, \forall\, A \subseteq X$, where $\overline{A}$ denotes the closure of A.

9. $f(A') \subseteq \overline{f(A)}, \forall\, A \subseteq X$ where A' is the derived set of A.
10. $\partial(f^{-1}(P)) \subseteq f^{-1}(\partial(P)), \forall\, P \subseteq \mathbb{R}$.
11. $\overline{f^{-1}(P)} \subseteq f^{-1}(\overline{P}), \forall\, P \subseteq \mathbb{R}$.

6.1.3 Example. Let a function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$, defined by $f(x) = \frac{1}{x}$, it can be seen to be continuous on its doain. For if Q is an open subset of $\mathbb{R}$, then we can show that $f^{-1}(Q)$ is open as follows. Suppose that $x \in f^{-1}(Q)$, i.e., $1/x \in Q$. Since Q is open, there is some $\epsilon > 0$ such that $\left(\frac{1}{x} - \epsilon, \frac{1}{x} + \epsilon\right) \subseteq Q$. We may assume that $\epsilon < |1/x|$, so that $\epsilon > 0$ so that $\left(\frac{1}{x} - \epsilon, \frac{1}{x} + \epsilon\right)$ does not contain 0. Then

$$f^{-1}\left(\frac{1}{x} - \epsilon, \frac{1}{x} + \epsilon\right) = \left(\frac{1}{\frac{1}{x} + \epsilon}, \frac{1}{\frac{1}{x} - \epsilon}\right) = \left(x - \frac{x^2\epsilon}{1 + x\epsilon}, x + \frac{x^2\epsilon}{1 - x\epsilon}\right).$$

Let $\delta = \min\left\{\frac{x^2\epsilon}{1+x\epsilon}, \frac{x^2\epsilon}{1-x\epsilon}\right\}$, so that

$$(x - \delta, x + \delta) \subseteq f^{-1}\left(\frac{1}{x} - \epsilon, \frac{1}{x} + \epsilon\right) \subseteq f^{-1}(Q).$$

Hence $f^{-1}(Q)$ is open.

6.2 Lipschitz Functions

6.2.1 Definition (Lipschitz functions). Let $X \subseteq \mathbb{R}$. A function $f : X \to \mathbb{R}$ is called **Lipschitz function** or f satisfies the **Lipschitz condition** if there exists a real constant $L \geq 0$ such that

$$|f(x) - f(y)| \leq L|x - y|\ \forall\, x, y \in X.$$

The **Lipschitz constant** of f is defined by the formula

$$Lip(f) = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|}$$

and represents the smallest number $L \geq 0$ for which f satisfies the Lipschitz condition.

6.2.2 Definition (Point-wise Lipschitz Functions). Let us say that a real-valued function f defined on an interval I is Lipschitz at $a \in I$ if there is a constant $C > 0$ such that $|f(x) - f(a)| \leq C|x - a|\ \forall\, x \in I$. We say that f is **pointwise Lipschitz** on I if this happens for each $a \in I$.

6.3 Uniform continuity

While continuity has a pointwise character, the property of being uniformly continuous has a global character, reflecting the behavior of the function on the entire domain. The simple characterization of the uniform continuity on bounded intervals does not have an analog for the case of unbounded intervals. In other words, we do not have a theorem that tells us which continuous functions on an unbounded interval are uniformly continuous and which ones are not. Therefore the study of uniform continuity on unbounded intervals is simply harder, we dicuss it in several problems below.

We consider the function $f(x) = \frac{1}{x}$ the function is continuous on (0,1), but as the point x_0 becomes closer to 0, the number

$$\delta(\epsilon) = \min\left\{\frac{x_0}{2}, \frac{\epsilon x_0^2}{2}\right\}$$

becomes even with ϵ becomes fixed. Furthermore, we see that no value of $\delta(\epsilon)$ can work for every $x_0 \in (0,1)$ even with ϵ becomes fixed.
Suppose that we restrict our set to be $[\frac{1}{2}, 1)$ instead of (0,1). Then

$$\delta(\epsilon) = \min\left\{\frac{x_0}{2}, \frac{\epsilon x_0^2}{2}\right\} \leq \min\left\{\frac{1}{4}, \frac{\epsilon}{8}\right\}$$

(taking $x_0 = \frac{1}{2}$, the smallest it can be.) Now we have a situation where it is possible to get $\delta(\epsilon)$ that works for every x in the set, once ϵ is chosen. This difference of behavior turns out to be very important. One use of it occurs in the study of Riemann theory of integration, but we shall not discuss it here.

6.3.1 Definition (uniform continuity). Let $f : S \to \mathbb{R}$ is said to be **uniformly continuous** on S if for every $\epsilon > 0$ there exists $\delta > 0$ such that whenever $x, y \in S$ and $|x - y| < \delta$ it follows that $|f(x) - f(y)| < \epsilon$.
Equivalently, f said to be uniformly continuous on a set S if for every sequences (x_n) and (y_n) in S such that $|x_n - y_n| \to 0$, implies $|f(x_n) - f(y_n)| \to 0$.

Now, by attaching to a function $f : A \to \mathbb{R}$ the so called **modulus of continuity**,

$$\omega_f(\delta) = \sup_{x,y \in A\ |x-a|<\delta} |f(x) - f(y)|$$

we can express the property of uniform continuity of f as

$$\lim_{\delta \to 0+} \omega_f(\delta) = 0$$

Geometrically, the property of uniform continuity means that whenever B is a subset of A with diameter less than δ, the diameter of its image should be less than ϵ. Clearly, the Lipschitz functions (particularly, the polynomial functions of the form $ax + b$) are uniformly continuous. The set of all uniformly continuous functions $f : A \to \mathbb{R}$ is a vector space with respect to pointwise operations of addition and multiplication by scalars. However, it is not generally an algebra. In fact, the product of two uniformly continuous functions may not be an uniformly continuous function. For example, the function $f(x) = x^2, x \in \mathbb{R}$, is not uniformly continuous since

$$\omega_f(\delta) = \sup_{x \in \mathbb{R}} \left|(x + \delta)^2 - x^2\right| = \sup_{x \in \mathbb{R}} \left|2x\delta + \delta^2\right| = \infty$$

for every $\delta > 0$.

- (**Non-uniform Continuity**) f fails to be uniformly continuous on a set S if there exists $\epsilon > 0$ such that for any $\delta > 0$, tere exist two points $x_1, x_2 \in S$ such that $|x_1 - x_2| < \delta$, but $|f(x_1) - f(x_2)| \geq \epsilon$. Equivalently,
  f fails to be uniformly continuous on a set S if there exist two sequences (x_n) and (y_n) in S such that $|x_n - y_n| \to 0$, but $|f(x_n) - f(y_n)| \nrightarrow 0$.

6.3.2 Remark. Note that uniform continuity is truly a stronger property than continuity, consider the map $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$. It is continuous but not uniformly continuous. One can easily check that f is not uniformly continuous. The existence of a function on $\mathbb{R}$ that is continuous but not uniformly continuous is directly linked to the fact that $\mathbb{R}$ is non-compact.

A few results are given below:

1. A real valued function f on (a, b) is uniformly continuous on (a, b) if and only if it can be extended to a continuous function $\tilde{f}$ on $[a, b]$.

2. Let f be a continuous function on an interval I [I may be bounded or unbounded]. Let I^0 be the interval obtained by removing the end points of I. If f is differentiable on I^0 and if f' is bounded on I^0, then f is uniformly continuous on I.

3. Suppose that $f : [1, \infty) \to \mathbb{R}$ is uniformly continuous. Then there is a real number $M > 0$ such that $\frac{|f(x)|}{x} \leq M$ for $x \in [1, \infty)$.

   Proof. By uniform continuity of f on $[1, \infty)$ there exists $\delta > 0$ such that $|f(x) - f(y)| < 1$ if $|x - y| < \delta$. Any $x \geq 1$ can be written in the form $x = 1 + n\delta + r$ where $n \in \mathbb{N} \cup \{0\}$ and $0 \leq r < \delta$. Hence

   $$\begin{aligned} |f(x)| &\leq |f(1)| + |f(x) - f(1)| \\ &\leq |f(1)| + (n + 1)|. \end{aligned}$$

   Dividing by x gives

   $$\frac{|f(x)|}{x} \leq \frac{|f(1)| + n + 1}{1 + n\delta + r} \leq \frac{|f(1)| + 2}{\delta} = M.$$

   □

4. If a function $f : (0, \infty) \to \mathbb{R}$ be continuous and satisfies the condition that $\lim_{x\to 0} f(x)$ and $\lim_{x\to\infty} f(x)$ are finite, then the function f is bounded on the interval $(0, \infty)$.

5. If a function $f : (a, \infty) \to \mathbb{R}$ be continuous and $\lim_{x\to\infty} f(x)$ is finite, then the function f is uniformly continuous on the interval $(0, \infty)$.

6. If f is uniformly continuous on a set S and (x_n) is a Cauchy sequence in S, then $(f(x_n))$ is also a Cauchy sequence.

6.3.3 Theorem. If f is continuous on a closed and bounded interval I, then on this interval

1. f is bounded,

2. f has a maximum and miniInum,

3. f is uniformly continuous.

Each has the form: **Local property** of f on $I \Rightarrow$ **Global property** of f on I.

6.3.4 Theorem (Bolzano). Let f be real-valued and continuous on a compact interval $[a, b]$ in $\mathbb{R}$, and suppose that $f(a)$ and $f(b)$ have opposite signs: that i.e., assume $f(a)f(b) < 0$. Then there is at least one point c in the open interval (a, b) such that $f(c) = 0$.

6.3.5 Definition (The Intermediate Value Property). Let $f : S \to \mathbb{R}$ and $a, b \in S$. Let γ be a number that lies between $f(a)$ and $f(b)$. If there is a number c between a and b such that $f(c) = \gamma$, then we say that f satisfies the **intermediate value property**.

6.3.6 Theorem. Assume f is real-valued and continuous on a compact interval S in $\mathbb{R}$. Suppose there are two points $\alpha, \beta;\ \alpha < \beta$ in S such that $f(\alpha) \neq f(\beta)$. Then f takes every value between $f(\alpha)$ and $f(\beta)$ in the interval (α, β).

6.3.7 Lemma. (**Cousin**) Let C be a collection of closed subintervals of $[a, b]$ with the property that for each $x \in [a, b]$ there exists $\delta = \delta_x > 0$ such that C contains all intervals $[c, d] \subseteq [a, b]$ that contain x and have length smaller than δ. Then there exists a partition $a = x_0 < x_1 < .... < x_n = b$ of $[a, b]$ such that $[x_{i-1}, x_i] \in C$ for $i = 1, 2, ..., n$.

This lemma makes precise the statement that if a collection of closed intervals contains all "sufficiently small" ones for $[a, b]$, then it contains a partition of $[a, b]$. We shall frequently see the usefulness of such a partition. This is the most elementary of a collection of tools called covering theorems. Roughly, a cover of a set is a family of intervals covering the set in the sense that each point in the set is contained in one or more of the intervals. We formalize the assumption in Cousin's lemma in this language:

6.3.8 Definition. (**Cousin Cover**) A collection C of closed intervals satisfying the hypothesis of Cousin's lemma is called a Cousin cover of $[a, b]$.

6.4 Darboux Continuous functions

6.4.1 Definition. Let $f : [a, b] \to \mathbb{R}$, then f is called **Darboux continuous** if for any p, q with $a \leq p < q \leq b$ and any $c \in \mathbb{R}$ between $f(p)$ and $f(q)$ there is an s between p and q such that $f(s) = c$.

6.4.2 Definition. A function $s : [a, b] \to \mathbb{R}$ is called a **step function** if $[a, b]$ is the union of a finite number of nonoverlapping intervals $I_1, I_2, ..., I_n$ such that s is constant on each interval, that is, $s(x) = c_k$ for all $x \in I_k,\ k = 1, 2, ..., n$.

6.4.3 Definition. Let $\mathcal{F}$ be a set of functions, each defined on a set $E \subseteq \mathbb{R}$. Let f be a given function defined on E. Then f is said to be **uniformly approximated** on E by the functions of $\mathcal{F}$, if $\forall\ \epsilon > 0\ \exists\ g \in \mathcal{F}$ such that $|f(x) - g(x)| < \epsilon\ \forall\ x \in E$.

6.4.4 Definition. Let $f : [a, b] \to \mathbb{R}$ be a continuous function. We say that f has a **horizontal chord** of length λ if there is a point x such that both of $x, x + \lambda \in [a, b]$ and $f(x) = f(x + \lambda)$.

6.4.5 Theorem (The Horizontal Chord Theorem). Suppose that $f : [0, 1] \to \mathbb{R}$ has a horizontal chord of length λ; say $f(0) = f(1)$, i.e., $\lambda = 1$. Then there are horizontal chords of lengths 1/2,1/3,1/4,..., but not necessarily a horizontal chord of any other length.

6.5 Semicontinuity

6.5.1 Definition. Let $D \subseteq \mathbb{R}$. A real-valued function $f : D \to \mathbb{R}$ is said to be **upper semicontinuous** if for each $\alpha \in \mathbb{R}$, the **super level set** $\{x; f(x) \geq \alpha\}$ is closed. It is **lower semicontinuous**

if every **sublevel set** $\{x; f(x) \leq \alpha\}$ is closed. We can also define about **semicontinuity** at a point. The real-valued function f is **upper semicontinuous** at the point a if

$$\forall\, \epsilon > 0\; \exists\, \delta > 0 \text{ such that } |x-a| < \delta \Rightarrow f(x) < f(a) + \epsilon.$$

Similarly, The real-valued function f is **lower semicontinuous** at the point a if

$$\forall\, \epsilon > 0\; \exists\, \delta > 0 \text{ such that } |x-a| < \delta \Rightarrow f(x) > f(a) - \epsilon.$$

Equivalently, f is **upper semicontinuous** at the point a if

$$f(a) \geq \limsup_{x\to a} f(x) = \inf_{\epsilon>0} \sup_{0<|x-a|<\epsilon} f(x).$$

Similarly, f is **lower semicontinuous** at the point a if

$$f(a) \leq \liminf_{x\to a} f(x) = \sup_{\epsilon>0} \inf_{0<|x-a|<\epsilon} f(x).$$

6.6 Discontinuity

6.6.1 Definition. The function f is said to be **discontinuous** at a point $a \in X$ it fails to be continuous at the point a, i.e. if there exists an $\epsilon > 0$ such that for every $\delta > 0$ there exists $x \in X$ such that $|x-a| < \delta \Rightarrow |f(x) - f(a)| \geq \epsilon$.

We denote the limits $\lim_{x\to c-} f(x)$ and $\lim_{x\to c+} f(x)$ by $f(c+)$ and $f(c-)$ respectively.

6.6.2 Definition. Let $f : [a,b] \to \mathbb{R}$. If $f(c+)$ and $f(c-)$ both exist at some interior point c, then:

1. $f(c) - f(c-)$ is called the lefthand jump of f at c,
2. $f(c+) - f(c)$ is called the righthand jump of f at c,
3. $f(c+) - f(c-)$ is called the jump of f at c.

If any one of these three numbers is different from 0, then c is called a jump discontinuity of f.

6.6.3 Theorem. A function $f : [a,b] \to \mathbb{R}$ is continuous at a point $c \in (a,b)$ if and only if both one sided limits of f at c exist, the limit of the function f at c also exists, and all these three limits are equal to $f(c)$, i.e.,

$$f(c+) = f(c-) = f(c).$$

In view of this theorem, one can classify the discontinuities of a function f defined on an open interval. Namely, if a function $f : (a,b) \to \mathbb{R}$ is discontinuous at a point $c \in (a,b)$, then there are three possibilities.

- If the limit $\lim_{x\to c} f(x)$ exists and equals to some number $L \neq f(c)$, then f has a **removable** discontinuity at the point c;

- If both one sided limits
$$\lim_{x\to c-} f(x) = L_1 \text{ and } \lim_{x\to c+} f(x) = L_2.$$
exist, but are unequal, $L_1 \neq L_2$, then f has a **first order discontinuity** at the point c; we call c an **irremovable** discontinuity because the discontinuity cannot be removed by redefining f at c.

- If at least one of the two one sided limits in the above does not exist, then f has a **second order discontinuity** at the point c.

6.7 Oscillation of a function at a point, on a set

6.7.1 Definition. If $p \in [a,b]$, then the **oscillation** $\omega_f(p)$ of f **at a point** p is defined by

$$\omega_f(p) = \lim_{h\to 0+} \omega_f(N(p;h)\cap[a,b]).$$

where $N(p;h) = \{x \in \mathbb{R}; |x-p| < h\} = (p-h, p+h)$. Equivalently,

$$\omega_f(p) = \inf\{\omega_f(N(p;h)\cap[a,b]); h > 0\}.$$

6.7.2 Definition. Let $f : [a,b] \to \mathbb{R}$ and $D \subseteq [a,b]$. The **oscillation** $\omega_f(D)$ of a function f on a set D is defined by

$$\begin{aligned}\omega_f(D) &= \sup\{|f(x)-f(y)|; x,y \in D\}\\ &= \sup_{x\in D} f(x) - \inf_{x\in D} f(x)\end{aligned}$$

6.7.3 Note. Suppose $S = \{|f(x)-f(y)|; x,y \in D\}$. Let $\sup_{x\in D} f(x) = M$, and $\inf_{x\in D} f(x) = m$. Now

$$\begin{aligned}&f(x) \leq M \text{ and } f(y) \geq m \;\forall\; x,y \in D\\ \Rightarrow&|f(x)-f(y)| \leq M-m\end{aligned}$$

i.e. $M-m$ is an upper bound of S. Let $\epsilon > 0$. Then $\exists\; x', y' \in D$ such that $f(x') > M - \epsilon/2$ and $f(y') < m + \epsilon/2$, so $|f(x') - f(y')| > M - m - \epsilon$. Hence $M-m$ is the supremum of S.

6.7.4 Note. This limit always exists, since $\omega_f(N(p;h)\cap[a,b])$ is a decreasing function of h. In fact, $A \subseteq B$ implies $\omega_f(A) \leq \omega_f(B)$. Also, $\omega_f(x) = 0 \Leftrightarrow f$ is continuous at x.

6.8 Category

6.8.1 Definition. A subset A of $\mathbb{R}$ is said to be of **first category** if it is a countable union of nowhere dense sets. If the set A is not of first category, it is said to be of **second category**. Observe that a subset of a set of first category is itself of first category. In other words, B is a **second category** set in $\mathbb{R}$ if, whenever we write $B = \bigcup_{n=1}^{\infty} E_n$, some E_n fails to be nowhere dense in $\mathbb{R}$; that is, $\left(\overline{E_n}\right)^\circ \neq \emptyset$ for some n. The complement of a first-category set is called a **residual set**.

6.8.2 Note. Consider the subspace $\mathbb{N}$ of $\mathbb{R}$. As a subset of $\mathbb{R}$, $\mathbb{N}$ is of the first category, since $\{n\}$ is nowhere dense in $\mathbb{R}$ for each $n \in \mathbb{N}$. But as a space in itself, $\mathbb{N}$ cannot be expressed as a countable union of nowhere dense sets, since each set $\{n\}$ is dense in $B(n; \frac{1}{2})$. In fact, the only residual set in $\mathbb{N}$ is $\mathbb{N}$ itself.

For example, it follows that $\mathbb{Q}$ is a first category set in $\mathbb{R}$. Some authors refer to first category sets as "meager" or "sparse" sets. In the language of category, we say that $\mathbb{R}$ is a second category set in itself. And by saying that $\mathbb{R} \setminus \mathbb{Q}$ is a second category set in $\mathbb{R}$. The two categories of subsets of $\mathbb{R}$ provide yet another measure of "big" versus "small". A first category set in $\mathbb{R}$, such as $\mathbb{Q}$, is "small" while a second category set in $\mathbb{R}$, such as $\mathbb{R} \setminus \mathbb{Q}$, is "big."

6.8.3 Definition. A point x is called **a point of the second category** of a set S if for every neighborhood U of x the set $U \cap S$ is of the second category.

6.8.4 Theorem (Baire's Theorem). Let (G_n) be a sequence of open dense subsets of $\mathbb{R}$. Then $\bigcap_{n=1}^{\infty} G_n$ is dense in $\mathbb{R}$.

6.8.5 Theorem (Baire Category Theorem). Let F be a nonempty closed subset of $\mathbb{R}$. Let (G_n) be a sequence of open dense subsets of F. Then $\bigcap_{n=1}^{\infty} G_n$ is dense in F.

6.8.6 Definition (Borel set). A set that can be obtained as the union and intersection of an enumerable collection of closed sets and open sets in $\mathbb{R}$ is said to be a **Borel set**.

6.9 Points of continuity and discontinuity, $C(f)$ and $D(f)$

Given an arbitrary function, how can we describe the nature of the set of points where f is continuous? Can it be any set? Given a set $E \subseteq \mathbb{R}$, how can we know whether there is a function that is continuous at every point of E and discontinuous at every point not in E?
We shall see that a function (Thomae's function) exists whose set of points continuity is exactly the irrationals. Can a function exist whose set of points continuity is exactly the rationals? By characterizing the set of such points we can answer this and other questions about the structure of functions.

Let $f : \mathbb{R} \to \mathbb{R}$, and we denote $D(f)$ for the set of points at which f is discontinuous. We investigate that what kind of set it is.
Now we see that

$$\begin{aligned} a \in D(f) &\Leftrightarrow \exists\, \epsilon > 0 \text{ such that } \forall\, \delta > 0\ \exists\, x \in B(a; \delta) \Rightarrow f(x) \notin B(f(a); \epsilon) \\ &\Leftrightarrow \exists\, \epsilon > 0 \text{ such that } \forall\, \delta > 0, \\ &\quad\ \text{we have } |f(x) - f(a)| \geq \epsilon \text{ for some } x \text{ with } |x - a| < \delta. \end{aligned}$$

Which implies that given any bounded, open interval I containing a, we always have

$$\begin{aligned} \text{oscillation of } f \text{ on } I = &\sup\{|f(x) - f(y)|; x, y \in I\} \geq \epsilon \\ \Rightarrow &diam(f(I)) \geq \epsilon \end{aligned}$$

We now state an important result of this section using primarily the notion of oscillation introduced in (6.7).

Result: Let f be defined on a closed interval I (which may be $\mathbb{R}$). Then the set $C(f)$ of points of continuity of f is of type G_δ, and the set $D(f)$ of points of discontinuity of f is of type F_σ. Conversely, if H is a set of type G_δ, then there exists a function f defined on $\mathbb{R}$ such that $C(f) = H$.

6.9.1 Note. For each $\epsilon > 0$, define the set A_ϵ as $A_\epsilon = \{x; x \in [a,b],\ \omega_f(x) \geq \epsilon\}$. If f is discontinuous at some $c \in [a,b]$, then $c \in A_s$ for some $s > 0$. So for $f : [a,b] \to \mathbb{R}$, the set of discontinuities $D(f)$ is can be written as $D(f) = \bigcup_{n=1}^{\infty} A_{1/n}$. That is, the set of discontinuities of $f : [a,b] \to \mathbb{R}$ is a countable union of closed sets. Such a set is said to be F_σ. One can also show that, more generally, if $f : \mathbb{R} \to \mathbb{R}$, then the set of discontinuities is an F_σ set.
An F_σ set will be discussed in the next section.

6.10 F_σ, G_δ sets:

6.10.1 Definition. Let $E \subset \mathbb{R}$. Then

A set E is a G_δ **set** if E is a countable intersection of open sets.

A set E is an F_σ **set** if E is a countable union of closed sets.

6.11 Problems and Solutions on Chapter 6

Warning! *Never Never Ever read this solution unless you tried problems quite long time and gave up. Doing so may impair your ability to think and solve problems.*

6.11.1 Problem. Let $f : X \to \mathbb{R}, X \subseteq \mathbb{R}$. Then the following are equivalent conditions for continuity of f:

1. f is continuous on X.
2. $\forall\, x \in X$, and $\forall\, V$ a nbhd. of $f(x)$ in $\mathbb{R} \implies f^{-1}(V)$ is nbhd. of $x \in X$.
3. For $x \in X$ and for each nbhd. V of $f(x)$ in $\mathbb{R}$, there exists a nbhd. U of $x \in X$, such that $f(U) \subseteq V$.
4. V is open in $\mathbb{R} \Rightarrow f^{-1}(V)$ is open X.
5. Each sequence (x_n) in X converging to x implies the sequence $(f(x_n))$ in $\mathbb{R}$ converges to $f(x)$.
6. $f^{-1}(P^\circ) \subseteq (f^{-1}(P))^\circ\ \forall\, P \subseteq \mathbb{R}$.
7. V is closed in $\mathbb{R} \Rightarrow f^{-1}(V)$ is closed in X.
8. $f(\bar{A}) \subseteq \overline{f(A)}, \forall\, A \subseteq X$, where $\overline{A}$ denotes the closure of A.
9. $f(A') \subseteq \overline{f(A)}, \forall\, A \subseteq X$ where A' is the derived set of A.
10. $\partial(f^{-1}(P)) \subseteq f^{-1}(\partial(P)), \forall\, P \subseteq \mathbb{R}$.
11. $\overline{f^{-1}(P)} \subseteq f^{-1}(\overline{P}), \forall\, P \subseteq \mathbb{R}$.

6.11.1.1 Solution.

1. (1)$\Rightarrow$(2): Let V be a nbhd. of $f(x)$ in $\mathbb{R}$ i.e.$\exists\ \epsilon > 0$ such that $B(f(x);\epsilon) \subseteq V$ and by (a), $\exists\ \delta > 0$ such that $f(B(x;\delta)) \subseteq B(f(x);\epsilon) \subseteq V$ i.e. $B(x;\delta) \subseteq f^{-1}(V)$, which shows that $f^{-1}(V)$ is a nbhd. of x.

2. (2)$\Rightarrow$(3): For $x \in X$, let V be a nbhd. of $f(x)$ in $\mathbb{R}$ by (2), $f^{-1}(V)$ is a nbhd. of x, this means that $\exists\ \epsilon > 0$ such that $B(x;\epsilon) \subseteq f^{-1}(V)$. Set $U = B(x;\epsilon)$, so $U \subseteq f^{-1}(V)$, which shows $f(U) \subseteq V$.

3. (3)$\Rightarrow$(4): Let V be open in $\mathbb{R}$, and $x \in f^{-1}(V)$ implies $f(x) \in V$, and V is a nbhd. of $f(x)$, so by (3) $\exists$ a nbhd. U of x such that $x \in U \subseteq f^{-1}(V)$ implies that x is an interior point of $f^{-1}(V)$, i.e. $f^{-1}(V)$ is open.

4. (4)$\Rightarrow$(5): For all $\epsilon > 0, B(f(x);\epsilon)$ is open in $\mathbb{R}$. By (4), $f^{-1}(B(f(x);\epsilon))$ is open in X, so $\exists\ \delta > 0$ such that $B(x;\delta) \subseteq f^{-1}(B(f(x);\epsilon)) \Rightarrow f(B(x;\delta)) \subseteq B(f(x);\epsilon)$. Let (x_n) be a sequence converges to x, so $\forall\ \delta > 0\ \exists\ m \in \mathbb{N}$ such that $n \geq m \Rightarrow x_n \in B(x;\delta)$ which shows that $x_n \in B(x;\delta) \subseteq f^{-1}(B(f(x);\epsilon))$ which implies $f(x_n) \in B(f(x);\epsilon)\ \forall\ n \geq m$. Hence $f(x_n) \to f(x)$.

5. (5)$\Rightarrow$(6): Let $P \subseteq \mathbb{R}$, and $x \in f^{-1}(P^\circ)$. Then $f(x) \in P^\circ \Rightarrow \exists\ \epsilon > 0$ such that $B(f(x);\epsilon) \subseteq P$. Suppose (x_n) is a sequence converges to x. By (5), $f(x_n) \to f(x)$, so $\exists\ m \in \mathbb{N}$ such that $n \geq m \Rightarrow f(x_n) \in B(f(x);\epsilon) \subseteq P$ implies $f(x_n) \in P\ \forall\ n \geq m \Rightarrow \{n; x_n \in f^{-1}(P)\}$ is finite $\Rightarrow x \in (f^{-1}(P))^\circ$.
   (A sequential criterion for open sets: A set $E \subseteq \mathbb{R}$ is open iff $\forall\ (x_n)$ in X that converges to $x \in E$, the set $\{n; x_n \notin E\}$ is finite.)

6. (6)$\Rightarrow$(7): Let V be closed in $\mathbb{R}$, then V^C is open in $\mathbb{R}$, hence $V^C = intV^C$. So by (6)
$$\begin{aligned}
&f^{-1}(intV^C) \subseteq intf^{-1}(V^C)\\
\Rightarrow\ & f^{-1}(V^C) \subseteq intf^{-1}(V^C) \subseteq f^{-1}(V^C)\\
\Rightarrow\ & f^{-1}(V^C) = intf^{-1}(V^C)\\
\Rightarrow\ & f^{-1}(V^C) \text{ is open}\\
\Rightarrow\ & (f^{-1}(V))^C \text{ is open}\\
\Rightarrow\ & f^{-1}(V) \text{ is closed.}
\end{aligned}$$

7. (7)$\Rightarrow$(8): Since $f(A) \subseteq \overline{f(A)} \Rightarrow A \subseteq f^{-1}(\overline{f(A)})$ and $f^{-1}(\overline{f(A)})$ is closed by (7) and contains A, so $\overline{A} \subseteq f^{-1}(\overline{f(A)})$, hence $f(\overline{A}) \subseteq \overline{f(A)}$.

8. (8)$\Rightarrow$(9): Since $A' \subseteq \overline{A}$ so $f(A') \subseteq f(\overline{A}) \subseteq \overline{f(A)}$ by (8).

9. (9)$\Rightarrow$(10): Let $B \subseteq \mathbb{R}$ and $f^{-1}(B) = A$, then $f(A) \subseteq B$ and
$$\begin{aligned}
&\partial A \subseteq \overline{A} = A \cup A'\\
\Rightarrow\ & f(\partial A) \subseteq f(A \cup A') \subseteq f(A) \cup f(A') \subseteq f(A) \cup \overline{f(A)} = \overline{f(A)} \text{ by (9)}\\
\Rightarrow\ & f(\partial A) \subseteq \overline{f(A)} \subseteq intf(A) \cup \partial f(A)\\
\Rightarrow\ & \partial A \subseteq f^{-1}(intf(A)) \cup f^{-1}(\partial f(A)) \subseteq intA \cup f^{-1}(\partial f(A)) \text{ by (6)}\\
\Rightarrow\ & \partial A \subseteq f^{-1}(\partial f(A))\\
\Rightarrow\ & \partial f^{-1}(B) \subseteq f^{-1}(\partial B).
\end{aligned}$$

10. (10)$\Rightarrow$(11): Let $P \subseteq \mathbb{R}$, and $f^{-1}(P) = Q$ then

$$\begin{aligned}\overline{f^{-1}(P)} &= int f^{-1}(P) \cup \partial f^{-1}(P)\\ &\subseteq f^{-1}(P) \cup f^{-1}(\partial P) = f^{-1}(P \cup \partial P) \text{ by (10)}\\ &= f^{-1}(\overline{P}).\end{aligned}$$

11. (11)$\Rightarrow$(4): Let V be open in $\mathbb{R}$ then V^C is closed in $\mathbb{R}$ and $V^C = \overline{V^C}$, so by (11) $\overline{f^{-1}(V^C)} \subseteq f^{-1}(V^C) \Rightarrow \overline{f^{-1}(V^C)} = f^{-1}(V^C) \Rightarrow f^{-1}(V^C)$ is closed which shows that $X \setminus f^{-1}(V)$ is closed, i.e. $f^{-1}(V)$ is open.

12. (4)$\Rightarrow$(1): Let $\epsilon > 0$, and $a \in X$ then by (4), $f^{-1}(B(f(a);\epsilon))$ is open in X as $B(f(a);\epsilon)$ is open in Y, so $a \in f^{-1}(B(f(a);\epsilon))$ is an interior point, therefore, $\exists\ \delta > 0$ such that $B(a;\delta) \subseteq f^{-1}(B(f(a);\epsilon)) \Rightarrow f(B(a;\delta)) \subseteq (B(f(a);\epsilon))$ shows that f is continuous at a. Since a is arbitrary, so f is continuous on X. □

6.11.2 Problem. Show that $f : \mathbb{R} \to \mathbb{R}$ is continuous iff for each $a \in \mathbb{R}$, $\{x \in \mathbb{R}; f(x) < a\}$ and $\{x \in \mathbb{R}; f(x) > a\}$ are open sets in $\mathbb{R}$.

6.11.2.1 Solution. Note that $\{x \in \mathbb{R}; f(x) < a\} = f^{-1}(-\infty, a)$ and $\{x \in \mathbb{R}; f(x) > a\} = f^{-1}(a, \infty)$. Note also that every nonempty open set S in $\mathbb{R}$ is the union of a countable collection of disjoint intervals of the type $[I = (-\infty, a), (a, b)$ or $(b, \infty)]$ of S. We also observe that if $a < b$ then $(a, b) = (-\infty, a) \cap (b, \infty)$. Thus if U be any open set in $\mathbb{R}$, then $U = \cup_{i=1}^{\infty} I_i$ implies $f^{-1}(U) = \cup_{i=1}^{\infty} f^{-1}(I_i)$. Hence $f^{-1}(U)$ is open, since each $f^{-1}(I_i)$ is open. Thus f is continuous. □

6.11.3 Problem. Can $\mathbb{R}$ be the continuous image of [0,1)? Justify the answer.

6.11.3.1 Solution. Yes. Consider the function $f : [0, 1) \to \mathbb{R}$, defined by

$$f(x) = \frac{1}{1-x} \sin \frac{1}{1-x}.$$

Evidently f is continuous and $f([0, 1)) = \mathbb{R}$.

6.11.4 Problem. Show that a surjective continuous function $f : \mathbb{R} \to \mathbb{R}$ carries a dense set to a dense set.

6.11.4.1 Solution. Let D be dense in $\mathbb{R}$. We show that $f(D)$ is dense in $\mathbb{R}$. Let V be an open set in $\mathbb{R}$, since f is continuous, so $f^{-1}(V)$ is open in $\mathbb{R}$ and $f^{-1}(V) \cap D \neq \emptyset$. Hence $f\left(f^{-1}(V) \cap D\right) = V \cap f(D) \neq \emptyset$ implies $f(D)$ is dense in $\mathbb{R}$. □

6.11.5 Problem. If two real-valued continuous functions f and g with a common domain $D \subseteq \mathbb{R}$ agree on a dense subset S of D, then $f = g$ on D.

6.11.5.1 Solution. Let $x \in D$ then $\exists$ a sequence (s_n) in S such that $s_n \to x$ which implies $f(s_n) \to f(x)$ and $g(s_n) \to g(x)$ and since $f(s_n) = g(s_n)$ so

$$\begin{aligned}&(f-g)(s_n) = f(s_n) - g(s_n) \to 0 \text{ and}\\ &(f-g)(s_n) \to (f-g)(x)\\ \Rightarrow&(f-g)(x) = f(x) - g(x) = 0\\ \Rightarrow&f(x) = g(x).\end{aligned}$$

Thus $f(x) = g(x)\ \forall\ x \in D \Rightarrow f = g$. □

6.11.6 Problem. Let f, g be strictly increasing on an interval $I \subseteq \mathbb{R}$ and let $f(x) > g(x)\ \forall\ x \in I$. If $y \in f(I) \cap g(I)$, show that $f^{-1}(y) < g^{-1}(y)$.

6.11.6.1 Solution. Let $y \in f(I) \cap g(I)$, then $\exists\ x_1, x_2 \in I$ such that $f(x_1) = y = g(x_2)$. Thus we get

$$g(x_1) < f(x_1) = y = g(x_2) < f(x_2).$$

$\square$

6.11.7 Problem. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. If $S = \{f(x); x \in \mathbb{R}\}$ is neither bounded above nor bounded below, prove that $S = \mathbb{R}$.

6.11.7.1 Solution. Obviously $S \subseteq \mathbb{R}$. We now show the reverse inclusion. Let $y \in \mathbb{R}$. As S is not bounded above, so, y is not an upper bound of S, that is, there exists an $x_0 \in \mathbb{R}$ such that $f(x_0) < y$. Similarly, since S is not bounded below, so, y is not a lower bound of S, and so there exists an $x_1 \in \mathbb{R}$ such that $f(x_1) > y$. Now consider the restriction of f to the interval with endpoints x_0 and x_1 with the endpoints included in the interval. Applying the intermediate value theorem to this continuous function, it follows that there exists a real number c such that $f(c) = y$. This shows that $S = \mathbb{R}$. $\square$

6.11.8 Problem. Suppose that $f : \mathbb{R} \to \mathbb{R}$ and there exists a $M > 0$ such that for all $x \in \mathbb{R}, |f(x)| \leq M|x|$. Prove that f is continuous at 0.

6.11.8.1 Solution. Hint: Find $f(0)$. $\square$

6.11.9 Problem. If f is continuous at 0 and $f(0) = 0$, then the inequality $|f(x)| \leq |x|$ holds at least in some neighborhood of 0. True or false?

6.11.9.1 Solution. If $f(x) = \sqrt[3]{x}$ then f is continuous at 0 (since $\lim_{x\to 0} \sqrt[3]{x} = 0$,) however $\sqrt[3]{x} > x$ for any $x \in (0, 1)$. $\square$

6.11.10 Problem. If f is continuous on $\mathbb{R}$ and $\lim_{n\to\infty} f(n) = A$, then $\lim_{x\to\infty} f(x) = A$. True or false?

6.11.10.1 Solution. The function $f(x) = \sin \pi x$ is continuous on $\mathbb{R}$ and

$$\lim_{n\to\infty} f(n) = \lim_{n\to\infty} \sin n\pi = 0,$$

but this is only one of the partial limits when x approaches $+\infty$.
Another partial limit $\lim_{n\to\infty} f\left(2n + \frac{1}{2}\right) = \lim_{n\to\infty} \sin\left(2n\pi + \frac{\pi}{2}\right) = 1$ gives a different result, so $\lim_{x\to\infty} f(x)$ does not exist.
Remark: The converse is true: if $\lim_{x\to\infty} f(x) = A$ and f is defined on $\mathbb{N}$, then $\lim_{n\to\infty} f(n) = A$. $\square$

6.11.11 Problem. A function f is said to be **symmetrically continuous** at a point x if

$$\lim_{h\to 0+} [f(x + h) - f(x - h)] = 0.$$

Show that if f is continuous at a point, then it must be symmetrically continuous there and that the converse does not hold. (Example: Consider $f(x) = |x|$, and it is symmetrically continuous.)

6.11.11.1 Solution.

- Left to the reader.

- Counterexample: Let

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Z} \\ 0, & \text{if } x \notin \mathbb{Z}. \end{cases}$$

We see that

$$\lim_{h \to 0+} [f(n+h) - f(n-h)] = 0 \; \forall \, n \in \mathbb{Z}.$$

But it is not continuous at any integer.

6.11.12 Problem. If f is continuous and bounded on an interval and $f(x) \neq 0$ for every x in this interval, then $\frac{1}{f(x)}$ is also continuous and bounded on this interval. True or false?

6.11.12.1 Solution. Consider the continuous function f defined by $f(x) = x$ which is bounded and different from zero on (0,1). Evidently, $\frac{1}{f(x)} = \frac{1}{x}$ is unbounded on (0,1). □

6.11.13 Problem. Let $f : \mathbb{R} \to \mathbb{R}$ be an increasing function. Suppose there are sequences (x_n) and (y_n) such that $y_n < 0 < x_n$ for all $n \geq 1$ and $f(x_n) - f(y_n) \to 0$ as $n \to \infty$. Prove that f is continuous at 0.

6.11.13.1 Solution. We first prove that $f(x_n) \to f(0)$ and $f(y_n) \to f(0)$. Let $\epsilon > 0$. So $\exists \, n_0 > 0$ such that $n \geq n_0 \Rightarrow f(x_n) - f(y_n) < \epsilon$. since $y_n < 0$ and f is increasing, $f(y_n) \leq f(0)$ implies $f(0) - f(y_n) \geq 0$. Hence for

$$n \geq n_0, f(x_n) - f(0) \leq f(x_n) - f(0) + f(0) - f(y_n) = f(x_n) - f(y_n) < \epsilon.$$

Therefore $f(x_n) \to f(0)$. Similarly, $f(y_n) \to f(0)$. We next show that if (z_n) is a sequence such that $z_n > 0$ and $z_n \to 0$ then $f(z_n) \to f(0)$. This will prove that f is right continuous at 0. Let $\epsilon > 0$. Since $f(x_n) \to f(0), \exists \, m$ such that $f(x_m) - f(0) < \epsilon$. Since $z_n \to 0, \exists \, n_o$ such that $n \geq n_0 \Rightarrow z_n < x_m$ and since f is increasing this implies $f(z_n) \leq f(x_m) \Rightarrow f(z_n) - f(0) \leq f(x_m) - f(0) < \epsilon$. This shows that $f(z_n) \to f(0)$. Similarly, for left continuity at 0, and thus f is continuous at 0. □

6.11.14 Problem. If f is continuous and unbounded on $(0, \infty)$, and $f(x) > 0, \forall \, x \in (0, \infty)$, then $\lim_{x \to \infty} \frac{1}{f(x)} = 0$. True or false?

6.11.14.1 Solution. Let f be defined by $f(x) = x \sin x + x + 1$ on $(0, \infty)$. Evidently, f is continuous on $(0, \infty)$ due to the arithmetic properties of continuous functions and $f(x) = x(\sin x + 1) + 1 > 0$ for all $x \in (0, \infty)$. Also, choosing $x_n = n\pi$, $\forall \, n \in \mathbb{N}$ we obtain the following partial limit

$$\lim_{x_n \to \infty} f(x_n) = \lim_{n \to \infty} (n\pi(\sin n\pi + 1) + 1) = \infty,$$

implying that f is unbounded on $(0, \infty)$. On the other hand, for the same sequence $x_n = n\pi$ we obtain

$$\lim_{x_n \to \infty} \frac{1}{f(x_n)} = \lim_{n \to \infty} \frac{1}{(n\pi(\sin n\pi + 1) + 1)} = 0,$$

while for $x_k = 2k\pi - \frac{\pi}{2}, k \in \mathbb{N}$, the partial limit is different:

$$\begin{aligned} \lim_{x_k \to \infty} \frac{1}{f(x_k)} &= \lim_{k \to \infty} \frac{1}{(2k\pi - \pi/2)(\sin(2k\pi - \pi/2) + 1) + 1} \\ &= \lim_{k \to \infty} \frac{1}{0 + 1} = 1. \end{aligned}$$

It means that the limit $\lim_{x \to \infty} \frac{1}{f(x)}$ does not exist. □

6.11.15 Problem. If f is continuous on a bounded set, then its image is also a bounded set. True or false?

6.11.15.1 Solution. The function f defined by $f(x) = \tan x$ is continuous on $(0, \pi/2)$, but its image is the interval $(0, \infty)$, which is not bounded. $\square$

6.11.16 Problem. If f is continuous on S and its image $f(S)$ is a compact set, then S is also a compact set. True or false?

6.11.16.1 Solution. False. Let $S = (0, 1)$ and $f : (0, 1) \to [0, 1]$ be a function defined by

$$f(x) = \begin{cases} 0 & \text{if } 0 < x \leq \frac{1}{3} \\ 3x - 1 & \text{if } \frac{1}{3} < x \leq \frac{2}{3} \\ 1 & \text{if } \frac{2}{3} < x < 1 \end{cases}$$

is continuous and $f(S) = [0, 1]$ is compact but S is not compact. $\square$

6.11.17 Problem. If f and g are not uniformly continuous on a set S, then $f.g$ is uniformly continuous on S. True or false?

6.11.17.1 Solution. True. Let $f, g : (-1, 1) \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x - 1 & \text{if } x \in (-1, 0) \\ x + 1 & \text{if } x \in [0, 1) \end{cases} \text{ and } g(x) = \begin{cases} -1 & \text{if } x \in (-1, 0) \\ 1 & \text{if } x \in [0, 1) \end{cases}$$

These functions are uniformly continuous on $(-1, 0)$ and [0,1), but they are not uniformly continuous on $(-1, 0) \cup [0, 1) = (-1, 1)$, because both have a jump discontinuity at the point 0. At the same time,

$$h(x) = f(x)g(x) = \begin{cases} -x + 1 & \text{if } x \in (-1, 0) \\ x + 1 & \text{if } x \in [0, 1) \end{cases} = 1 + |x|$$

is a uniformly continuous function on $(-1, 1)$. In fact, for any pair $x_1, x_2 \in (-1, 1)$ such that $|x_1 - x_2| < \delta$, it follows that

$$|h(x_1) - h(x_2)| = |1 + |x_1| - 1 - |x_2|| \leq |x_1 - x_2| < \delta = \epsilon$$

shows that h is uniformly continuous on S. $\square$

6.11.18 Problem. If f is continuous on X, and g is uniformly continuous on $f(X)$, then the composite function $g(f(x))$ is uniformly continuous on X. True or false?

6.11.18.1 Solution. Consider the functions $f(x) = \frac{1}{x}$ defined on $X = (0, \infty)$ and $g(x) = \sin x$ defined on $f(X) = (0, \infty)$. The first function is not uniformly continuous on X, can be verified. The second function is uniformly continuous on $\mathbb{R}$ and, consequently, it is uniformly continuous on $(0, \infty)$. At the same time, the composite function $h(x) = g(f(x)) = \sin \frac{1}{x}$ is not uniformly continuous on $(0, \infty)$, because for $x_k = 1/2k\pi$ and $y_k = 2/(4k + 1)\pi$, one gets

$$|x_k - y_k| = \left| \frac{1}{2k\pi} - \frac{2}{(4k + 1)\pi} \right| \to 0 \text{ but } |h(x_k) - h(y_k)| = 1 > 0. \quad \square$$

6.11.19 Problem. Find two discontinuous functions f and g so that $f \circ g$ is continuous everywhere.

6.11.19.1 Solution. Hint. In $[0,1], g = \chi_{\{1\}}, f = \chi_{(0,1)}$, where χ stands for the characteristic function. $\square$

6.11.20 Problem. Find two sets $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}^2$, and an everywhere continuous function $f : A \to B$ that is one-to-one and onto, such that $f^{-1} : B \to A$ is not everywhere continuous.

6.11.20.1 Solution. Hint. Consider $f : [0, 2\pi) \to S^1$ defined by $f(t) = (\cos t, \sin t)$. Note that $S^1 = \{(x, y) \in \mathbb{R}^2; x^2 + y^2 = 1\}$. $\square$

6.11.21 Problem. Find a function $f : \mathbb{R} \to \mathbb{R}$ such that f is continuous nowhere but $|f|$ is continuous everywhere.

6.11.21.1 Solution. Consider a function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ -1 & \text{if } x \text{ is irrational.} \end{cases} \quad \square$$

6.11.22 Problem. Let f and g be two continuous functions mapping the interval [0,1] into itself. Show that if $f \circ g = g \circ f$, then f and g agree at some point of [0,1].

6.11.22.1 Solution. Let $x \in [0, 1]$. Now consider the sequence

$$\left(x, f(x), f^2(x), ..., f^n(x)....\right)$$

in [0,1] and then it converges to α for some $\alpha \in [0, 1]$. Again, since g is continuous, so the $g(f^n(x))$ converges to $g(\alpha)$, but $g(f^n(x)) = f^n(g(x))$ implies $g(\alpha) = \alpha$. And $f^n(x)$ converges to α implies $f(f^n(x)) = f\left(f^{n+1}(x)\right)$ converges to $f(\alpha)$ which again implies $f(\alpha) = \alpha$. i.e. $f(\alpha) = \alpha = g(\alpha)$. Thus f and g agree at $\alpha \in [0, 1]$. (Here $f^n = f \circ f \circ ... \circ f(n$ copies of $f)$.) $\square$

6.11.22.2 Solution. Hint. Argue by contradiction. Show that we may suppose, without loss of generality, that $f(x) - g(x) > 0 \; \forall \, x \in [0, 1]$. Now try to show that there is a number $a > 0$ such that $f^n(x) \geq g^n(x) + na$ for every natural number n and every $x \in [0, 1]$. $\square$

6.11.23 Problem. Let $f, g : [0, 1] \to [0, 1]$ be two continuous functions. If g is non-decreasing and $f \circ g = g \circ f$, prove that f and g have a common fixed point.

6.11.23.1 Solution. Let $F = \{x; x \in [0, 1]$ such that $f(x) = x\}$ and $G = \{x; x \in [0, 1], \; g(x) = x\}$. It can be shown that F, G are closed non-empty subsets of [0,1]. Since $f \circ g = g \circ f$, we have $g(F) \subseteq F$. Let $\alpha = \sup F$. If $g(\alpha) = \alpha$, we are done. If not, $g(\alpha) < \alpha$ and so $(g^n(\alpha))$ is a decreasing sequence in F. Hence $\beta = \lim_{n\to\infty} g^n(\alpha)$ is a common fixed point of f and g. $\square$

6.11.23.2 Solution. Hint: Use previous problem. $\square$

6.11.24 Problem. Let f be continuous on $[a, b]$. Suppose that f has a local maximum at x_1 and a local maximum at x_2. Show that there must be a third point between x_1 and x_2 where f has a local minimum.

6.11.24.1 Solution. Let $x_1 < x_2$. Suppose that, no points in (x_1, x_2) can be a local minimum of f. Since f is continuous on $[x_1, x_2]$, then $\inf\{f(x); x \in [x_1, x_2]\} = f(x_1)$ or $f(x_2)$ by hypothesis. We consider two cases as follows:

1. If $\inf\{f(x); x \in [x_1, x_2]\} = f(x_1)$, then
   (a) f has a local maximum at x_1 and $f(x) \geq f(x_1)\ \forall\ x \in [x_1, x_2]$.
   (b) By (a), there exists a $\delta > 0$ such that $x \in [x_1, x_1 + \delta) \subseteq [x_1, x_2]$, we have $f(x) \leq f(x_1)$.

   So, by (b), as $x \in [x_1, x_1 + \delta)$ we have $f(x) = f(x_1)$ which contradicts the hypothesis that no points in (x_1, x_2) can be a local minimum of f.
2. If $\inf\{f(x); x \in [x_1, x_2]\} = f(x_2)$, it is similar to above, so, we omit it.

Hence, from (1) and (2), we have there has a third point between x_1 and x_2 where f has a local minimum. □

6.11.25 Problem. Let I be a nondegenerate interval and let $f : I \to \mathbb{R}$ be an injective function with intermediate value property. Then, f is strictly monotone.

6.11.25.1 Solution. Suppose that f is not strictly monotone. Then, there exist $a, b, c \in I$ such that $a < b < c$ and $f(b)$ is not between $f(a)$ and $f(c)$. In other words, one of the following cases may occur:

1. $f(b) < f(a) < f(c)$
2. $f(a) < f(c) < f(b)$
3. $f(b) < f(c) < f(a)$
4. $f(c) < f(a) < f(b)$.

Suppose the case (1) and let $\lambda = f(a)$. Since f has the intermediate value property, there exists $\alpha \in (b, c)$ such that $\lambda = f(\alpha)$. Since $\alpha \neq a$, this fact contradicts the injectivity of f. The other cases treat similarly. □

6.11.26 Problem. Let f be defined in the open interval (a, b) and assume that for each interior point $x \in (a, b)$ there exists a ball $B(x; r)$ in which f is increasing. Prove that f is an increasing function throughout (a, b).

6.11.26.1 Solution. Suppose that, there exist p, q with $p < q$ such that $f(p) > f(q)$. Consider $[p, q] \subseteq (a, b)$, and since for each interior point $x \in (a, b)$ there exists a ball $B(x, \delta_x)$ in which f is increasing. Then $[p, q] = \bigcup_{x \in [p,q]} B(x; \delta_x)$, (The choice of balls comes from the hypothesis). It implies that $[p, q] = \bigcup_{k=1}^{n} B(x_k; \delta_k) = B_n$(by compactness.) Note that if $B_i \subseteq B_j$, we remove such B_i and make one left. Without loss of generality, we assume that $x_1 \leq x_2 \leq .. \leq x_n$. Then

$$f(p) \leq f(x_1) \leq .... \leq f(x_n) \leq f(q)$$

which is absurd. So, we know that f is an increasing function throughout (a, b). □

6.11.27 Problem. Let f be continuous on $[a, b]$ and assume that f does not have a local maximum or a local minimum at any interior point. Prove that f must be monotonic on $[a, b]$.

6.11.27.1 Solution. Since f is continuous on $[a, b]$, we have

$$\sup\{f(x); x \in [a, b]\} = f(p) \text{ where } p \in [a, b] \text{ and}$$
$$\inf\{f(x); x \in [a, b]\} = f(q) \text{ where } q \in [a, b].$$

So, we have $\{p, q\} = \{a, b\}$ by hypothesis that f does not have a local maximum or a local minimum at any interior point. Without loss of generality, we assume that $p = a$, and $q = b$. Claim that f is decreasing on $[a, b]$ as follows. Suppose f is increasing, then there exist $x, y \in [a, b]$ with $x < y$ such that $f(x) < f(y)$. Consider $[x, y]$ and by hyothesis, we know that $f|_{[x,y]}$ has the maximum at y, and $f|_{[x,y]}$ has the minimum at x. Then it implies that there exists a ball $B(x; \delta)$ such that f is constant on $B(x; \delta) \cap [x, y]$, which contradicts to the hypothesis. Hence, we have proved that f is decreasing on $[a, b]$. □

6.11.28 Problem. Let f be a continuous function on the interval $[a, b]$. Then there is a continuous function g such that $g(f(x)) = x$ for all $x \in [a, b]$ if and only if f is strictly monotone. In this case we also have $f(g(y)) = y$ for each y between $f(a)$ and $f(b)$.

6.11.28.1 Solution. Let f be a continuous function on $[a, b]$. Suppose f is strictly monotone. We may assume f is strictly monotone increasing. Then f is one-to-one. Also, by the Intermediate Value Theorem, f maps $[a, b]$ onto $[f(a), f(b)]$. Let $g = f^{-1} : [f(a), f(b)] \to [a, b]$. Then $g(f(x)) = x\ \forall\ x \in [a, b]$. Let $y \in [f(a), f(b)]$. Then $y = f(x)$ for some $x \in [a, b]$. Given $\epsilon > 0$, choose $\delta > 0$ such that $\delta < \min(f(x) - f(x - \epsilon), f(x + \epsilon) - f(x))$. When $|y - z| < \delta, z = f(x_0)$ for some $x_0 \in [a, b]$ with $|g(y) - g(z)| = |g(f(x)) - g(f(x_0))| = |x - x_0| < \delta$. Thus g is continuous on $[f(a), f(b)]$.
Conversely, suppose there is a continuous function g such that $g(f(x)) = x\ \forall\ x \in [a, b]$. If $x, y \in [a, b]$ with $x < y$, then $g(f(x)) < g(f(y))$ so $f(x) \neq f(y)$. We may assume $x \neq a$ and $f(a) < f(b)$. If $f(x) < f(a)$, then by the Intermediate Value Theorem, $f(a) = f(x')$ for some $x' \in [x, b]$ and $a = g(f(a)) = g(f(x')) = x'$, a contradiction. Thus $f(a) < f(x)$. Now if $f(x) \geq f(y)$, then $f(a) < f(y) \leq f(x)$ so $f(y) = f(x'')$ for some $x'' \in [a, x]$ and $y = g(f(y)) = g(f(x'')) = x''$, a contradiction. Thus $f(x) < f(y)$. Hence f is strictly monotone increasing. □

6.11.29 Problem. If f is one-to-one and continuous on $[a, b]$, prove that f must be strictly monotonic on $[a, b]$.

6.11.29.1 Solution. Since f is continuous on $[a, b]$, we have

$$\sup\{f(x); x \in [a, b]\} = f(p) \text{ where } p \in [a, b] \text{ and}$$
$$\inf\{f(x); x \in [a, b]\} = f(q) \text{ where } q \in [a, b].$$

Assume that $p \in (a, b)$, then there exists $\delta > 0$ such that $f(y) \leq f(p)$ for all $y \in (p - \delta, p + \delta)$. Choose $y_1 \in (p - \delta, p)$ and $y_2 \in (p, p + \delta)$ then we have by 1-1, $f(y_1) < f(p)$ and $f(y_2) < f(p)$. And thus choose r so that $f(y_1) < r < f(p) \Rightarrow f(z_1) = r$, where $z_1 \in (y_1, p)$ by Intermediate Value Theorem, and $f(y_2) < r < f(p) \Rightarrow f(z_2) = r$, where $z_2 \in (p, y_2)$ by Intermediate Value Theorem, which contradicts to 1-1. So, we have that $p \in \{a, b\}$. Similarly, we have $q \in \{a, b\}$. Without loss of generality, we assume that $p = a$ and $q = b$. Claim that f is strictly decreasing on $[a, b]$.
Suppose not, then there exist $x, y \in [a, b]$ with $x < y$ such that $f(x) < f(y)$ (= does not hold since f is 1-1.) Consider $[x, y]$ and by above method, we know that $f|_{[x,y]}$ has the maximum at y, and $f|_{[x,y]}$ has the minimum at x.Then it implies that there exists a nbhd. $B(x; \delta)$ such that f is constant on $B(x; \delta) \cap [x, y]$, which contradicts to 1-1. Hence, for any $x < y$ we have $f(x) > f(y)$ does not hold since f is 1-1. So, we have proved that f is strictly decreasing on $[a, b]$. □

6.11.29.2 Solution. Suppose not, it suffices to show that 1-1 and continuity imply that f does not have a local maximum or a local minimum at any interior point. Suppose not, it means that f has a local extremum at some interior point x. Without loss of generality, we assume that f has a local minimum at the interior point x. Since x is an interior point of $[a, b]$, then there exists an

open interval $(x-\delta, x+\delta)$ such that $f(y) \geq f(x)\ \forall\ y \in (x-\delta, x+\delta)$. Note that f is 1-1, so we have $f(y) > f(x)$ for all $y \in (x-\delta, x+\delta) \setminus \{x\}$. Choose $y_1 \in (x-\delta, x)$ and $y_2 \in (x, x+\delta)$, then we have $f(y_1) > f(x)$ and $f(y_2) > f(x)$. And thus choose r so that so that $f(y_1) > r > f(x) \Rightarrow f(z_1) = r$, where $z_1 \in (y_1, x)$ by Intermediate Value Theorem, and $f(y_2) > r > f(x) \Rightarrow f(z_2) = r$, where $z_2 \in (x, y_2)$ by Intermediate Value Theorem, which contradicts to the hypothesis that f is 1-1. Hence, we have proved that f is 1-1 and continuity implies that f does not have a local maximum or a local minimum at any interior point. □

6.11.30 Problem. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and $f(f(f(x))) = x\ \forall\ x \in \mathbb{R}$. Prove that $f(x) = x\ \forall\ x \in \mathbb{R}$.

6.11.30.1 Solution. Let $f(x) = f(y)$ then $f^3(x) = f^3(y) \Rightarrow x = y$. So f is injective. Since f is continuous, by the above problem, f must be strictly monotonic. Suppose f is increasing and that $f(x) \neq x$, then $f(x) > x$ or $f(x) < x$. Let $f(x) > x$ then $f^2(x) > f(x) \Rightarrow f^3(x) > f^2(x) > f(x)$ i.e. $x > f(x)$, a contradiction. Similarly for other cases. Hence $f(x) = x\ \forall\ x \in \mathbb{R}$. □

6.11.31 Problem. Show that there exists a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f(f(x)) = x\ \forall\ x \in \mathbb{R}$, but $f(x) \neq x\ \forall\ x \in \mathbb{R}$.

6.11.31.1 Solution. Consider $f(x) = 1 - x$. □

6.11.32 Problem. Show that there exist a continuous functions $f, g, h : \mathbb{R}' \to \mathbb{R}'$ such that $f(f(x)) = x, g(g(x)) = x, h(h(h(x))) = x\ \forall\ x \in \mathbb{R}'$, but $f, g, h \neq \iota$ where $\mathbb{R}' = \mathbb{R} \setminus \{0, 1\}$.

6.11.32.1 Solution. Consider $f(x) = \frac{1}{x}$, and $g(x) = \frac{x}{x-1}, h(x) = \frac{x-1}{x}$. □

6.11.33 Problem. Show that, if $f : [a, b] \to [a, b]$ is continuous and its inverse is also continuous, then either a and b are fixed points or $f(a) = b$ and $f(b) = a$.

6.11.33.1 Solution. Suppose that $f(a) = a$ and $f(b) \neq b$, so $f(b) < b$, then by surjectivity $\exists\ c \in (a, b)$ such that $f(c) = b$. Thus we get $f(a) < f(b) < b = f(c)$, hence by IVP $\exists\ p \in (a, c)$ such that $f(p) = f(b)$ and by injectivity $p = b$, but $p \neq b$ a contradiction. Therefore $f(b) = b$. □

6.11.34 Problem. Let f be an increasing function defined on $[a, b]$ and let $x_1, ..., x_n$ be n points in the interior such that $a \leq x_1 \leq x_2 \leq .. \leq x_n \leq b$.

1. Show that
$$\sum_{i=0}^{n} [f(x_k+) - f(x_k-)] \leq f(b-) - f(a+).$$
2. Deduce from part (1) that the set of dicontinuities of f is countable.

6.11.34.1 Solution. Let $a = x_0$ and $b = x_{n+1}$; since f is an increasing function defined on $[a, b]$, we have that both $f(x_k+)$ and $f(x_k-)$ exist for $1 \leq k \leq n$. Assume that $y_k \in (x_k, x_{k+1})$, then we have $f(y_k) \geq f(x_k+)$ and $f(x_{k-1}) \geq f(y_{k-1})$. Hence,

1.
$$\begin{aligned}\sum_{k=1}^{n} [f(x_k+) - f(x_k-)] &\leq \sum_{k=1}^{n} [f(y_k) - f(y_{k-1})] \\ &\leq f(y_n) - f(y_0) \\ &\leq f(b-) - f(a+).\end{aligned}$$

2. Let D denote the set of points of discontinuities of f. Consider

$$D_m = \left\{x; x \in [a, b],\ f(x+) - f(x-) \geq \frac{1}{m}\right\}$$

then $D = \bigcup_{m=1}^{\infty} D_m$. Note that $|D_m|$ is finite, so we have D is countable. That is, the set of points of discontinuities of f is countable. $\square$

6.11.35 Problem.

1. Let f be defined on $\mathbb{R}$ and assume that there exists at least one point x_0 in $\mathbb{R}$ at which f is continuous. Suppose also that, for every x and y in $\mathbb{R}$, f, satisfies the equation $f(x+y) = f(x) + f(y)$. Prove that there exists a constant a such that $f(x) = ax$ for all $x \in \mathbb{R}$.
2. Give an example of a function f such that f satisfies the equation $f(x+y) = f(x) + f(y)$, but f is not continuous on $\mathbb{R}$.

6.11.35.1 Solution.

1. Left to the reader.
2. **Note:** If a function f satisfies the equation $f(x+y) = f(x) + f(y)$, then it is called an additive function. An example of an additive function, let $f_a : \mathbb{R} \to \mathbb{R}$ where $f_a(x) = a.x$ for all $x \in \mathbb{R}$, and $a \in \mathbb{R}$ is fixed.
   Consider the set $\mathbb{R}(\mathbb{Q})$ of all real numbers as a vector space over the field of rational numbers $\mathbb{Q}$. This vector space has a basis $\mathcal{B}$, called a **Hamel basis** for $\mathbb{R}$. Now, we show that: " There exists an additive function $f : \mathbb{R}(\mathbb{Q}) \to \mathbb{R}(\mathbb{Q})$ such that $f \neq f_a$ for all $a \in \mathbb{R}$."

   Proof. Let $\mathcal{B} = \{x_i; i \in I\}$ be a Hamel basis for $\mathbb{R}$. Choose a fixed $\tilde{x} \in \mathcal{B}$. Define

$$f(x) = \begin{cases} r_i & \text{if } x = r_1x_1 + ... + r_ix_i + ... + r_nx_n \text{ and } x_i = \tilde{x} \\ 0 & \text{otherwise.} \end{cases}$$

   It is easy to check that f is additive. Notice also that the null vector $0 \notin \mathcal{B}$ and $\mathcal{B}$ is infinite (actually, $|\mathcal{B}| = 2^{\aleph_0} = \mathfrak{c}$). We have $f(\tilde{x}) = 1$ (because $\tilde{x} = 1.\tilde{x}$ is the basis representation of $\tilde{x}$), while $f(x') = 0$ for any $x' \in \mathcal{B}$, $x' \neq \tilde{x}$ (because x does not occur in the basis representation of $x' = 1.x'$ of x'. If $f = f_a$ were to hold for some $a \in \mathbb{R}$, we would have $f(\tilde{x}) = 1 = a\tilde{x}$, showing $a \neq 0$, and, on the other hand, $f(x') = 0 = ax'$, showing $a = 0$. $\square$

   Again, since $\mathcal{B} = \{x_i; i \in I\}$ be a basis for $\mathbb{R}$, note that $\mathcal{B} = \{x_i; i \in I\}$ is an uncountable set, so there exists a convergent sequence (s_n), where $\{s_n; n \in \mathbb{N}\} \subseteq \mathcal{B}$. Hence

$$\Gamma = (\mathcal{B} \setminus \{s_n; n \in \mathbb{N}\}) \bigcup \left\{\frac{s_n}{n}; n \in \mathbb{N}\right\}$$

   is a new basis of $\mathbb{R}$ over $\mathbb{Q}$. Given $x, y \in \mathbb{R}$, and we can find the some $N \in \mathbb{N}$ such that

$$x = \sum_{k=1}^{N} r_k x_k \text{ and } y = \sum_{k=1}^{N} t_k x_k$$

where $x_k \in \Gamma$, and $r_k, t_k \in \mathbb{Q}$. Define the sum

$$x + y = \sum_{k=1}^{N} (r_k + t_k) x_k.$$

By uniqueness, we define $f(x)$ to be the sum of coefficients, i.e.,

$$f(x) = \sum_{k=1}^{N} r_k.$$

$$\text{Now, } f(x+y) = \sum_{k=1}^{N} (r_k + t_k) = \sum_{k=1}^{N} r_k + \sum_{k=1}^{N} t_k = f(x) + f(y).$$

Suppose f is continuous and note that

$$\frac{s_n}{n} \to 0 \text{ as } n \to \infty \text{ and } f\left(\frac{s_n}{n}\right) = 1,\ \forall\, n \in \mathbb{N}.$$

$$\begin{aligned} \text{Hence } 1 &= \lim_{n\to\infty} f\left(\frac{s_n}{n}\right) \\ &= f\left(\lim_{n\to\infty} \frac{s_n}{n}\right) \text{ by continuity of } f \\ &= f(0) = 0, \end{aligned}$$

which is absurd. Hence, f is not continuous on $\mathbb{R}$. □

6.11.36 Problem. Let A be a non-empty subset of real numbers, and let the function $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \inf\{|t - x|; t \in A\}.$$

Prove that

1. $|f(x) - f(y)| \leqq |x - y|\ \forall\, x, y \in \mathbb{R}$, and f is uniformly continuous on $\mathbb{R}$.
2. $f(x) = 0$, $\forall x \in \mathbb{R}$ iff $A \cap (a, b) \neq \emptyset$ for every open interval (a, b).

6.11.36.1 Solution.

1. Let $x, y \in \mathbb{R}$ then for any $t \in A$, we have

$$\begin{aligned} |x - t| &\le |x - y| + |y - t| \\ \Rightarrow \inf_{t\in A} |x - t| &\le |x - y| + \inf_{t\in A} |y - t| \\ \Rightarrow f(x) - f(y) &\le |x - y|, \\ \text{similarly } f(y) - f(x) &\le |x - y| \\ \text{hence } |f(x) - f(y)| &\le |x - y|. \end{aligned}$$

   Now, for any $\epsilon > 0$ taking $\delta = \epsilon$, we conclude that f is uniformly continuous on $\mathbb{R}$.

2. Suppose $A \cap (a, b) \neq \emptyset$ for every open interval (a, b), this means that A is dense in $\mathbb{R}$, since f is continuous in a dense set A and $f(a) = 0$, $\forall\, a \in A$, so, $f(x) = 0\ \ \forall\, x \in \mathbb{R}$. □

6.11.37 Problem. For each $\alpha \in (0,1)$, define $f_\alpha : (0,\infty) \to \mathbb{R}$ by $f_\alpha(x) = x^\alpha \log x$. For which values of α, f_α is uniformly continuous? Justify your answer.

6.11.37.1 Solution. We claim that f_α is uniformly continuous for $0 < \alpha < 1$ and not uniformly continuous for $\alpha > 1$. Note that f_α is differentiable on $(0,\infty)$ $\forall$ $\alpha > 0$. The derivative is

$$f'_\alpha = x^{\alpha-1}(1 + \log x).$$

First suppose $\alpha > 1$. Let $\epsilon > 0$. To disprove uniform continuity, it suffices to show that for any $\delta > 0$, there exist $x, y \in (0\infty)$ such that $|f(x) - f(y)| \geq \epsilon$ with $|x-y| < \delta$. Fix $\delta > 0$. Let x be so large that $y > x$ implies that $|f'_\alpha(y)| > 2\epsilon/\delta$. This is possible since $f'_\alpha \to \infty$ as $x \to \infty$ for $\alpha > 1$. Let $y = x+\delta/2$. By the mean value theorem, there exists $c \in (x,y)$ such that $f_\alpha(y) - f_\alpha(x) = f'_\alpha(c)(y-x)f'(c)\delta/2$. Since $c > x$,

$$|f_\alpha(y) - f_\alpha(x)| = f'_\alpha(c)(y-x)|f'(c)|\delta/2 \geq \epsilon.$$

But $|x-y| = \delta/2 < \delta$. Thus f_α is not uniformly continuous.
Now suppose $0 < \alpha < 1$. To prove uniform continuity, it suffices to show that $|f'_\alpha|$ is bounded. This is because by the mean value theorem,

$$|f(x) - f(y)| < M|x-y|.$$

Given $\epsilon > 0$, we may choose $\delta < 1/M$, to see that $|f(x) - f(y)| < \epsilon$ if $|x-y| < 1/M$. To show that $|f_\alpha|$ is bounded on $(0,\infty)$, it suffices to show that is bounded as $x \to 0^+$ and $x \to \infty$. In this case, f'_α will be bounded for $0 < x < y_1$ for some y_1 and also for $x > y_2 > y_1$ for some y_2. By the continuity f'_α of on $[y_1, y_2], f'_\alpha$ is bounded on $[y_1, y_2]$. Taking the maximum of these three bounds will give a bound of f'_α on all of $(0,\infty)$.
To bound $f'_\alpha(x)$ as $x \to \infty$, it suffices to bound $x^{\alpha-1}\log x$ for $x \geq 1$. Note that for any $y, e^y \geq y$. Then for $x > 1$,

$$0 \leq x^{\alpha-1}\log x = \frac{\log x}{e^{(1-\alpha)\log x}} \leq \frac{1}{1-\alpha}.$$

Because $x^{\alpha-1} \to 0$ as $x \to \infty$, to bound $f'_\alpha(x)$ as x approaches zero from above, it suffices to bound $x^{\alpha-1}\log x$ for $0 < x < 1$. Cearly $0 > x^{\alpha-1}\log x$. But as above,

$$0 < -x^{\alpha-1}\log x \leq -\frac{1}{1-\alpha}.$$

Therefore f'_α is also bounded near zero. This completes the proof that f'_α is bounded on all of $(0,\infty)$, which implies that f_α is uniformly continuous in the case $0 < \alpha < 1$. □

6.11.38 Problem. Let $f : [0,2] \to \mathbb{R}$ be continuous, given that $f(0) = f(2) = -1$ and $f(1) = 1$. Show that $\exists$ $c \in (0,1)$ such that $f(c) = f(c+1)$.

6.11.38.1 Solution. Consider $g(x) = f(x+1) - f(x)$. Then, we see that $g(0) = f(1) - f(0) = 1+1 = 2$ and $g(1) = f(2) - f(1) = -2$. So, $g(0)g(1) < 0$. Hence, $\exists$ $c \in (0,1)$ such that $g(c) = 0$. Thus $f(c) = f(c+1)$. □

6.11.39 Problem. (**Cousin**) Let C be a collection of closed subintervals of $[a,b]$ with the property that for each $x \in [a,b]$ there exists $\delta = \delta_x > 0$ such that C contains all intervals $[c,d] \subseteq [a,b]$ that contain x and have length smaller than δ. Then there exists a partition $a = x_0 < x_1 < .... < x_n = b$ of $[a,b]$ such that $[x_{i-1}, x_i]$ for $i = 1, 2, ..., n$.

6.11.39.1 Solution. Let us suppose that C does not contain a partition of the interval $[a, b]$. Let c be the midpoint of that interval and consider the two subintervals $[a, c]$ and $[c, b]$. If C contains a partition of both intervals $[a, c]$ and $[c, b]$, then by putting those partitions together we can obtain a partition of $[a, b]$, which we have supposed is impossible. Let $I_1 = [a, b]$ and let I_2 be either $[a, c]$ or $[c, b]$ chosen so that C contains no partition of I_2. Inductively we can continue in this fashion, obtaining a shrinking sequence of intervals $I_1 \supset I_2 \supset I_3 \supset ...$ so that the length of I_n is $(b-a)/2^{n-1}$ and C contains no partition of I_n. By the Cantor intersection theorem there is a single point z in all of these intervals. For sufficiently large n, the interval In contains z and has length smaller than $\delta(z)$. Thus, by definition, $I_n \in C$. In particular, C does indeed contain a partition of that interval I_n since the single interval $\{I_n\}$ is itself a partition. But this contradicts the way in which the sequence was chosen and this contradiction completes our proof. □

6.11.40 Problem. Suppose that a function f is locally bounded at each point of a closed and bounded set E. Then f is bounded on the whole of the set E.

6.11.40.1 Solution. (Cousin compactness argument) The set E is bounded and so is contained in some interval $[a, b]$. Let us say that an interval $[c, d] \supset [a, b]$ is "black" if the following statement is true: There is a number M (which may depend on $[c, d]$) so that $|f(t)| \leq M \ \forall\, t \in E$ that are in the interval $[c, d]$. The collection of all black intervals is a Cousin cover of $[a, b]$. This is because of the local boundedness assumption on f. Consequently, by Cousin's lemma, there is a partition of the interval $[a, b]$ consisting of black intervals. The function f is bounded in E on each of these finitely many black intervals and so, since there are only finitely many of them, f must be bounded on E in $[a, b]$. But $[a, b]$ includes all of E and so the proof is complete. □

6.11.41 Problem. (**Baire**) Let F be a nonempty closed subset of $\mathbb{R}$. Then, every nonempty open subset O of F is of second category in F.

6.11.41.1 Solution. Let O be a nonempty open subset of F. Assume that O is of first category in F, i.e., $O = \bigcup_{n=1}^{\infty} A_n$, where each A_n is nowhere dense in F. It follows easily that $G_n = \left(\overline{A_n}\right)^C$ is open and dense in F. By the Baire category theorem, $\bigcap_{n=1}^{\infty} G_n = \bigcap_{n=1}^{\infty} \left(\overline{A_n}\right)^C = \left(\bigcup_{n=1}^{\infty} \overline{A_n}\right)^C$ is dense. However, $\left(\bigcup_{n=1}^{\infty} \overline{A_n}\right)^C \bigcap O = \emptyset$, a contradiction. □

6.11.1 Remark. The above problem applies to non-empty subsets of F that are open relatively to F. Note that the set F itself is open relatively to F, since $F = \mathbb{R} \cap F$, and the set $\mathbb{R}$ is open in $\mathbb{R}$. Then the above problem applies also to $O = F$. Of course, this means that F (an arbitrary nonempty closed subset of $\mathbb{R}$) cannot be written as a countable union of subsets of F that are nowhere dense relatively to F. As an application, we recover the fact that [0,1] is uncountable. If [0,1] is countable, then for at least one of its points x we should have that the interior (relatively to [0,1]) of $\{x\}$ will be nonempty. Since $\{x\}$ is closed, this means that x should be isolated in [0,1], and this is false.

6.11.42 Problem. The characteristic function of the rationals $\chi_{\mathbb{Q}}$ is not the limit of a sequence of continuous functions.

6.11.42.1 Solution. Suppose, to the contrary, that there is a sequence (f_n) of continuous functions such that $\chi_{\mathbb{Q}}(x) = \lim_n f_n(x)$ for each $x \in \mathbb{R}$. Then, the set $A_n = \left\{x; f_n(x) > \frac{1}{2}\right\}$ is open for each n, and, hence, so is

$$G_n = \bigcup_{k=1}^{\infty} A_k = \left\{x; f_k(x) > \frac{1}{2} \text{ for some } k \geq n\right\}.$$

But then, $\bigcap_{n=1}^{\infty} G_n = \left\{x; f_n(x) > \frac{1}{2} \text{ for infinitely many } n\right\} = \mathbb{Q}$ (why?), and this contradicts the fact that $\mathbb{Q}$ cannot be written as the countable intersection of open subsets of $\mathbb{R}$. This example illustrates a special case of a deep result, due to both Baire and Osgood, stating that any function $f : \mathbb{R} \to \mathbb{R}$ that is the limit of a sequence of continuous functions must have a point of continuity. $\square$

6.11.43 Problem. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and $\lim_{n\to\infty} f(nx) = 0$ for every $1 \leq x \leq 2$. Show that $\lim_{n\to\infty} f(x) = 0$.

6.11.43.1 Solution. Fix $\epsilon > 0$ and for each $k \geq 1$ define

$$E_k = \{x \in [1, 2];\ |f(nx)| < \epsilon\ \forall\, n \geq k\}.$$

Evidently each E_k is closed and $[1, 2] = \cup_{k=1}^{\infty} E_k$. Therefore, by Baire's Category Theorem, we can find a k and an open interval I such that $I \subseteq E_k$. It is easy to see that the union $\bigcup_{n=k}^{\infty}(nI) = \{y; y = nx \text{ for some } x \in I \text{ and some } n \geq k\}$ contains an interval of the form (a, ∞). It follows that for every $y > a, |f(y)| \leq \epsilon$. $\square$

6.11.44 Problem. There exists a subset of $\mathbb{R}$ that is dense, has cardinality $\mathfrak{c}$, and is of the first category.

6.11.44.1 Solution. For such a set, consider the union of the Cantor ternary set and the rationals. $\square$

6.11.45 Problem. Prove that every set $B \subseteq \mathbb{R}$ contains a countable set A that is dense in B.

6.11.45.1 Solution. $A \subseteq \overline{A} \Rightarrow A$ is dense in $\overline{A}$. $\square$

6.11.46 Problem. A set is of type G_δ iff its complement is of type F_σ.

6.11.46.1 Solution. A set G is of type G_δ iff

$$G = \bigcap_{n=1}^{\infty} U_n \Leftrightarrow G^C = \bigcup_{n=1}^{\infty} U_n^C.$$

where U_n is open. $\square$

6.11.47 Problem. A half-open interval $(a, b]$ is both of type G_δ and of type F_σ.

6.11.47.1 Solution.

$$(a, b] = \bigcap_{n=1}^{\infty} \left(a, b + \frac{1}{n}\right) = \bigcup_{n=1}^{\infty} \left[a + \frac{1}{n}, b\right]. \quad \square$$

6.11.48 Problem. Show that the set of rationals is F_σ.

6.11.48.1 Solution. If $q_1, q_2, q_3, ...$ is an enumeration of $\mathbb{Q}$, then

$$\mathbb{Q} = \bigcup_{k=1}^{\infty} \{q_k\}$$

Thus the set $\mathbb{Q}$ is a countable union of closed sets. Hence the set of rational numbers F_σ. $\square$

6.11.49 Problem. Show that the set of irrationals is G_δ.

6.11.49.1 Solution. If $q_1, q_2, q_3, ...$ is an enumeration of $\mathbb{Q}$, then

$$\mathbb{R} \setminus \mathbb{Q} = \bigcap_{k=1}^{\infty} (\mathbb{R} \setminus \{q_k\})$$

Thus the set $\mathbb{R} \setminus \mathbb{Q}$ is a countable intersection of open sets. Hence the set of irrational numbers G_δ. $\square$

6.11.50 Problem. A closed interval $[a, b]$ or a half-open interval $(a, b]$ is of type G_δ.

6.11.50.1 Solution. Since

$$[a, b] = \bigcap_{k=1}^{\infty} \left(a - \frac{1}{n}, b + \frac{1}{n}\right)$$
$$(a, b] = \bigcap_{k=1}^{\infty} \left(a, b + \frac{1}{n}\right). \quad \square$$

6.11.51 Problem. Show that [0,1) is simultaneously G_δ and F_σ.

6.11.51.1 Solution. Hint. Consider $(-\frac{1}{n}, 1)$ and $[0, 1 - \frac{1}{n}]$. $\square$

6.11.52 Problem. Show that a set in $\mathbb{R}$ is G_δ , if and only if its complement is F_σ.

6.11.52.1 Solution. Left to the reader.

6.11.53 Problem. Show that if F is an F_σ-set, then there is an increasing sequence (F_i) of closed sets such that $F = \cup_{i=1}^{\infty} F_i$.

6.11.53.1 Solution. Hint: If $F = \cup_{n=1}^{\infty} F_i$, then, let $G_n = \cup_{i=1}^{n} F_i$ and $F = \cup_{n=1}^{\infty} G_n$.

6.11.54 Problem. Show that if G is a G_δ-set, then there is a decreasing sequence (G_i) open sets G_i such that $G = \cap_{n=1}^{\infty} G_n$. .

6.11.54.1 Solution. Similar to the above.

6.11.55 Problem. Show that a countable union and a finite intersection of F_σ-sets is an F_σ-set.

6.11.55.1 Solution. Hint: Write

$$\bigcup_{i=1}^{\infty} \left(\bigcup_{j=1}^{\infty} F_{i,j} \right) = \bigcup \{F_{i,j}; (i,j) \in \mathbb{N} \times \mathbb{N}\}.$$

If the family is not empty, write

$$\bigcap_{i=1}^{n} \left(\bigcup_{j=1}^{\infty} F_{i,j} \right) = \bigcup \{F_{1,j_1} \cap ... \cap F_{n,j_n}; (j_1, ..., j_n) \in \mathbb{N} \times ... \times \mathbb{N}\}. \quad \square$$

6.11.56 Problem. Every open set and every closed set in $\mathbb{R}$ is of type G_δ.

6.11.56.1 Solution. Let G be an open set in $\mathbb{R}$. It is clear that G is of type G_δ. We also show that G can be expressed as a countable union of closed sets. Express G in the form

$$G = \bigcup_{k=1}^{\infty} (a_k, b_k)$$

where the intervals (a_k, b_k) are pairwise disjoint. Now for each $k \in \mathbb{N}$ there exist sequences (c_{k_j}) and (d_{k_j}) such that the sequence (c_{k_j}) decreases to a_k, the sequence (d_{k_j}) increases to b_k and (c_{k_j}) and (d_{k_j}) for each $j \in \mathbb{N}$. Thus

$$(a_k, b_k) = \bigcup_{k=1}^{\infty} \left[c_{k_j}, d_{k_j}\right].$$

We have expressed each component interval of G as a countable union of closed sets. It follows that

$$G = \bigcup_{k=1}^{\infty} \bigcup_{j=1}^{\infty} \left[c_{k_j}, d_{k_j}\right] = \bigcup_{j,k=1}^{\infty} \left[c_{k_j}, d_{k_j}\right].$$

is also a countable union of closed sets. Now take complements. This shows that $\mathbb{R} \setminus G$ can be expressed as a countable intersection of open sets (by using the de Morgan laws). Since every closed set F can be written

$$F = \mathbb{R} \setminus G$$

for some open set G, we have shown that any closed set is of type G_δ. □

6.11.57 Problem. Every open set and every closed set in $\mathbb{R}$ is both of type F_σ and G_δ.

6.11.57.1 Solution. Left to the reader. □

6.11.58 Problem. Find a set in $\mathbb{R}$ that is neither a G_δ-set nor an F_σ-set.

6.11.58.1 Solution. Hint. The union of the set of rational numbers in $(-\infty, 0)$ and the set of irrational numbers in $(0, \infty)$. □

6.11.59 Problem. Let H be of type G_δ and be dense in $\mathbb{R}$. Then H is residual.

6.11.59.1 Solution. Write

$$H == \bigcap_{k=1}^{\infty} G_k$$

with each of the sets G_k open. Since H is dense by hypothesis and $H \subseteq G_k$ for each $k \in \mathbb{N}$, each of the open sets G_k is also dense. Thus $\mathbb{R} \setminus G$ is nowhere dense for every $k \in \mathbb{N}$, and so each G_k is residual. Hence result follows.

6.11.60 Problem. Let E be closed, nowhere dense set. Suppose that $[u, v]$ is a closed, bounded interval, then there exists an interval $[c, d] \subseteq [u, v]$ such that $[c, d] \cap E = \emptyset$.

6.11.60.1 Solution. Since E is a nowhere dense set, it contains no open intervals. In particular, there is a point $p \in (u, v) \setminus E$. Since E is closed and $p \notin E$, there exists $\epsilon > 0$ such that $(p - \epsilon, p + \epsilon) \cap E = \emptyset$, and $[p - \epsilon, p + \epsilon] \subseteq [u, v]$. We see that the interval $[p - \epsilon, p + \epsilon]$ satisfies the required properties. □

6.11.61 Problem. If (E_n) is a sequence of nowhere dense sets, then every closed interval $[a, b]$ contains a point x that does not belong to any of the sets E_n.

6.11.61.1 Solution. Without loss of generality, we may assume that each E_n is closed. By the previous problem $\exists$ an interval $[a_1, b_1] \subseteq [a, b]$ such that $[a_1, b_1] \cap E_1 = \emptyset$. Similarly, $\exists$ an interval $[a_2, b_2] \subseteq [a_1, b_1]$ such that $[a_2, b_2] \cap E_2 = \emptyset$. Continuing the process, we get a nested sequence $([a_n, b_n])$ of closed intervals such that $[a_n, b_n] \cap E_n = \emptyset \ \forall \ n \in \mathbb{N}$. By the nested interval theorem, there exists a point x such that $x \in \bigcap_{n=1}^{\infty}[a_n, b_n]$ and the point x does not belong to any of the sets E_n.
One simple consequence of the preceding problem is the fact that the set of all real numbers cannot be expressed as a countably infinite union of nowhere dense sets. We can use this fact to prove that the set of rational numbers is not a G_δ set. Therefore, there is no function $f : \mathbb{R} \to \mathbb{R}$ that is continuous on $\mathbb{Q}$ and discontinuous on $\mathbb{Q}^C$. □

6.11.62 Problem. Show that the set of irrational numbers is not a countable union of closed subsets of $\mathbb{R}$.

6.11.62.1 Solution. Let $\mathbb{Q} = \{r_1, r_2, ..\}$ be an enumeration of the rational numbers of $\mathbb{R}$. Assume that there exists a sequence of closed sets (E_n) of $\mathbb{R}$ such that $\mathbb{Q}^C = \bigcup_{n=1}^{\infty} E_n$. Then

$$\mathbb{R} = \mathbb{Q}^C \cup \mathbb{Q} = \left(\bigcup_{n=1}^{\infty} E_n\right) \bigcup \left(\bigcup_{n=1}^{\infty} \{r_n\}\right)$$

and by the Baire Category Theorem, we must have $(E_n)^\circ \neq \emptyset$ for some n. Thus, some E_n contains an interval. However, since $E_n \subseteq \mathbb{Q}^C$ holds and each interval contains rational numbers, this is impossible, and the conclusion follows. □

6.11.63 Problem. The set of rationals $\mathbb{Q}$ is not a G_δ set.

6.11.63.1 Solution. Suppose that $\mathbb{Q}$ is a G_δ set. Then $\mathbb{Q}^C$ is an F_σ set. Let

$$\mathbb{R} \setminus \mathbb{Q} = \bigcup_{n=1}^{\infty} E_n,$$

where each E_n is a closed set. Since $E_n = \overline{E_n}$ contains no rational numbers, it contains no open intervals. Therefore, each E_n is a nowhere dense set. Let $\mathbb{Q} = \{r_n; n \in \mathbb{N}\}$, then the equation

$$\mathbb{R} = (\mathbb{R} \setminus \mathbb{Q}) \cup \mathbb{Q} = \left(\bigcup_{n=1}^{\infty} E_n\right) \bigcup \left(\bigcup_{n=1}^{\infty} \{r_n\}\right)$$

shows that the set of all real numbers can be expressed as a countably infinite union of nowhere dense sets. This is a contradiction. So, the set of rationals $\mathbb{Q}$ is not a G_δ set. □

6.11.64 Problem. Let f be defined on a closed interval I (which may be $\mathbb{R}$). Then the set $C(f)$ of points of continuity of f is of type G_δ, and the set $D(f)$ of points of discontinuity of f is of type F_σ. Conversely, if H is a set of type G_δ, then there exists a function f defined on $\mathbb{R}$ such that $C(f) = H$.

6.11.64.1 Solution. Let $f : I \to \mathbb{R}$. We show that the set

$$C(f) = \{x; \omega_f(x) = 0\}$$

is of type G_δ. For each $k \in \mathbb{N}$, let

$$B_k = \left\{x; \omega_f(x) \geq \frac{1}{k}\right\}$$

and each of the sets B_k is closed. Thus the set

$$B = \bigcup_{k=1}^{\infty} B_k$$

is of type F_σ, and $D(f) = B$. Therefore, $C(f) = I \setminus B$. Since the complement of an F_σ is a G_δ, the set $C(f)$ is a G_δ. To prove the converse, let H be any subset of $\mathbb{R}$ of type G_δ. Then H can be expressed in the form

$$H = \bigcap_{k=1}^{\infty} G_k$$

with each of the sets G_k being open. We may assume without loss of generality that $G_1 = \mathbb{R}$ and that $G_i \supset G_{i+1}$ for each $i \in \mathbb{N}$. (Verify this.) Let (α_k) and (β_k) be sequences of positive numbers, each converging to zero, with

$$\alpha_k > \beta_k > \alpha_{k+1} \; \forall \; k \in \mathbb{N}.$$

Define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0, & \text{if } x \in H \\ \alpha_k, & \text{if } x \in (G_i \setminus G_{i+1}) \cap \mathbb{Q} \\ \beta_k, & \text{if } x \in (G_i \setminus G_{i+1}) \cap \mathbb{Q}^C. \end{cases}$$

We show that f is continuous at each point of H and discontinuous at each point of $\mathbb{R} \setminus H$. Let $x_0 \in H$ and let $\epsilon > 0$. Choose $n \in \mathbb{N}$ such that $\alpha_n < \epsilon$. Since

$$x_0 \in H = \bigcap_{k=1}^{\infty} G_k,$$

we see that $x_0 \in G_n$. The set G_n is open, so there exists $\delta > 0$ such that $B(x_0; \delta) \subseteq G_n$. From the definition of f on G_n, we see that

$$0 \leq f(x) \leq \alpha_n < \epsilon \; \forall \; x \in B(x_0; \delta).$$

Thus

$$|f(x) - f(x_0)| = |f(x) - 0| = |f(x)| < \epsilon$$

if $|x - x_0| < \delta$, so f is continuous at x_0.

Now let $x_0 \in \mathbb{R} \setminus H$. Then there exists $k \in \mathbb{N}$ such that x_0 belongs to the set $G_k \setminus G_{k+1}$. Thus $f(x_0) = \alpha_k$ or $f(x_0) = \beta_k$. Let us suppose that $f(x_0) = \alpha_k$. If x_0 is an interior point of $G_k \setminus G_{k+1}$, then x_0 is a limit point of

$$\{x; x \in (G_i \setminus G_{i+1}) \cap \mathbb{Q}^C\} = \{x; f(x) = \beta_k\},$$

so f is discontinuous at x_0.
The argument is similar if x_0 is a boundary point of $G_k \setminus G_{k+1}$. Again, assume $f(x_0) = \alpha_k$. Arbitrarily close to x_0 there are points of the set

$$\mathbb{R} \setminus (G_k \setminus G_{k+1}).$$

At these points, f takes on values in the set

$$S = \{0\} \bigcup_{i \neq k} \{\alpha_i\} \bigcup_{j \neq k} \{\beta_j\}.$$

The only limit point of this set is zero and so S is closed. In particular, α_k is not a limit point of this set and does not belong to the set. Let ϵ be half the distance from the point α_k to the closed set S; that is, let

$$\epsilon = \frac{1}{2} d(\alpha_k, S),$$

where $d(\alpha_k, S) = \inf\{|\alpha_k - s|\}; s \in S$. Now, arbitrarily close to x_0 there are points x such that $f(x) \in S$. For such a point,

$$|f(x) - f(x_0)| = |f(x) - \alpha_k| > \epsilon,$$

so f is discontinuous at x_0. $\square$

6.11.65 Problem. Let $A = \left\{\frac{1}{i}; i \in \mathbb{N}\right\}$. Show that A is a G_δ set in $\mathbb{R}$. Construct a real-valued function on $\mathbb{R}$ that is continuous exactly at the points of A.

6.11.65.1 Solution. Choose $\epsilon_i >$ such that $\epsilon_i < \frac{1}{i}$ for each $i \in \mathbb{N}$, and that the intervals $\left(\frac{1}{i} - \epsilon, \frac{1}{i} + \epsilon\right)$ are pairwise disjoint. Call the family of these intervals G_1. Form the same with intervals of radius $\epsilon_i/2$, and call the family of these intervals G_2. Keep on going to form G_3, G_4, etc. Then $\bigcap_{n=1}^{\infty} G_n$ is the set A. The function is, for example,

$$f(x) = \begin{cases} \mathfrak{D}(x) \sin\left(\frac{\pi}{x}\right) & \text{if } x \neq 0 \\ 1 & \text{if } x = 0. \end{cases}$$

where $\mathfrak{D}$ is the Dirichlet function. Show that $x\mathfrak{D}(x)$ does not have one-sided derivative at 0.

6.11.66 Problem. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. the set

$$S = \{a \in \mathbb{R}; f \text{ is not increasing at } a\}.$$

is a G_δ-set without isolated points.

6.11.66.1 Solution. For $n \in \mathbb{N}$, let

$$U_n = \left\{a \in \mathbb{R};\ \exists\, x \in \left(a - \frac{1}{n}, a\right), f(x) > f(a)\right\}$$

and

$$V_n = \left\{a \in \mathbb{R};\ \exists\, x \in \left(a, a + \frac{1}{n}\right), f(x) < f(a)\right\}.$$

Then U_n and V_n, are open for each $n \in \mathbb{N}$, hence so is $W_n = U_n \cup V_n$. Let $a \in \mathbb{R}$. f is not increasing at a if and only if a lies in every W_n. The theorem follows. $\square$

As a consequence, we get the following

6.11.67 Problem. There is no continuous function $f:\mathbb{R}\to\mathbb{R}$ for which the set

$$\{a\in\mathbb{R}; f \text{ is increasing at } a\}=\mathbb{R}\setminus\mathbb{Q}.$$

6.11.67.1 Solution. $\mathbb{Q}$ is not a G_δ-set. $\square$

6.11.68 Problem. There is no continuous function $f:\mathbb{R}\to\mathbb{R}$ for which the set

$$\{a\in\mathbb{R}; f \text{ is continuous at } a\}=\mathbb{R}\setminus\mathbb{Q}.$$

6.11.68.1 Solution. It is sufficient to prove that the set

$$D(f)=\{x; x\in\mathbb{R}, f \text{ is discontinuous at } x\}$$

is an F_σ set. Note that $x\in D$ iff $\omega_f(x)>0$. For each $n\in\mathbb{N}$, let $E_n=\{x;\omega_f(x)>\frac{1}{n}\}$, and we can prove that each E_n is a closed set. Since $D=\cup_{n=1}^{\infty}E_n$, the set D is an F_σ set. $\square$

6.11.69 Problem. If f is a real-valued function defined on $\mathbb{R}$,

1. then the set of its points of continuity is a G_δ-set.
2. given a G_δ-set G in $\mathbb{R}$, there is a function on $\mathbb{R}$ whose set of points of continuity is exactly G.

6.11.69.1 Solution.

1. Let

$$G_n=\left\{x\in\mathbb{R}; \exists \text{ a nbhd. } U \text{ of } x \text{ such that } |f(x)-f(y)|<\frac{1}{n}\ \forall\, x,y\in U\right\}.$$

Then each G_n is open and $\cap_n G_n$ is exactly the set of continuity points of f. Indeed, if $x\in\cap_n G_n$, then clearly x is a point of continuity of f. If x_0 is a point of continuity of f and $n\in\mathbb{N}$ is given, choose $\delta>0$ so that for each point x of $(x_0-\delta,x_0+\delta)$, we have $|f(x)-f(x_0)|<\frac{1}{2n}$. Then, if $x,y\in(x_0-\delta,x_0+\delta)$,

$$|f(x)-f(y)|\le|f(x)-f(x_0)|+|f(y)-f(x_0)|<\frac{1}{n}.$$

Thus $x_0\in\cap_n G_n$.

2. If G is a G_δ-set in $\mathbb{R}$, write $G=\cap_{n=1}^{\infty}G_n$, where each G_n is an open set in $\mathbb{R}$ and $G_1=\mathbb{R}$. Assume, without loss of generality, that $G_n\supseteq G_{n+1}$ for each $n\in\mathbb{N}$. Define the function f on $\mathbb{R}$ by

$$f(x)=\begin{cases}0, & \text{if } x\in G\\ \frac{1}{n} & \text{if } x\in G_n\setminus G_{n+1}, n\in\mathbb{N}, \text{ and } x \text{ is rational,}\\ -\frac{1}{n} & \text{if } x\in G_n\setminus G_{n+1}, n\in\mathbb{N}, \text{ and } x \text{ is irrational.}\end{cases}$$

6.11.70 Problem. Define $\chi_{\mathbb{Q}}:\mathbb{R}\to\mathbb{R}$ by $\chi_{\mathbb{Q}}(x)=\begin{cases}1, & \text{if } x\in\mathbb{Q}\\ 0, & \text{if } x\in\mathbb{Q}^C\end{cases}$. Show that $\chi_{\mathbb{Q}}$ is not continuous on $\mathbb{R}$.

6.11.70.1 Solution. We see that $\chi_{\mathbb{Q}}^{-1}\left(B\left(1;\frac{1}{3}\right)\right)=\mathbb{Q}$ and $\chi_{\mathbb{Q}}^{-1}\left(B\left(0;\frac{1}{3}\right)\right)=\mathbb{Q}^C$. Thus $\chi_{\mathbb{Q}}$ cannot be continuous at any point of $\mathbb{R}$ because neither $\mathbb{Q}$ nor $\mathbb{R}\setminus\mathbb{Q}$ is open set. $\square$

6.11.71 Problem. Let $f:\mathbb{N}\to\mathbb{R}$ be any function. Then f is continuous.

6.11.71.1 Solution. Let $n\in\mathbb{N}$, then $f\left(B\left(n;\frac{1}{2}\right)\cap\mathbb{N}\right)=\{f(n)\}\subseteq B\left(f(n),\epsilon\right)\ \forall\ \epsilon>0$. Hence f is continuous. $\square$

6.11.72 Problem. Let $f:\mathbb{R}\to\mathbb{N}$. Then f is continuous iff f is constant.

6.11.72.1 Solution. If f were not constant on $\mathbb{R}$, then $f(\mathbb{R})$ contains more than one element, let $f(\mathbb{R})=\{p,q\}$ where $p\neq q$. Let $0<\epsilon=\frac{|p-q|}{2}$, since f is continuous then both $f^{-1}\left(B\left(p;\epsilon\right)\right)$ and $f^{-1}\left(B\left(q;\epsilon\right)\right)$ are open, disjoint with

$$f^{-1}\left(B\left(p;\epsilon\right)\right)\cup f^{-1}\left(B\left(q;\epsilon\right)\right)=\mathbb{R},$$

which is impossible, as $\mathbb{R}$ is connected.
If f is constant, then $f(x)=k\ \forall\ x\in\mathbb{R}$, for some $k\in\mathbb{N}$, hence $\forall\ \epsilon>0\ f^{-1}\left(B\left(k;\epsilon\right)\right)=\mathbb{R}$. Thus f is continuous on $\mathbb{R}$. $\square$

6.11.73 Problem (The Baire Category Theorem for $\mathbb{R}$)**.** If (G_n) is a sequence of dense, open sets in $\mathbb{R}$, then $\bigcap_{n=1}^{\infty}G_n\neq\emptyset$. In fact, $\bigcap_{n=1}^{\infty}G_n$ is dense in $\mathbb{R}$.

1. Use the above theorem to prove that $\mathbb{R}$ is uncountable.
2. A dense G_δ subset of $\mathbb{R}$ must also be an uncountable set.
3. If $\mathbb{R}=\bigcup_{n=1}^{\infty}E_n$ where each E_n is closed, then some E_n contains an open interval.

6.11.73.1 Solution.

1. If $\mathbb{R}$ were countable then $\mathbb{R}=\{x_1,x_2,....\}$ and then each $G_n=\mathbb{R}\setminus\{x_n\}$ is open and dense but $\cap_{n=1}^{\infty}G_n=\emptyset$, which is a contradiction.
2. Let (G_n) be open dense sets in $\mathbb{R}$ and $\bigcap_{n=1}^{\infty}G_n=\{x_1,x_2,....\}$, then the sets $H_n=G_n\setminus\{x_n\}$ are open and dense but $\bigcap_{n=1}^{\infty}G_n=\emptyset$, which is a contradiction. Therefore, $\bigcap_{n=1}^{\infty}G_n$ is uncountable.
3. Each set $G_n=\mathbb{R}\setminus E_n$ is open in $\mathbb{R}$ and $\bigcap_{n=1}^{\infty}G_n=\emptyset$. Therefore, by Baire's theorem, some G_n is not dense implies some G_n contains an open interval which implies some E_n conatins an interval. $\square$

6.11.74 Problem (Upper and lower envelopes of a function). Let f be a real-valued function defined on $[a,b]$. We define the lower envelope g of f to be the function g defined by

$$g(y)=\underline{\lim}_{x\to y}f(x)=\sup_{\delta>0}\inf_{x\in\hat{N}(y;\delta)}f(x).$$

and the upper envelope h by

$$h(y)=\overline{\lim}_{x\to y}f(x)=\inf_{\delta>0}\sup_{x\in\hat{N}(y;\delta)}f(x)$$

Let g and h be the lower and upper envelopes of f, then show that

1. For each $x \in [a,b], g(x) \leq f(x) \leq h(x)$, and $g(x) = f(x)$ if and only if f is lower semicontinuous at x, while $g(x) = h(x)$ if and only if f is continuous at x.

2. If f is bounded, the function g is lower semicontinuous, while h is upper semicontinuous.

3. If ψ is any lower semicontinuous function such that $\psi(x) \leq f(x)$ for all $x \in [a,b]$, then $\psi(x) \leq g(x)$ for all $x \in [a,b]$.

6.11.74.1 Solution.

1. Let $x \in [a,b]$. Then $\inf_{|y-x|<\delta} f(y) \leq f(x) \leq \sup_{|y-x|<\delta} f(y)$ for any $\delta > 0$. Hence we have
$$g(x) = \sup_{\delta>0} \inf_{|y-x|<\delta} f(y) \leq f(x) \leq \inf_{\delta>0} \sup_{|y-x|<\delta} f(y) = h(x).$$
Suppose $g(x) = f(x)$. Then given $\epsilon > 0$, there exists $\delta > 0$ such that $f(x) - \epsilon = g(x) - \epsilon < \inf_{|y-x|<\delta} f(y)$. Thus $f(x) - \epsilon < f(y)$ whenever $|y-x| < \delta$ and f is lower semicontinuous at x. Conversely, suppose f is lower semicontinuous at x. $f(x)$ is an upper bound for $\left\{\inf_{|y-x|<\delta} f(y); \delta > 0\right\}$ and $\epsilon > 0$, there exists $\delta > 0$ such that $f(x) - \epsilon \leq f(y)$ whenever $|y-x| < \delta$. Thus $f(x) \leq \inf_{|y-x|<\delta} f(y)$ so $f(x) = \sup_{\delta>0} \inf_{|y-x|<\delta} f(y) = g(x)$. By a similar argument, $f(x) = h(x)$ if and only if f is upper semicontinuous at x. Thus f is continuous at x if and only if f is both upper semicontinuous and lower semicontinuous at x if and only if $f(x) = h(x)$ and $g(x) = f(x)$ if and only if $g(x) = h(x)$.

2. Let $\lambda \in \mathbb{R}$. Suppose that $g(x) > \lambda$. Then there exists $\delta > 0$ such that $f(y) > \lambda$ whenever $|y-x| < \delta$. Hence $\{x; g(x) > \lambda\}$ is open in $[a,b]$ and g is lower semicontinuous. Suppose $h(x) < \lambda$. Then there exists $\delta > 0$ such that $f(y) < \lambda$ whenever $|y-x| < \delta$. Hence $\{x; h(x) < \lambda\}$ is open in $[a,b]$ and h is upper semicontinuous.

3. Let ψ be a lower semicontinuous function such that $\psi(x) \leq f(x)$ for all $x \in [a,b]$. Suppose $\psi(x) > g(x)$ for some $x \in [a,b]$. Then there exists $\delta > 0$ such that $\psi(x) \leq \psi(y) + \psi(x) - g(x)$ whenever $|x-y| < \delta$. i.e. $g(x) \leq \psi(y)$ whenever $|x-y| < \delta$. In particular, $g(x) \leq \psi(x)$. Contradiction. Hence $\psi(x) \leq g(x)$ for all $x \in [a,b]$. □

6.11.75 Problem. Prove that the characteristic function χ_S of a subset S of $\mathbb{R}$ is lower semicontinuous, if and only if, the set S is open.

6.11.75.1 Solution. Hint. Assume first that χ_S is lower semicontinuous. Then, $\{x \in \mathbb{R}; \chi_S(x) \leq 1/2\} = \mathbb{R} \setminus S$, and this set is closed. Hence S is open. Assume now that S is open. Observe that, for $r \in \mathbb{R}$, the set $\{x \in \mathbb{R}; \chi_S(x) \leq 1/2\} = \mathbb{R} \setminus S$ is either $\mathbb{R}$ or $\mathbb{R} \setminus S$, or, finally, the empty set (all of them closed in $\mathbb{R}$). Thus the function χ_S is lower semicontinuous. □

6.11.76 Problem. Discuss the continuity of the Dirichlet's function:
$$f(x) = \lim_{m\to\infty} \left(\lim_{n\to\infty} \cos^n(\pi m! x) \right), x \in \mathbb{R}.$$

6.11.76.1 Solution. Assume first that x is a rational number, i.e., $x = \frac{p}{q}$ for some $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. Then for $m > q$ it holds
$$m!x = m!\frac{p}{q} = 1.2....(q-1)(q+1)..m.p$$
hence $m!x$ is an even number. This implies $\cos(\pi m! x) = 1$, hence $f(x) = 1$ for every $x \in \mathbb{Q}$. If x is an irrational number, then for no m can the number $m!x$ be an integer. But then $|\cos(\pi m! x)| < 1$, which implies $\lim_{n\to\infty} \cos^n(\pi m! x) = 0$. Passing to the limit in m, the last equality gives $(\forall\ x \in \mathbb{R} \setminus \mathbb{Q})$, $f(x) = 0$. It is proved that f is discontinuous at every real x. □

6.11.76.2 Solution. If a is an arbitrary real number, then there exist two sequences, one of rational, and the other of irrational numbers, both converging to a. Denoting the first by (r_n) and the second by (i_n), we have

$$\lim_{n\to\infty} r_n = a \Rightarrow \lim_{n\to\infty} f(r_n) = 1 \text{ and}$$
$$\lim_{n\to\infty} i_n = a \Rightarrow \lim_{n\to\infty} f(i_n) = 0$$

Hence for no $x \in \mathbb{R}$ does the limit $\lim_{n\to\infty} f(x)$ exist, which means that f has a second order discontinuity at every real a. ☐

6.11.77 Problem. A function is uniformly continuous on A and on B, is it true that it is uniformly continuos on $A \cup B$?

6.11.77.1 Solution. Let $f : [1, 2) \to \mathbb{R}$ be defined by $f(x) = 1\ \forall\ x \in [1, 2)$ and $f : [2, 3] \to \mathbb{R}$ be defined by $f(x) = 2\ \forall\ x \in [2, 3]$, we see that f is uniformly continuous on $[1, 2)$ and on $[2, 3]$, but f is discontinuous on $[1, 3]$. ☐

6.11.78 Problem. Prove that the function $f(x) = \frac{|\sin x|}{x}, x \neq 0$, is uniformly continuous on each of the intervals $(-1, 0)$ and $(0, 1)$, but is not uniformly continuous on their union $(-1, 0) \cup (0, 1)$.

6.11.78.1 Solution. The function $f : (0, 1) \to \mathbb{R}$ is the restriction $F : [0, 1] \to \mathbb{R}$ on (0,1) given by

$$F(x) = \begin{cases} 1 & \text{if } x = 0 \\ \frac{\sin x}{x} & \text{if } 0 < x < 1 \\ \sin 1 & \text{if } x = 1 \end{cases}$$

Clearly, F is continuous on the closed interval [0,1], hence it is uniformly continuous there. This also gives the uniform continuity of f on the interval (0,1). Analogously, the function G given by

$$G(x) = \begin{cases} -\sin 1 & \text{if } x = -1 \\ \frac{-\sin x}{x} & \text{if } -1 < x < 0 \\ -1 & \text{if } x = 0 \end{cases}$$

is uniformly continuous on the closed interval $[-1, 0]$ and equals to f on $(-1, 0)$. Hence f is uniformly continuous on $(-1, 0)$.
However, we shall prove next that f is not uniformly continuous on the union $(-1, 0) \cup (0, 1)$. Let $x_n = -\frac{1}{1+n}$ and $y_n = \frac{1}{1+n}$, then both $x_n, y_n \to 0$. But

$$\begin{aligned} |f(x_n) - f(y_n)| &= \left| \frac{\left|\sin\left(\frac{-1}{n+1}\right)\right|}{\frac{-1}{n+1}} - \frac{\left|\sin\left(\frac{1}{n+1}\right)\right|}{\frac{1}{n+1}} \right| \\ &= 2(n+1)\left|\sin\left(\frac{1}{n+1}\right)\right| \geq 2(n+1)\left(\frac{2}{\pi}\frac{1}{n+1}\right) = \frac{4}{\pi}. \end{aligned}$$

Thus $|f(x_n) - f(y_n)|$ does not tend to 0. Hence f is not uniformly continuous. ☐

6.11.79 Problem. Characterise all functions from $\mathbb{R} \to \mathbb{R}$ which can be uniformly approximated on the real line by a sequence of polynomials.

6.11.79.1 Solution. Let $f : \mathbb{R} \to \mathbb{R}$ be the uniform limit of a sequence $(P_n(x))$ of polynomials. Since $(P_n(x))$ is a Cauchy sequence, so $\exists\ N$ such that $\sup_{x\in\mathbb{R}} |P_n(x) - P_N(x)| < 1$ if $n \geq N$. This implies $P_n(x) - P_N(x)$ is a bounded polynomial on $\mathbb{R}$, so $P_n(x) = c_n + P_N(x)$ where $c_n \in \mathbb{R}$. Hence

$$f = \lim_{n\to\infty} P_n(x) = \lim_{n\to\infty} c_n + P_N(x) = c + P_N(x)$$

where $c = \lim_{n\to\infty} c_n$. Therefore f must be a polynomial. □

6.11.80 Problem. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and suppose that $f(U)$ is open for every open set $U \subseteq \mathbb{R}$. Prove that f is monotonic.

6.11.80.1 Solution. Suppose that f is not monotonic, then $\exists\ x < y < z$ such that either $f(x) < f(y)$ and $f(y) > f(z)$ or $f(x) > f(y)$ and $f(y) < f(z)$. We shall consider the former case, as later case is analogous. Since f is continuous, it attains a maximum on $[x, z]$, say $f(u) = v = \sup\{f(t); t \in [x, z]\}$ for some $u \in [x, z]$. Since $y \in (x, z)$ and $f(y) > f(x), f(y) > f(z)$, we must actually have $u \in (x, z)$. Now let $U = (x, z)$ and note that $v \in f(U)$. But for any $\epsilon > 0, (v - \epsilon, v + \epsilon) \nsubseteq f(U)$: for example $v + \frac{\epsilon}{2} \notin f(U)$, so $f(U)$ is not open, a contradiction. □

6.11.81 Problem. Suppose $f : D \to \mathbb{R}$ is uniformly continuous and D is a bounded set of real numbers, then $f(D)$ is bounded.

6.11.81.1 Solution. Here we use the property that uniform continuous function carries Cauchy sequences to Cauchy sequences. Suppose $f(D)$ is not bounded, then $\exists\ d_n \in D$ such that $f(d_n) > n, \forall n \in \mathbb{N}$, and the sequence (d_n) is bounded as D is bounded. So by B-W theorem the sequence (d_n) has a convergent subsequence (d_{n_k}). Now $f(d_{n_k}) > n_k$ shows that $(f(d_{n_k}))$ is unbounded, which contradicts that $(f(d_{n_k}))$ is a Cauchy sequence. □

6.11.82 Problem. If the absolute value of the function f is continuous on (a, b), then the function is also continuous on (a, b). True or false?

6.11.82.1 Solution. False. Counterexample The absolute value of the function of f defined by

$$f(x) = \begin{cases} -1, & \text{if } 0 \leq x \\ 1, & \text{if } x > 0 \end{cases}$$

is $|f(x)| = 1$ for all real x and it is continuous, but the function f is discontinuous $x = 0$. □

6.11.83 Problem. If both functions f and g are discontinuous at $x = a$, then $f + g$ is also discontinuous at $x = a$. True or false?

6.11.83.1 Solution. False. Counterexample:

$$\begin{aligned} f(x) &= -\frac{1}{x-a} \text{ if } x \neq a \\ g(x) &= x + \frac{1}{x-a} \text{ if } x \neq a \\ f(x) &= g(x) = \frac{a}{2} \text{ if } x = a \end{aligned}$$

Both functions f and g are discontinuous at $x = a$, but the function $f + g$

$$(f+g)(x) = f(x) + g(x) = \begin{cases} x & \text{if } x \neq a \\ a & \text{if } x = a \end{cases}$$

is continuous at $x = a$. □

6.11.84 Problem. If both functions f and g are discontinuous at $x = a$, then fg is also discontinuous at $x = a$. True or false?

6.11.84.1 Solution. False. Counterexample: Both functions

$$f(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 2 & \text{if } x = 0 \end{cases} \text{ and } g(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ \frac{1}{2} & \text{if } x = 0 \end{cases}$$

are discontinuous at the point $x = 0$, but their product

$$(fg)(x) = f(x)g(x) = \begin{cases} \frac{\sin^2 x}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

is continuous at the point $x = 0$. □

6.11.85 Problem. A function always has a local maximum between any two local minima. True or false?

6.11.85.1 Solution. False. Counterexample: The functions $f(x) = \sec^2 x$ and $g(x) = \frac{x^4+1}{x^2}$ have no maximum between two local minima. □

6.11.86 Problem. For a continuous function there is always a strict local maximum between any two local minima. True or false?

6.11.86.1 Solution. False. Counterexample: The continuous function

$$f(x) = \begin{cases} (x+2)^2 + 1, & \text{if } -\infty < x \leq -1 \\ 2, & \text{if } -1 \leq < x \leq 1 \\ (x-2)^2 + 1, & \text{if } 1 \leq x < \infty \end{cases}$$

does not have a strict local maximum between its two local minima.
Note: A function f has a strict local maximum at the point $x = a$ if $f(a) > f(x)$ for all x within a certain neighbourhood $(a - \delta, a + \delta)$, $\delta > 0$ of the point $x = a$. □

6.11.87 Problem. If a function is defined in a certain neighborhood of point $x = a$ including the point itself and is increasing for all $x < a$ and decreasing for all $x > a$, then there is a local maximum at $x = a$. True or false?

6.11.87.1 Solution. False. Counterexample: The function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \dfrac{1}{(x-3)^2}, & \text{if } x \neq 3 \\ 1 & \text{if } x = 3 \end{cases}$$

is increasing for all $x < 3$ and decreasing for all $x > 3$, but it has no local maximum at the point $x = 3$. □

6.11.88 Problem. If a function is defined on $[a, b]$ and continuous on (a, b), then it takes its extreme values on $[a, b]$. True or false?

6.11.88.1 Solution. False. Counterexample: The function $f : [-\pi/2, \pi/2] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \tan x, & \text{if } x \in (-\pi/2, \pi/2) \\ 0 & \text{if } x = -\pi, \pi \end{cases}$$

is continuous on $(-\pi/2, \pi/2)$, but it has no extreme values on $[-\pi/2, \pi/2]$. □

6.11.89 Problem. Every continuous and bounded function on $(-\infty, \infty)$ takes on its extreme values. True or false?

6.11.89.1 Solution. False. Counterexample: The function f defined by $f(x) = \tan^{-1} x$ is continuous and bounded on $(-\infty, \infty)$, but takes no extreme values. □

6.11.90 Problem. If a function f is continuous on $[a, b]$, the tangent line exists at all points on its graph and $f(a) = f(b)$, then there is a point $c \in (a, b)$ such that the tangent line at the point $(c, f(c))$ is horizontal. True or false?

6.11.90.1 Solution. False. Counterexample: Consider the function $f : [-1, 1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \sqrt{-x}, & \text{if } -1 \le x \le 0 \\ \sqrt{x}, & \text{if } 0 \le x \le 1 \end{cases}$$

Here we see that $f'(c) \neq 0 \ \forall \ c \in [-1, 1]$ and the tangent is vertical at (0,0) and $f(-1) = f(1)$, so there is no horizontal tangent. □

6.11.91 Problem. If on the closed interval $[a, b]$ a function is:

1. bounded;
2. takes its maximum and minimum values;
3. takes all its values between the maximum and minimum values; then this function is continuous on $[a, b]$. True or false?

6.11.91.1 Solution. False. Counterexample: Consider the function $f : [0, 1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1, & \text{if } x = 0 \\ x, & \text{if } 0 < x < 1 \\ 0, & \text{if } x = 1. \end{cases} \quad \square$$

6.11.92 Problem. If on the closed interval $[a, b]$ a function is:

1. bounded;
2. takes its maximum and minimum values;
3. takes all its values between the maximum and minimum values; then this function is continuous at one or more points or subintervals on $[a, b]$. True or false?

6.11.92.1 Solution. False. Counterexample: The function $f : [-1, 1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x = 0 \\ x & \text{if } x \text{ is rational } x \neq 0, x \neq 1 \\ -x & \text{if } x \text{ is irrational} \\ 0 & \text{if } x = 1 \end{cases}$$

satisfies all three conditions above, but it is discontinuous at every point on $[-1, 1]$. $\square$

6.11.93 Problem. If a function is continuous on $[a, b]$, then it cannot take its absolute maximum or minimum value infinitely many times. True or false?

6.11.93.1 Solution. False. Counterexample: The function f is defined on [1,4] by

$$f(x) = \begin{cases} 3 & \text{if } x \in [1, 2] \\ 7 - 2x & \text{if } x \in [2, 3] \\ 1 & \text{if } x \in [3, 4]. \end{cases} \quad \square$$

6.11.94 Problem. If a function f is defined on $[a, b]$, and $f(a)f(b) < 0$, then there is some point $c \in (a, b)$ such that $f(c) = 0$. True or false?

6.11.94.1 Solution. False. Counterexample: The function f is defined on $[-1, 1]$ by

$$f(x) = \begin{cases} \frac{1}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

and $f(-1)f(1) = -1 < 0$ but there is no point $c \in [-1, 1]$ such that $f(c) = 0$. $\square$

6.11.95 Problem. If a function f is defined on $[a, b]$, and continuous on (a, b), then for any $N \in (f(a), f(b))$ there is some point $c \in (a, b)$ such that $f(c) = N$. True or false?

6.11.95.1 Solution. False. Counterexample: The function f below is defined on [3,4] by

$$f(x) = \begin{cases} 1 & \text{if } x \in (3, 4) \\ x^2; & \text{if } x = 3, 4. \end{cases}$$

and continuous but for any $N \in (f(a), f(b))$ there is no point c such that $f(c) = N$. $\square$

6.11.96 Problem. If a function is discontinuous at every point in its domain, then the square and the absolute value of this function cannot be continuous. True or false?

6.11.96.1 Solution. False. Counterexample The function

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

is discontinuous at every point in its domain, but both the square and the absolute value

$$f^2(x) = |f(x)|^2 = 1$$

are continuous. $\square$

6.11.97 Problem. A function cannot be continuous at only one point in its domain and discontinuous everywhere else. True or false?

6.11.97.1 Solution. False. Counterexample: The function

$$f(x) = \begin{cases} x; & \text{if } x \text{ is rational} \\ -x; & \text{if } x \text{ is irrational} \end{cases}$$

is continuous at the point $x = 0$ and discontinuous at all other points on $\mathbb{R}$. It is impossible to draw the graph of the function $y = f(x)$. □

6.11.98 Problem. Let $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \lim_{t\to\infty} \frac{(1 + \sin \pi x)^t - 1}{(1 + \sin \pi x)^t + 1}.$$

Show that f is discontinuous at the points 0, 1, 2,...,n,..... □

6.11.98.1 Solution. For $x = n$, we have

$$f(n) = \lim_{t\to\infty} \frac{(1 + 0)^t - 1}{(1 + 0)^t + 1} = 0.$$

For $x = x_0$, where $2n < x_0 < 2n + 1$, we have $0 < \sin \pi x_0 < 1$ and hence

$$f(x_0) = \lim_{t\to\infty} \frac{(1 + \sin \pi x_0)^t - 1}{(1 + \sin \pi x_0)^t + 1} = \lim_{t\to\infty} \frac{1 - \frac{1}{(1+\sin \pi x_0)^t}}{1 + \frac{1}{(1+\sin \pi x_0)^t}} = 1.$$

On the other hand for $x = x_0$, where $2n + 1 < x_0 < 2n + 2$, we have $-1 \le \sin \pi x_0 < 0$ and hence

$$f(x_0) = \lim_{t\to\infty} \frac{(1 + \sin \pi x_0)^t - 1}{(1 + \sin \pi x_0)^t + 1} = -1.$$

Thus f has discontinuities of the first kind at the points 0, 1, 2,...,n,..... and the result follows.

6.11.99 Problem. Let $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \lim_{n\to\infty} \psi_n(x) \text{ where}$$
$$\psi_n(x) = \lim_{t\to\infty} \frac{\sin^2 n!\pi x}{\sin^2 n!\pi x + t^2}.$$

Show that f is totally discontinuous.[1]

6.11.99.1 Solution. Let us first consider $f(x)$ for rational values of $x = p/q$ where $p, q \in \mathbb{Z} \setminus \{0\}$ and prime to each other. By taking n sufficiently large $n!\pi x$ is an integral multiple of π for any

[1]First discussed by Hankel, who was the first to classify functions as continuous, point-wise discontinuous and totally discontinuous and to discuss the characteristic properties of discontinuous functions. See *Math. Ann.* Vol. 20 (1882), page 89.

given value of x. We need at most to put $n = q$. For any such value of x, we have $\sin n!\pi x = 0$ and hence

$$f(x) = \lim_{n\to\infty} \lim_{t\to 0} \frac{\sin^2 n!\pi x}{\sin^2 n!\pi x + t^2} = \lim_{n\to\infty} \lim_{t\to 0} \frac{0}{0 + t^2} = 0.$$

If x is irrational, we have $x \neq p/q$ and hence for all values of n however large we have $\sin^2 n!\pi x > 0$, hence

$$f(x) = \lim_{n\to\infty} \lim_{t\to 0} \frac{\sin^2 n!\pi x}{\sin^2 n!\pi x + t^2} = \lim_{n\to\infty} \lim_{t\to 0} \frac{1}{1 + \frac{t^2}{\sin^2 n!\pi x}} = 1.$$

Thus

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational.} \end{cases}$$

f is **totally discontinuous**, that is, it is discontinuous for every value of x. □

A discontinuous function is said to be **point-wise discontinuous** in a given interval if the points of continuity are everywhere dense but do not form a closed set.

6.11.100 Problem. A sequence of continuous functions on $[a, b]$ always converges to a continuous function on $[a, b]$. True or false?

6.11.100.1 Solution. False. Counterexample: The sequence of continuous functions

$$f_n(x) = x^n, \text{ for } n \in \mathbb{N}$$

on $[a, b]$ converges to a discontinuous function when $n \to \infty$

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 0 & \text{if } x \in [0, 1) \\ 1 & \text{if } x = 1. \end{cases} \quad \square$$

6.11.101 Problem. A monotone function $f : [a, b] \to \mathbb{R}$ has only countably many discontinuities. Conversely, if $S \subseteq [a, b]$ is countable, then there is a monotone $f : [a, b] \to \mathbb{R}$ such that

$$S = \{x \in [a, b]; f \text{ is discontinuous at } x\}.$$

6.11.101.1 Solution. Let f be increasing. Thus for every $x, f(x+)$ and $f(x-)$ exist. Let $\omega(x) = f(x+) - f(x-)$ (the 'jump' of f at x). Then f is discontinuous at x if and only if $\omega(x)$ is positive. For $n \in \mathbb{N}$, let $S_n = \{x; \omega(x) \geq 1/n\}$. If $p_1, .., p_k$ are k distinct points of S_n, then the sum of the 'jumps' in $p_1, ..., p_k$ is at most $f(b) - f(a)$. Thus surely $k/n \leq f(b) - f(a)$ i.e. $k \leq n(f(b) - f(a))$. So S_n consists of only finitely many points. Consequently, $\bigcup_{n\in\mathbb{N}} S_n$ the set of points of discontinuity of f is countable.
Conversely, let S be a countable set in $[a, b]$ with an enumeration $(x_1, x_2, ...)$. Define functions $f_1, f_2, ...f_n, ..$ as follows:

$$f_n(x) = \begin{cases} -n^{-2}, & \text{if } x < x_n \\ 0, & \text{if } x = x_n \\ n^{-2}, & \text{if } x > x_n \end{cases}$$

and let $f = \sum_{n=1}^{\infty} f_n$. (Since $|f_n(x)| \leq n^{-2}$ for all x, the series converges uniformly.) Each f_n, is increasing, hence so is f. Let $x \notin S$. Then each function $f_1 + f_2 + .. + f_k$ is continuous at x and so is the uniform limit, f. Now let $x \in S$. Then $x = x_i$ for some i and $f = f_i + \sum_{j \neq i} f_j$. By the same argument, $\sum_{j \neq i} f_j$ is continuous at x. But f is not. Then f is not continuous at x. □

6.11.102 Problem. Let

$$A = \left\{ \frac{1}{i}; i \in \mathbb{N} \cup \{0\} \right\}.$$

Show that A is a G_δ set in $\mathbb{R}$. Construct a real-valued function on $\mathbb{R}$ that is continuous exactly at the points of A.

6.11.102.1 Solution. Choose $\epsilon_i > 0$ such that $\epsilon_i < \frac{1}{i}$ for each $i \in \mathbb{N}$, and that the intervals $\left(\frac{1}{i} - \epsilon_i, \frac{1}{i} + \epsilon_i\right)$ are pairwise disjoint. Call the family of these intervals G_1. Form the same with intervals of radius $\epsilon_i/2$, and call the family of these intervals G_2. Continue to form $G_3, G_4, ...$ etc. Then $\bigcap_{i=1}^{\infty} G_i$ is the set A.
The function is, for example,

$$f(x) = \begin{cases} \mathfrak{D}(x) \sin\left(\frac{\pi}{x}\right), & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases}$$

where $\mathfrak{D}(x)$ is the Dirichlet function. Show that $x\mathfrak{D}(x)(x)$ does not have one-sided derivative at 0. Dirichlet function is the characteristic function of the set of irrationals. □

6.11.103 Problem. Show that the function defined on $\mathbb{R}$ by

$$f(x) = \begin{cases} x\mathfrak{D}(x) \sin\left(\frac{\pi}{x}\right), & \text{for } x \neq 0 \\ 0, & \text{for } x = 0. \end{cases}$$

where $\mathfrak{D}$ is the Dirichlet function and is continuous exactly at the points in the set

$$\left\{ \frac{1}{i}; i \in \mathbb{N} \cup \{0\} \right\}.$$

6.11.103.1 Solution. Follow the previous problem. □

6.11.104 Problem. Give an example showing that the condition "to be open" for the sets G_n in the statement of the Baire Category Theorem cannot be relaxed.

6.11.104.1 Solution. Consider, for each natural number N, the set

$$Q_N = \left\{ \frac{p}{q} \in [0,1]; p \in \mathbb{Z}, q \in \mathbb{N} \text{ and } q \geq N \right\};$$

where each rational number is represented by its irreducible fraction (i.e., numerator and denominator have no common factors). Observe that Q_N is dense in [0,1]. However,

$$\bigcap_{N=1}^{\infty} Q_N = \emptyset. \quad \square$$

6.11.105 Problem. Prove that if $f : \mathbb{R} \to \mathbb{R}$, then the set of points of continuity is Borel.

6.11.105.1 Solution. For every positive integer n let

$$\mathcal{I}_n = \left\{ I \subseteq \mathbb{R}; I \text{ is an open interval and } \sup_I f - \inf_I f < \frac{1}{n} \right\}$$

and let

$$A_n = \bigcup \mathcal{I}_n = \bigcup_{I \in \mathcal{I}_n} I$$

By Cauchy's criterion, any $a \in \mathbb{R}$ is a point of continuity of f if and only if

$$\forall\, n \in \mathbb{N}\ \exists\, I \in \mathcal{I}_n, a \in I,$$

or equivalently

$$n \in \mathbb{N},\ a \in A_n.$$

Therefore, the set of points of continuity is $\bigcap_{n\in\mathbb{N}} A_n$, that is in G_δ. □

6.11.106 Problem. Give an example of a function f which maps a Cauchy sequence to a Cauchy sequence, but f is not continuous

6.11.106.1 Solution. Let $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \begin{cases} 1 & \text{if } x \leq 0 \\ x & \text{if } x > 0. \end{cases}$ □

6.11.107 Problem. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous periodic function of period 1.

1. Prove that the function is bounded above and below and that moreover, it attains its bounds.

2. Show that there exists x_0 such that $f(x_0 + \pi) = f(x_0)$.

6.11.107.1 Solution.

1. Note that $f(1) = f(0 + 1) = f(0)$ and $f(2) = f(1 + 1) = f(1)$. Thus by periodicity, bounds of f on $\mathbb{R}$ and f on $[0, 1]$ are same. Again, as f is continuous on $[0, 1]$, hence it is bounded and attains its bounds.

2. Let $M = \sup\{f(x); x \in \mathbb{R}\}$ and $m = \inf\{f(x); x \in \mathbb{R}\}$, then $\exists\ a, b \in \mathbb{R}$ such that $f(a) = M$ and $f(b) = m$. Consider the function ϕ defined by $\phi(x) = f(x + \pi) - f(x)$. So, we see that $\phi(a) = f(a + \pi) - f(a) < 0$ and $\phi(b) = f(b + \pi) - f(b) > 0$, hence by IVP, $\exists\ x_0$ lying between a and b such that $\phi(x_0) = 0 \Rightarrow f(x_0 + \pi) = f(x_0)$. □

6.11.108 Problem.

1. Let $f : [0, 1] \to \mathbb{R}$ be a function with the following property: For every real number y, either there is no x in $[0, 1]$ for which $f(x) = y$ or that takes on each of its values exactly twice. Show that f cannot be continuous at every point.

2. Construct a function f which has the above property.

6.11.108.1 Solution.

1. Suppose that f is continuous [0,1], so it will attain maximum and minimum values, and since it takes on each of its values exactly twice, hence $\exists\, c_1, c_2, d_1, d_2$ all distinct. Without loss of generality, consider $c_1 < d_1 < c_2 < d_2$ and

$$f(c_1) = f(c_2) = \max_{x\in[0,1]} f(x),\ f(d_1) = f(d_2) = \min_{x\in[0,1]} f(x),$$

$$\text{hence } f(d_1) < \frac{f(c_1)+f(d_1)}{2} < f(c_1),\ f(d_1) < \frac{f(c_2)+f(d_1)}{2} < f(c_2),$$

$$f(d_2) < \frac{f(c_1)+f(d_2)}{2} < f(c_1),\ f(d_2) < \frac{f(c_2)+f(d_2)}{2} < f(c_2),$$

$$\text{and } \frac{f(c_1)+f(d_1)}{2} = \frac{f(c_2)+f(d_1)}{2} = \frac{f(c_1)+f(d_2)}{2} = \frac{f(c_2)+f(d_2)}{2}.$$

Hence by IVT $\exists\, m_1 \in [c_1, d_1]$, $\exists\, m_2 \in [d_1, c_2]$, and $\exists\, m_3 \in [c_2, d_1]$ such that

$$f(m_1) = \frac{f(c_1)+f(d_1)}{2} = f(m_2) = \frac{f(c_2)+f(d_1)}{2} = f(m_3) = \frac{f(c_2)+f(d_1)}{2}$$

then this implies f takes values at least thrice, a contradiction. Thus f cannot be continuous at every point.

2. Consider $[0,1] = \big(\mathbb{Q}^C \cap [0,1]\big) \cup (\mathbb{Q} \cap [0,1])$ and write $\mathbb{Q} \cap [0,1] = \{x_1, x_2, ..., x_n ...\}$. Define

$$f(x) = \begin{cases} x & \text{if } x \in \left(0, \frac{1}{2}\right) \cap \mathbb{Q}^C \\ 1-x & \text{if } x \in \left(\frac{1}{2}, 1\right) \cap \mathbb{Q}^C \end{cases}$$
$$\text{and } f(x_{2n-1}) = f(x_{2n}) = n.$$

Then if $x = y$, then it is clear that $f(x) = f(y)$. That is, f is well-defined. And from construction, we can observe that the function defined on [0,1] with the property in the problem. Now, we show that the set $D(f)$ of discontinuities of f is $[0,1]$. Given $a \in [0,1]$. Note that since $f(x) \in \mathbb{N}$ for all $x \in \mathbb{Q} \cap [0,1]$, and $\mathbb{Q}$ is dense in $\mathbb{R}$, for any ball $B(a;r) \cap [0,1] \cap \mathbb{Q}$ there is always a rational number $y \in B(a;r) \cap [0,1] \cap \mathbb{Q}$ such that $|f(y) - f(a)| \geq 1$. □

6.11.109 Problem. A function $f : \mathbb{R} \to \mathbb{R}$ is such that for every y in the range set of f, there are exactly three preimages of y in $\mathbb{R}$. Does there exist any such continuous function satisfying above property?

6.11.109.1 Solution. Yes, there is a continuous function defined as follows:
Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \cdots\cdots \\ x, & \text{if } -2 \leq x \leq 0 \\ -x, & \text{if } 0 \leq x \leq 1 \\ x-2, & \text{if } 1 \leq x \leq 3 \\ -x+4, & \text{if } 3 \leq x \leq 4 \\ x-4, & \text{if } 4 \leq x \leq 5 \\ \cdots\cdots\cdots \end{cases}$$

Sketch the graph of $f(x)$. □

6.11.110 Problem. Suppose that $f : (0, \infty) \to [a, b]$ is continuous and for any real y, either there is no $x \in (0, \infty)$ for which $f(x) = y$ or there are finitely many x in $(0, \infty)$ for which $f(x) = y$. Prove that $\lim_{x\to-\infty} f(x)$ exists.

6.11.110.1 Solution. We partition into n subintervals. Then, by continuity and the given property, as x is large enough, $f(x)$ is lying in one and only one subinterval. Given $\epsilon > 0$, there exists N such that $2/N < \epsilon$. For this N, we partition $[a, b]$ into N subintervals, then there is an $M > 0$ such that as $x, y \geq M$

$$|f(x) - (y)| \leq 2/N < \epsilon.$$

So, $\lim_{x\to\infty} f(x)$ exists. □

6.11.111 Problem. If $f : [0, \infty) \to \mathbb{R}$ is continuous, and $\lim_{x\to\infty} f(x)$ exists, show that f is bounded on $[0, \infty)$.

6.11.111.1 Solution. Since $\lim_{x\to\infty} f(x)$ exists, let $\lim_{x\to\infty} f(x) = L$, then $\forall\ \epsilon > 0, \exists\ M > 0$ such that $x > M \Rightarrow L - \epsilon < f(x) < L + \epsilon$. Thus f is bounded on (M, ∞). Since f is continuous on $[0, M]$ so, $\exists\ M_1 > 0$ such that $f(x) \leq M_1\ \forall\ x \in [0, M]$. Let $m = \max\{M_1, L + \epsilon\}$, Hence $|f(x)| \leq m\ \forall\ x \in [0, \infty)$. □

6.11.112 Problem. Justify the definition of the limit that can be reformulated as follows:
$\lim_{x\to a} f(x) = A$ if $\forall\ \delta > 0\ \exists\ \epsilon > 0$ such that $|f(x) - A| < \epsilon \Rightarrow |x - a| < \delta$.

6.11.112.1 Solution. However, it is not sufficient to guarantee the existence of the general limit. For example, if $f(x) = \begin{cases} x^2, & \text{if } x \in \mathbb{Q} \\ 0,, & \text{if } x \in \mathbb{Q}^C \end{cases}$, $a = 0$ and $A = 0$, then by choosing $\epsilon = \delta^2$ for any $\delta < 1$ one guarantees that for all $f(x)$ such that $|f(x)| = |x|^2 < \epsilon$, we obtain $|x| < \delta$ for the corresponding values of x. However, the general limit does not exist, because there are two different partial limits: $\lim_{x\to0, x\in\mathbb{Q}} f(x) = 0$ and $\lim_{x\to0, x\in\mathbb{Q}^C} f(x) = 1$. □

6.11.113 Problem. Justify the definition of the limit that can be reformulated as follows:
$\lim_{x\to a} f(x) = A$ if $\forall\ \delta > 0\ \exists\ \epsilon > 0$ such that $|x - a| < \delta \Rightarrow |f(x) - A| < \epsilon$.

6.11.113.1 Solution. For example, Dirichlet's function

$$\mathfrak{D}(x)(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \in \mathbb{Q}^C, \end{cases}$$

which actually does not have a limit at any point, satisfies the above "definition" with an arbitrary a and $A = 0$ if one chooses $\epsilon = 2$ for every $\delta > 0$. □

6.11.2 Remark. The condition of the statement implies boundedness of $f(x)$ in any deleted neighborhood of the point a.

6.11.114 Problem. Assume that the function $f : (a, b) \to \mathbb{R}$ has the following property at a point $c \in (a, b)$.

1. $\forall\ \epsilon > 0\ \exists\ \delta > 0$ such that $\forall\ x \in (a, b)\ |f(x) - f(c)| < \epsilon \Rightarrow |x - c| < \delta$.

2. $\forall\ \delta > 0\ \exists\ \epsilon > 0$ such that $\forall\ x \in (a, b)\ |f(x) - f(c)| < \epsilon \Rightarrow |x - c| < \delta$.

3. $\forall\, \delta > 0\ \exists\, \epsilon > 0$ such that $\forall\, x \in (a, b)\ |x - c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon$.

What can one say about the continuity of f at c?

6.11.114.1 Solution. We shall find three functions that will satisfy the stated implications in 1), 2) and 3) respectively, but neither of them will turn out to be continuous at the point c. This means that the order of the quantifiers as well as the direction of the implication in definition () of continuity are essential.

1. Let us define the function f by

$$f(x) = \begin{cases} x, & \text{if } x \leq 1 \\ x + 1, & \text{if } x > 1. \end{cases}$$

Clearly, this function is continuous on the set $\mathbb{R} \setminus \{1\}$, while it has a first order discontinuity at the point $c = 1$. Let us show, however, that f does have the stated property at the point $c = 1$. To that end, for $\epsilon > 1$ we put $\delta = \epsilon - 1$. Then for every $x \in \mathbb{R}$, we get

$$|f(x) - f(1)| < \epsilon \Rightarrow |x - 1| < \delta.$$

For $\epsilon \leq 1$, we put simply $\delta = \epsilon$. Then the set of points $x > 1$ that satisfy the inequality $|f(x) - f(1)| < \epsilon$ is empty, hence the implication above is true. For $x < 1$, we get

$$|f(x) - f(1)| < \epsilon \Leftrightarrow |x - 1| < \epsilon.$$

.

2. Let us define the function f by

$$f(x) = \begin{cases} x^2, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases}$$

has a removable discontinuity at $c = 0$, while it is continuous on the set $\mathbb{R} \setminus \{0\}$. Let us show that it satisfies the stated condition at the point $c = 0$. For given $\delta > 0$ let us choose ϵ such that $1 < \epsilon < 1 + \delta^2$ (say $\epsilon = 1 + \delta^2/2$). Then for every $x \in \mathbb{R} \setminus \{0\}$, we get

$$\begin{aligned} &|f(x) - f(0)| = |x^2 - 1| < x^2 + 1 < \epsilon \\ \Rightarrow &|x - 0| = \sqrt{|x|^2 + 1 - 1} < \sqrt{\epsilon - 1} < \delta. \end{aligned}$$

3. Let us define the function f by

$$f(x) = \begin{cases} x^2 - 4x + 5, & \text{if } x > 2 \\ 0, & \text{if } x = 2 \\ -x^2 + 4x - 5, & \text{if } x < 2. \end{cases}$$

This function has a first order discontinuity at the point $c = 2$, and is continuous on the set $\mathbb{R} \setminus \{2\}$. Let us show that it satisfies the stated condition at the point $c = 2$. For given $\delta > 0$ let us choose ϵ such that $\epsilon = 1 + \delta^2$. Then for every $x \in \mathbb{R}$, we get

$$\begin{aligned} &|x - 2| < \delta \\ \Rightarrow &|f(x) - f(2)| = 1 + (x - 2)^2 < 1 + \delta^2 < \epsilon. \quad \square \end{aligned}$$

6.11.115 Problem. Let $f : \mathbb{R} \to \mathbb{R}$ be bounded, and in every closed interval it attains its supremum and infimum there. Is it continuous on $\mathbb{R}$? Prove your assertion.

6.11.115.1 Solution. Consider the function $f(x) = \begin{cases} 0, & \text{if } x \in \mathbb{Q} \\ 1, & \text{if } x \in \mathbb{Q}^C. \end{cases}$ □

6.11.116 Problem. Let $f : [0,1] \to \mathbb{R}$ be continuous, and $f(0) = f(1)$. Prove that for each $n \in \mathbb{N} \; \exists \; x \in \left[0, \frac{1}{n}\right]$ such that $f(x) = f\left(x + \frac{1}{n}\right)$.

6.11.116.1 Solution. This problem is a part of the "Universal Chord Theorem". Let $n \geq 1$, and $g : \left[0, 1 - \frac{1}{n}\right] \to \mathbb{R}$ be a function defined by $g(x) = f\left(x + \frac{1}{n}\right) - f(x)$. If $g(x) \neq 0 \; \forall \; x\imath \left[0, 1 - \frac{1}{n}\right]$ then either $g(x) > 0$ or $g(x) < 0$. Hence the sum $\sum_{r=0}^{n-1} g(\frac{r}{n})$ is either positive or negative, but the sum

$$\sum_{r=0}^{n-1} g\left(\frac{r}{n}\right) = f\left(\frac{1}{n}\right) - f(0) + f\left(\frac{2}{n}\right) - f\left(\frac{1}{n}\right) + .. + f(1) - f\left(\frac{n-1}{n}\right) = 0,$$

gives a contradiction. Hence $\exists \; c \in \left[0, \frac{1}{n}\right]$ such that $f(c) = f\left(c + \frac{1}{n}\right)$. □

6.11.117 Problem. Prove that, if f is an increasing continuous function from $[a,b] \to [a,b]$ such that $f(a) = a$, and if $E = \{x \in [a,b]; f(x) \geq x\}$, then $f(E) = E$.

6.11.117.1 Solution. If $x \in E$, then $f(x) \geq x \Rightarrow f(f(x)) \geq f(x)$, as f is increasing. Hence $f(x) \in E \Rightarrow f(E) \subseteq E$. Now we show that $E \subseteq f(E)$. Let $y \in E$, then $f(y) \geq y$. If $f(y) = y$, then $y \in f(E)$. If $f(y) > y$, then $f(a) = a < y < f(y)$, so $\exists x \in [a,y]$ such that $f(x) = y$, (by Intermediate Value Theorem). Then $f(x) = y \geq x \Rightarrow x \in E$ and $y \in f(E)$. Thus $E \subseteq f(E)$. Hence $f(E) = E$. □

6.11.118 Problem. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous, such that $\lim_{x\to-\infty} f(x) = 0$ and $\lim_{x\to\infty} f(x) = 0$. Prove that f is bounded on $\mathbb{R}$ and attains either a maximum or a minimum or both on $\mathbb{R}$.

6.11.118.1 Solution. Let $\epsilon > 0$ then $\exists \; M_1, M_2 > 0$ such that $|f(x)| < \epsilon \; \forall \; x < -M_1$ and $|f(x)| < \epsilon \; \forall \; x > M_2$, Since f is continuous on $[-M_1, M_2]$, f is bounded on $[-M_1, M_2]$. Thus f is bounded on $\mathbb{R}$. □

6.11.3 Note. If $f(x) > 0 \; \forall x \in \mathbb{R}$, then f does not attain a minimum. If $f(x) < 0 \; \forall x \in \mathbb{R}$, then f does not attain a maximum. If $\exists \; x \in \mathbb{R}$ such that $f(x) = 0$, then f attains both maximum and minimum.

6.11.4 Example.

$$f(x) = \begin{cases} 1/x, & \text{if } |x| \geq 1 \\ x, & \text{if} -1 < x < 1 \end{cases}$$

6.11.119 Problem. Give an example of a bounded continuous function $f : \mathbb{R} \to \mathbb{R}$ with both $\sup\{f(x); x \in \mathbb{R}\}$ and $\inf\{f(x); x \in \mathbb{R}\} \notin \text{Range} f$.

6.11.119.1 Solution. $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \tan^{-1} x$ and $f(\mathbb{R}) = (-\pi/2, \pi/2)$ but $\sup_{x\in\mathbb{R}} f(x) = \pi/2$, $\inf_{x\in\mathbb{R}} f(x) = -\pi/2 \notin (-\pi/2, \pi/2)$. □

6.11.120 Problem. Give an example of a continuous bijective function $f : \mathbb{R} \to \mathbb{R}$ such that f^{-1} is not continuous.

6.11.120.1 Solution. Define $f : [0,1) \cup [2,3] \to [0,2]$ by

$$f(x) = \begin{cases} x, & \text{if } x \in [0,1) \\ 4-x, & \text{if } x \in [2,3]. \end{cases}$$

Then f is continuous and bijective, but

$$f^{-1}(x) = \begin{cases} x, & \text{if } x \in [0,1) \\ 4-x, & \text{if } x \in [1,2] \end{cases}$$

is not continuous. □

6.11.121 Problem. Let $a, b \in \left(0, \frac{1}{2}\right)$ and let g be continuous real valued function such that $g(g(x)) = ag(x) + bx \ \forall \ x \in \mathbb{R}$. Prove that $g(x) = cx$ for some $c \in \mathbb{R}$.

6.11.121.1 Solution. Note that $g(x) = g(y)$ implies

$$\begin{aligned} g(g(x)) = g(g(y)) &\Rightarrow ag(x) + bx = ag(y) + by \\ &\Rightarrow x = y \end{aligned}$$

by the given equation. That is g is injective. Since g is continuous, g is either strictly increasing or strictly decreasing. Moreover, g cannot tend to a finite limit L as $x \to \infty$ or else we would have $g(g(x)) - ag(x) = bx$, with the left hand side bounded and the right side is unbounded. Similarly g cannot tend to a finite limit as $x \to -\infty$. Together with monotonocity, this gives g is also surjective. Pick x_0 arbitrarily, and define $x_n \ \forall \ n \in \mathbb{Z}$ recursively by

$$\begin{aligned} x_{n+1} &= g(x_n), \ \text{if } n > 0 \\ x_{n-1} &= g^{-1}(x_n), \ \text{if } n < 0 \end{aligned}$$

Now, $g(g(x_n)) = ag(x_n) + bx_n \Rightarrow x_{n+2} = ax_{n+1} + bx_n$, if the α, β be the solutions of this equation so that $\alpha > 0 > \beta$ and $1 > |\alpha| > |\beta|$ then $x_n = c_1\alpha^n + c_2\beta^n \ \forall \ n \in \mathbb{Z}$. Suppose that g is strictly increasing. If $c_2 \neq 0$ for some choice of x_0, then x_n is dominated by β^n for sufficiently negative values of n. But taking x_n and x_{n+2} sufficiently negative of the right parity, we get $0 < x_n < x_{n+2}$ but $g(x_n) > g(x_{n+2})$ a contradiction. Thus $c_2 = 0$. Since $x_0 = c_1$ and $x_1 = c_1\alpha$, we have $g(x) = \alpha x \ \forall \ x$. Analogously, if g is strictly decreasing, then $c_2 = 0$ or else x_n is dominated by α^n for n sufficiently positive. But taking x_n and x_{n+2} sufficiently positive of the right parity, we get $0 < x_{n+2} < x_n$ but $g(x_{n+2}) < g(x_n)$, a contradiction. Thus in that case, $g(x) = \beta x \ \forall \ x \in \mathbb{R}$. □

6.11.122 Problem.

1. Prove that, if $f : \mathbb{R} \to \mathbb{R}$ is a continuous injection then it is either strictly decreasing or strictly increasing.
2. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous injection. Prove that if there exists n such that n-th iteration of f is identity, i.e. $f^n(x) = x, \ \forall \ x \in \mathbb{R}$, then
   (a) $f(x) = x, \ \forall \ x \in \mathbb{R}$, if f is strictly increasing,
   (b) $f^2(x) = x, \ \forall \ x \in \mathbb{R}$, if f is strictly decreasing.
3. Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f^2(x) = -x \ \forall \ x \in \mathbb{R}$, then f can never be a continuous function.

6.11.122.1 Solution.

1. Suppose that there are $a, b, c \in \mathbb{R}$ and such that $a < b < c$, and $f(a) > f(b); f(b) < f(c)$. By the intermediate value property, for every p such that $f(b) < p < \min\{f(a), f(c)\}$ there are $q \in (a, b)$ and $r \in (b, c)$ satisfying $f(q) = p = f(r)$. Since f is injective, $q = r$, contradicting the fact that $a < q < b < r < c$.

2. It follows from the result in the previous problem that f is either strictly decreasing or strictly increasing.

   (a) Let f be strictly increasing and there is $x \in \mathbb{R}$ such that $f(x) \neq x$ Let, $f(x) > x$. Then $f^2(x) > f(x) > x$ and ultimately $x = f^n(x) > f^{n-1}(x) > ... > f(x) > x$ implies $x > x$ contrary to our assumption. Similar arguments apply to the case $f(x) < x$.

   (b) If f is strictly decreasing, then $x > y$ implies $f(x) < f(y)$ and $f^2(x) > f^2(y)$ which shows that f^2 is strictly increasing. Since $f^n(x) = x$, we get $f^{2n}(x) = x$, which means that the n-th iteration of f^2 is the identity. Therefore, by (a), $f^2(x) = x$.

3. We show that f is injective. Indeed, if $f(x) = f(y)$, then $-x = f^2(x) = f^2(y) = -y$. Hence $x = y$. It follows from (1) that if f were continuous, then it would be either strictly increasing or strictly decreasing. If f is increasing, then

$$x > y \Rightarrow f(x) > f(y) \Rightarrow f^2(x) > f^2(y) \Rightarrow -x > -y \Rightarrow y > x,$$

   a contradiction. And if f is decreasing, then

$$x > y \Rightarrow f(x) < f(y) \Rightarrow f^2(x) > f^2(y) \Rightarrow -x > -y \Rightarrow y > x,$$

   a contradiction. Hence f is not continuous. □

6.11.123 Problem. Let $f : [a, b] \to \mathbb{R}$ is bounded above and $g : [a, b] \to \mathbb{R}$ is defined by

$$g(x) = \begin{cases} f(a) & \text{if } x = a \\ \sup\{f(t); t \in [a, x]\} & \text{if } x \in [a, b]. \end{cases}$$

Prove that

1. Show that g is a well-defined function.
2. g is increasing,
3. if f is increasing, then $g = f$,
4. if f is continuous, then g is continuous.
5. If $f : [-1, 1] \to \mathbb{R}$ is given by $f(x) = x - x^2$, then find g.

6.11.123.1 Solution.

1. Suppose that $x = y$, then

$$\begin{aligned} &[a, x] = [a, y] \\ \Rightarrow &\{f(t); t \in [a, x]\} = \{f(t); t \in [a, y]\} \\ \Rightarrow &\sup\{f(t); t \in [a, x]\} = \sup\{f(t); t \in [a, y]\} \\ \Rightarrow &g(x) = g(y). \end{aligned}$$

   Thus g is well-defined.

2. Let $x \geq y$ then $[a, y] \subseteq [a, x]$ implies

$$\begin{aligned}
&\{f(t); t \in [a, x]\} \supseteq \{f(t); t \in [a, y]\} \\
\Rightarrow &\sup\{f(t); t \in [a, x]\} \geq \sup\{f(t); t \in [a, y]\} \\
\Rightarrow &g(x) \geq g(y).
\end{aligned}$$

   Thus g is increasing.

3. As f is increasing, $g(x) = \sup\{f(t); t \in [a, x]\} = f(x)$, so $g(x) = f(x)\ \forall\, x \in [a, b]$. Thus $g = f$.

4. To prove that g is continuous, let $c \in [a, b]$, then $f(c) \leq g(c)$. Then $\exists\ \delta > 0$ such that $\forall\ t \in (c - \delta, c + \delta)$, $g(t) = g(c)$. Let (x_n) be a sequence such that $x_n \to c$, then $\exists\, m \in \mathbb{N}$ such that $n > m$ implies $x_n \in (c - \delta, c + \delta)$, so, $g(x_n)(= g(c)) \to g(c)$. Thus g is continuous.

5. We see that $f(x) = x - x^2 = 1/4 - (1/2 - x)^2$ i.e. $\sup\{f(x); x \in [-1, 1]\} = 1/4$ which occurs at $x = 1/2$, and $g(x)$ is increasing in $[-1, 1/2]$ and decreasing in $[1/2, 1]$, hence

$$g(x) = \begin{cases} x - x^2 & \text{if } x \in [-1, 1/2] \\ 1/4 & \text{if } x \in [1/2, 1]. \end{cases} \qquad \square$$

6.11.124 Problem. If g is continuous at L and $f(x) \to L$ as $x \to a$, prove $g(f(x)) \to g(L)$ as $x \to a$.

6.11.124.1 Solution. Let $\epsilon > 0$, then $\exists\ \delta_0 > 0$ such that

$$g\left(N(L; \delta_0)\right) \subseteq N\left(g(L); \epsilon\right),$$

since $f(x) \to L$ so, $\exists\ \delta > 0$ such that

$$f(N(a; \delta)) \subseteq N(L; \delta_0).$$

Then

$$\begin{aligned}
&g\left(f(N(a; \delta))\right) \subseteq g\left(N(L; \delta_0)\right) \subseteq N\left(g(L); \epsilon\right) \\
\Rightarrow &(g \circ f)\left(N(a; \delta)\right) \subseteq g\left(N(L; \delta_0)\right) \subseteq N\left(g(L); \epsilon\right) \\
\Rightarrow &(g \circ f)(x) \to g(L) \text{ as } x \to a. \quad \square
\end{aligned}$$

6.11.125 Problem. Prove or provide a counterexample of each of the following:

1. A function $f\colon (a, b) \to \mathbb{R}$ that is differentiable on (a, b) but not uniformly continuous on (a, b).
2. A function $f\colon (a, b) \to \mathbb{R}$ that is uniformly continuous on (a, b) but not differentiable on (a, b).
3. A continuous function $f\colon \mathbb{R} \to \mathbb{R}$ and a closed subset $G \subseteq \mathbb{R}$ such that $f(G)$ is not closed.
4. A continuous function $f\colon \mathbb{R} \to \mathbb{R}$ and a open subset $H \subseteq \mathbb{R}$ such that $f(H)$ is not open.
5. An injective function $f\colon \mathbb{R} \to \mathbb{R}$ that is continuous on $\mathbb{R}$ and $G \subseteq \mathbb{R}$ is open such that $f(G)$ is not open.

6.11.125.1 Solution.

1. Example: $f(x) = 1/x$.
2. Example: $f(x) = |x|$ in $(-1, 1)$.
3. Example: $f(x) = e^x, G = \mathbb{R} \;\Rightarrow\; f(G) = (0, \infty)$.
4. Example: $f(x) = 0 \;\forall\; x \in \mathbb{R}, H = \mathbb{R} \Rightarrow\; f(H) = \{0\}$.
5. Ans: No such example exists. □

6.11.126 Problem. Give an example of a non-constant continuous function $f : \mathbb{R} \to \mathbb{R}$ and an open set G such that $f(G)$ is not open.

6.11.126.1 Solution. Define $f : (0, 1) \to [0, 1]$ by

$$f(x) = \begin{cases} 0. & \text{if } x \in \left(0, \frac{1}{3}\right], \\ 3x - 1, & \text{if } x \in \left[\frac{1}{3}, \frac{2}{3}\right], \\ 1, & \text{if } x \in \left[\frac{2}{3}, 1\right). \end{cases} \quad \square$$

6.11.127 Problem. Give an example (with proof) of a function $f : \mathbb{R} \to \mathbb{R}$ which is bounded and continuous but not uniformly continuous on $\mathbb{R}$.

6.11.127.1 Solution. Define $f : (0, 1) \to \mathbb{R}$ by $f(x) = \cos\,(\pi/x)$, we see that f is continuous, now consider the sequences $x_n = 1/n$ and $y_n = 1/(n + 1)$, then $|x_n - y_n| \to 0$ but $|f(x_n) - f(y_n)| = |\cos\, n\pi - \cos\,(n + 1)\pi| = 2 \nrightarrow 0$. □

6.11.128 Problem. Show that given any continuous function $f : \mathbb{R} \to \mathbb{R}$, there exists a $x_0 \in [0, 1]$ and an $m \in \mathbb{Z} \setminus \{0\}$ such that $f(x_0) = mx_0$. In other words, the graph of f intersects some nonhorizontal line $y = mx$ at some point $x_0 \in [0, 1]$.

6.11.128.1 Solution. The following three cases are possible:

1. $f(0) = 0$. Then let $x_0 = 0$ and let $m \in \mathbb{Z} \setminus \{0\}$. Clearly $f(x_0) = f(0) = 0 = m.0 = mx_0$.
2. $f(0) > 0$. Choose $N \in \mathbb{N}$ satisfying $N > f(1)$ (that such an N exists follows from the Archimedean property). Consider $g : [0, 1] \to \mathbb{R}$ defined by $g(x) = f(x) - Nx, x \in [0, 1]$. As f and $x \mapsto Nx$ are continuous, so is g. Note that $g(0) = f(0) - N.0 = f(0) > 0$, while $g(1) = f(1) - N < 0$. Applying the intermediate value theorem to g (with $y = 0$), it follows that there exists a $x_0 \in [0, 1]$ such that $g(x_0) = 0$, that is, $f(x_0) = Nx_0$.
3. $f(0) < 0$. Choose an $N \in \mathbb{N}$ such that $N > -f(1)$ (again the Archimedean property guarantees the existence of such an N), and consider the continuous function $g : [0, 1] \to \mathbb{R}$ defined by $g(x) = f(x) + Nx$. We observe that $g(0) = f(0) < 0$, and $g(1) = f(1) + N > 0$, and so by the intermediate value theorem, it follows that there exists an $x_0 \in [0, 1]$ such that $g(x_0) = 0$, that is, $f(x_0) = -Nx_0$. This completes the proof. □

6.11.129 Problem. Prove that there does not exist a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that assumes rational values at irrational numbers, and irrational values at rational numbers, that is,

$$f(\mathbb{Q}) \subseteq \mathbb{R} \setminus \mathbb{Q} \text{ and } f(\mathbb{R} \setminus \mathbb{Q}) \subseteq \mathbb{Q}.$$

6.11.129.1 Solution. Suppose that such a continuous function exists. From the part above, it follows that there exists a $x_0 \in \mathbb{R}$ and a $m \in \mathbb{Z} \setminus \{0\}$ such that $f(x_0) = mx_0$. We have the following two possible cases:

1. $x_0 \in \mathbb{Q}$. But then $f(x_0)$ is irrational, while mx_0 is rational, a contradiction.

2. $x_0 \notin \mathbb{Q}$. But then $f(x_0)$ is rational, while mx_0 is irrational, a contradiction.

So f cannot be continuous.

6.11.129.2 Solution. A direct proof of the above problem, using cardinality arguments goes as follows: Suppose to the contrary that an everywhere continuous function $f : \mathbb{R} \to \mathbb{R}$ exists with $f(\mathbb{Q}) \subseteq \mathbb{R} \setminus \mathbb{Q}$ and $f(\mathbb{R} \setminus \mathbb{Q}) \subseteq \mathbb{Q}$. It follows that f takes at least two values, one rational and one irrational, say p and q. By the intermediate value theorem, since f is continuous, f assumes all values in $[p, q]$, where without loss of generality, $p < q$, and since $p \neq q$, we get that the set $[p, q]$ is uncountable. On the other hand, the set $f(\mathbb{Q})$ is countable, since $\mathbb{Q}$ is countable. Further, since $f(\mathbb{Q}^C) \subseteq \mathbb{Q}$, $f(\mathbb{Q}^C)$ is countable and consequently $f(\mathbb{R})$ is countable. This contradiction shows that f can not exist.

6.11.129.3 Solution. We observe that $f(\mathbb{R})$ is countable. For $f(\mathbb{R}) = f\left(\mathbb{Q}^C\right) \cup f(\mathbb{Q})$ is countable. But then $f\left(\mathbb{Q}^C\right)$ must be a singleton, since if it contained two distinct points $a < b$, then it would also contain the whole interval $[a, b]$ by the intermediate value theorem, and $f(\mathbb{R})$ would then be uncountable. A similar result for $f(\mathbb{Q})$. Suppose that $f(\mathbb{R}) = \{x'\}$. But $f(0) = x' = f(\sqrt{2})$ gives a contradiction.

6.11.129.4 Solution. Consider the function f which is continuous at each irrational but discontinuous at each rational. Suppose, to the contrary, that a continuous function $\phi : \mathbb{R} \to \mathbb{R}$ exists that maps rationals to the irrationals and the irrationals to the rationals. Let $h(x) = f(\phi(x))$. Then for $x \in \mathbb{Q}$, $\phi(x)$ is irrational and so f is continuous at $\phi(x)$ and ϕ is continuous everywhere by hypothesis; hence we observe that h is continuous at each $x \in \mathbb{Q}$. For an irrational number y, let (x_n) be a sequence of rationals that tends to y. Then $\lim_{n\to\infty} h(x_n) = \lim_{n\to\infty} f(\phi(x_n)) = 0$, since $x_n \in \mathbb{Q}$ implies that $\phi(x_n)$ is irrational and hence $f(\phi(x_n)) = 0$. However, $h(y) = f(\phi(y)) \neq 0$, since y is irrational and so $\phi(y)$ is rational. Hence h is discontinuous at each irrational number, contradicts the above proof, thus completing the proof.

6.11.129.5 Solution. Assume that such a function f exists. Then, $f(\mathbb{R})\left(= f(\mathbb{Q}) \cup f(\mathbb{Q}^C)\right)$ is countable, say $f(\mathbb{R}) = \{x_n; n \in \mathbb{N}\}$. Then $\mathbb{R} = \bigcup_{n=1}^{\infty} f^{-1}(x_n)$, and the Baire Category Theorem implies that for some $n \in \mathbb{N}$, $f^{-1}(x_n)$ contains an interval J. However, J contains both rational and irrational numbers, and so its image cannot be a single number x_n. $\square$

6.11.130 Problem.

1. Does there exist a map from $\mathbb{Q}$ onto $\mathbb{Q}^C$?

2. Does there exist a continuous map from $\mathbb{R}$ onto $\mathbb{Q}^C$?

6.11.130.1 Solution.

1. No (cardinality reasons).

2. No (connectedness).

6.11.131 Problem. Suppose that $f : \mathbb{R} \to \mathbb{R}$ and there exists a $M > 0$ such that for all $x \in \mathbb{R}, |f(x)| \leq M|x|$. Prove that f is continuous at 0.

6.11.131.1 Solution. Hint: Find $f(0)$. □

6.11.132 Problem. Let $\mathcal{A}$ be the set of all sequences whose elements are the digits 0 and 1. Then $\mathcal{A}$ is uncountable.

6.11.132.1 Solution. Let $\mathcal{B}$ be a countable subset of $\mathcal{A}$, and let $\mathcal{A}$ consists of the sequences $s_1, s_2, ...$ We construct a sequence s as follows. If the n-th digit in s_n is 1 we let the n-th digit of s be 0, and vice versa. Then the sequence s differs from every member of $\mathcal{B}$ in at least one place; hence $s \notin \mathcal{A}$. But clearly $s \in \mathcal{A}$, so that $\mathcal{B}$ is a proper subset of $\mathcal{A}$. We have shown that every countable subset of $\mathcal{A}$ is a proper subset of $\mathcal{A}$. It follows that $\mathcal{A}$ is uncountable (for otherwise $\mathcal{A}$ would be a proper subset of $\mathcal{A}$, which is absurd).

6.11.132.2 Solution. Here, $\mathcal{A} = 2^{\mathbb{N}}$. Then show that $|\mathcal{P}(\mathbb{N})| = 2^{|\mathbb{N}|} > |\mathbb{N}|$. □

6.11.133 Problem. Construct a real-valued function on [0,1] that has a limit at each point but is not continuous at infinite countably many points.

6.11.133.1 Solution. Let

$$f(x) = \begin{cases} \frac{1}{n}, & \text{if } x = 1 - \frac{1}{n+1} \\ 0, & \text{otherwise.} \end{cases} \quad \square$$

6.11.5 Example. The function

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{if } x \in \mathbb{Q}^c \end{cases}$$

is called the Dirichlet function. It is discontinuous everywhere.

6.11.6 Note. $\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x))$, provided f is continuous.

6.11.134 Problem. Give an example of a function f defined and strictly increasing os a set $S \subseteq \mathbb{R}$, such that f^{-1} is not continuous on $f(S)$.

6.11.134.1 Solution. Let $f(x) = \begin{cases} x, & \text{if } x \in [0,1) \\ 1, & \text{if } x = 2. \end{cases}$ f strictly increasing os $[0,1]$.

And $f^{-1}(x) = \begin{cases} x, & \text{if } x \in [0,1) \\ 2, & \text{if } x = 1. \end{cases}$ is not continuous on $f(S) = [0,1]$. □

6.11.135 Problem. Let f be strictly increasing on a subset S of $\mathbb{R}$. Assume that the image $f(S)$ has the one of the following properties:

1. $f(S)$ is open.
2. $f(S)$ is connnected.
3. $f(S)$ is closed.

Prove that f must be continuous on S.

6.11.135.1 Solution.

1. Given that f is strictly increasing on $S \subseteq \mathbb{R}$. Since $f(S)$ is an open subset of $\mathbb{R}$, then $f(S)$ is the countable union of disjoint open intervals. Let $f(S) = \cup_{n=1}^{\infty} I_n$ (the open intervals.) We show that f is continuous on S.
   Let $\epsilon > 0$ and $a \in S$. Then $f(a) \in f(S)$. Since $f(S)$ is open $\exists\ 0 < \epsilon' < \epsilon$ such that $B(f(a); \epsilon') \subseteq f(S)$. Choose $y_1 = f(a) - \epsilon'/2$ and $y_2 = f(a) + \epsilon'/2$, then let $y_1 = f(x_1)$ and $y_2 = f(x_2)$ for some $x_1, x_2 \in S$. Since f is increasing, $x_1 < a < x_2$ implies $f(x_1) < f(a) < f(x_2)$, so for $x \in (x_1, x_2)$, $f(x_1) < f(x) < f(x_2)$, thus $f(x) \in B(f(a); \epsilon')$. Let $\delta = \min\{a - x_1, x_2 - a\}$, then $B(a; \delta) \cap S = (a - \delta, a + \delta) \cap S \subseteq (x_1, x_2) \cap S$ which shows that
   $$f(B(a; \delta) \cap S) \subseteq B(f(a); \epsilon') \subseteq B(f(a); \epsilon).$$
   Hence f is continuous.

2. Since $f(S) \subseteq \mathbb{R}$ is connnected, so $f(S)$ is an interval of the type
   $$I = (a, b), [a, b), (a, b] \text{ or}, [a, b].$$
   Here we consider only the two cases
   (a) for $x \in S$, $f(x)$ is an interior point and
   (b) $f(x)$ is the end point of I. For both cases, the proof is similar to (1).

3. Hint: Given $a \in S$ then $f(a) \in f(S)$. Since $f(S)$ is closed, we consider the cases
   (a) $f(a)$ is an isolated point, and
   (b) $f(a)$ is an accumulation point. □

6.11.136 Problem. Let $f\colon [a, b] \to \mathbb{R}$ be continuous, and assume that $f(a) < 0, f(b) > 0$. Let $W = \{x \in [a, b]; f(x) < 0\}$, and let $w = \sup W$. Prove that$f(w) = 0$.

6.11.136.1 Solution. If possible, let $f(w) > 0$, then $\exists\ \delta_1 > 0$ such that $f(x) > 0\ \forall\ x \in N(w; \delta_1)$ which implies $W \cap N(w; \delta_1) = \emptyset$ which means that there is no $x \in W$ such that $x > w - \delta_1$ contradicting that $w = \sup W$. Again, if $f(w) < 0$, then $\exists\ \delta_2 > 0$ such that $f(x) < 0\ \forall\ x \in N(w; \delta_2)$ which implies $\exists\ x \in W$ such that $x > w$, again a contradiction. Thus $f(w) = 0$. □

6.11.137 Problem. Let $f : \mathbb{R} \to \mathbb{R}$, consider the funtions f_i by
$$f_i(x) = \sup\{|f(u) - f(v)|; u, v \in B[x; 1/i]\},$$
where $i \in \mathbb{N}, x \in \mathbb{R}$. Define again,
$$A_{i,n} = \{x \in \mathbb{R}; f_i(x) < 1/n\}, n \in \mathbb{N},$$
and
$$A_n = \bigcup_{i=1}^{\infty} A_{i,n}, n \in \mathbb{N}.$$
Now let
$$C = \{x \in \mathbb{R};\ f \text{ is continuous at } x\}.$$
Express C in terms of A_n.

6.11.137.1 Solution. Left to the reader. □

6.11.138 Problem. Check for uniform continuity for the function $f(x) = \sin(\sin(x^2))$ on $\mathbb{R}$.

6.11.138.1 Solution. Consider the sequences $(x_n), (y_n)$ defined by $x_n = \sqrt{2n\pi}, y_n = \sqrt{2n\pi + \pi/2}$. □

6.11.139 Problem. Let $f : [a, b] \to \mathbb{R}$, be such that for every $x \in [a, b]$ $\exists$ $\delta_x > 0$ such that f is bounded on $N(x, \delta_x)$. Prove that f is bounded on $[a, b]$.

6.11.139.1 Solution.

Let $\mathcal{A} = \{N(x, \delta_x);\ x \in [a, b]$ such that f is bounded on $N(x, \delta_x)\}$. We see that $\mathcal{A}$ is an open cover of $[a, b]$. Since $[a, b]$ is compact, so by Heine-Borel theorem $\exists$ $x_1, x_2, .., x_n$, such that

$$[a, b] \subseteq \bigcup_{i=1}^{n} N(x_i, \delta_{x_i}).$$

Now, let $|f(x)| \le M_i$ for all $x \in N(x_i, \delta_{x_i})$ and let $\max_{1\le i\le n}\{M_i\} = M$, then $|f(x)| \le M$ for all $x \in [a, b]$. Thus f is bounded. □

6.11.139.2 Solution. Suppose that f is not bounded in $[a, b]$. So, for each $n \in \mathbb{N}$ $\exists$ $x_n \in [a, b]$ such that $f(x_n) \ge n$, thus (x_n) is a bounded sequence in $[a, b]$, hence there exists a convergent subsequence (x_{n_k}) of (x_n). Suppose, (x_{n_k}) converges to $c \in [a, b]$, then $\exists$ $\epsilon > 0$ such that f is bounded in $N(c; \epsilon) \subseteq [a, b]$. Suppose that for some $M > 0$, $|f(x)| \le M$ for all $x \in N(c; \epsilon)$. Since (n_k) is increasing $\exists$ $n_p > M$. And $N(c; \epsilon)$ contains a tail $x_{n_q}, x_{n_{q+1}}, x_{n_{q+2}}, ...$ of $(x_{n_k}) \Rightarrow x_{n_k} \in (c - \epsilon, c + \epsilon)$ $\forall$ $k \ge r = \max\{p, q\}$ i.e. $x_{n_r} \in N(c; \epsilon) \Rightarrow f(x_{n_r}) > n_r \ge n_p > M$, a contradiction. □

6.11.140 Problem. Show that $|\log x - \log y| < |x - y|$ $\forall$ $x, y \in [1, \infty)$. Use this inequality to prove that log x is uniformly continuous on $[1, \infty)$. Also show that $\log x$ is not uniformly continuous on $(0, 1]$.

6.11.140.1 Solution. Let$f(x) = \log x$ and assume that $x \ne y$. The MVT implies $\exists$ $\xi \in (x, y)$ such that

$$\frac{f(x) - f(y)}{x - y} = f'(\xi) = \frac{1}{\xi}.$$

When $\xi \ge 1, 0 \le \frac{1}{\xi} \le 1$. So $|f(x) - f(y)| = \frac{1}{\xi}|x - y| \le |x - y|$ $\forall$ $x, y \in [1, \infty)$, with $x \ne y$. And this inequality holds trivially when $x = y$. Hence $|\log x - \log y| < |x - y|$ $\forall$ $x, y \in [1, \infty)$.
Let $\epsilon > 0$ then let $\delta = \epsilon$. So when $x, y \in [1, \infty)$ and $|x-y| < \delta = \epsilon$, we have $|\log x - \log y| < |x-y| < \epsilon$. Hence $\log x$ is uniformly continuous on $[1, \infty)$.
Consider the Cauchy sequence $(\frac{1}{n})$ in $(0, 1]$ but $f(\frac{1}{n}) = -\log n$ is not Cauchy, hence $\log x$ is not uniformly continuous on $(0, 1]$. □

6.11.141 Problem. Give an example of a function on a closed set which is continuous but not uniformly continuous.

6.11.141.1 Solution. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \sin x^2$. □

6.11.142 Problem. Give an example (with proof) of a function $f : \mathbb{R} \to \mathbb{R}$ which is bounded and continuous but not uniformly continuous on $\mathbb{R}$.

6.11.142.1 Solution. Define $f : (0, 1) \to \mathbb{R}$ by $f(x) = \cos(\pi/x)$, we see that f is continuous, now consider the sequences $x_n = 1/n$ and $y_n = 1/(n + 1)$, then $|x_n - y_n| \to 0$ but $|f(x_n) - f(y_n)| = |\cos\ n\pi - \cos\ (n + 1)\pi| = 2 \nrightarrow 0$ □

6.11.143 Problem. Let $\mathcal{A}$ be the set of all sequences whose elements are the digits 0 and 1. Then $\mathcal{A}$ is uncountable.

6.11.143.1 Solution. Here, $\mathcal{A} = 2^{\mathbb{N}}$. Then show that $|\mathcal{P}(\mathbb{N})| = 2^{|\mathbb{N}|} > |\mathbb{N}|$. □

6.11.144 Problem. Prove or disprove: $\dfrac{\sin x}{x}; x > 0$, is not uniformly continuous.

6.11.144.1 Solution. Since $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$, so we can remove this discontinuity by taking $\frac{\sin x}{x} = 1$ at $x = 0$ and is continuous every where in $(0, \infty)$.Thus $\frac{\sin x}{x}$ is bounded in $[0, 1]$ and uniformly continuous on $[0, 1]$. But in $[1, \infty)$ we take $1 \le x < y$ then by MVT $\exists, \xi \in (x, y)$ such that

$$\left|\frac{\sin x}{x} - \frac{\sin y}{y}\right| = \left|\frac{\xi\cos\xi - \sin\xi}{\xi^2}\right| |x - y|$$

we see that $\left|\frac{\xi\cos\xi - \sin\xi}{\xi^2}\right| \le \left|\frac{|\xi\cos\xi| + |\sin\xi|}{\xi^2}\right| \le \frac{|\xi|+1}{\xi^2} < 2$, as $|\xi| > 1$. Thus for $\epsilon > 0$ and $0 < |x-y| < \frac{\epsilon}{2}$, we have $\left|\frac{\sin\ x}{x} - \frac{\sin\ y}{y}\right| < 2|x - y| < \epsilon$ taking $\delta = \frac{\epsilon}{2}$. Hence $\frac{\sin\ x}{x}$ is uniformly continuous on $(0, \infty)$. □

6.11.145 Problem. Give an example of a function f which is monotone increasing on a closed and bounded interval $[a, b]$ but does not satisfy the intermediate value property on $[a, b]$.

6.11.145.1 Solution. Consider the function $f : [0, 2] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x, & \text{if } 0 \le x < 1 \\ x + 1, & \text{if } 1 \le x \le 2. \end{cases} \qquad \square$$

6.11.146 Problem. Give an example of a continuous function on a closed interval I such that

1. f is not bounded on I.
2. $f(I)$ is not a closed interval.

6.11.146.1 Solution.

1. Let $I = [a, \infty)$. Define $f : [a, \infty) \to \mathbb{R}$ by $f(x) = x^2$, that is not bounded on I.
2. Let $I = (-\infty, \infty)$. Define $f : I \to \mathbb{R}$ by $f(x) = e^{-x^2}$, but $f(I) = (0, 1]$ is not a closed interval. □

6.11.147 Problem. Let $f : [0, 2] \to \mathbb{R}$ be continuous, and $f(0) = f(2)$. Prove that there exists a point $p \in [0, 1]$ such that $f(p) = f(p + 1)$.

6.11.147.1 Solution. If $f(0) = f(1)$, then $p = 0, 1$. If $f(0) \ne f(1)$, then consider a function $g : [0, 1] \to \mathbb{R}$ by $g(x) = f(x) - f(x + 1)$. Since f is continuous, so g is continuous. Now, suppose $f(1) > f(0)$, then $g(0) < 0$ and $g(1) = f(1) - f(2) = f(1) - f(0) > 0$, so by IVP there exists $p \in [0, 1]$ such that $g(p) = 0$ which shows that $f(p) = f(p + 1)$. □

6.11.148 Problem. Let $f : [a, b] \to \mathbb{R}$ be continuous, and $x_1, x_2, .., x_n \in [a, b]$. Prove that there is a point $p \in [a, b]$ such that

$$f(p) = \frac{1}{n}\left(\sum_{i=1}^{n} f(x_i)\right).$$

6.11.148.1 Solution. Since $f : [a, b] \to \mathbb{R}$ be continuous, so $\exists\ \lambda$ and μ such that $f(\lambda) = \sup_{x\in[a,b]} > f(\mu) = \inf_{x\in[a,b]}$ i.e. $f(\mu) < f(x) < f(\lambda)\ \forall\ x \in [a, b]$. Hence

$$f(\mu) < \frac{1}{n}\left(\sum_{i=1}^{n} f(x_i)\right) < f(\lambda),$$

and by IVP $\exists\ p \in [a, b]$ such that

$$f(p) = \frac{1}{n}\left(\sum_{i=1}^{n} f(x_i)\right). \quad \square$$

6.11.149 Problem. Show that the product of two uniformly continuous functions is not uniformly continuous.

6.11.149.1 Solution. Consider $f, g : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x$ and $g(x) = \sin x$. Now $h(x) = x \sin x$ is not uniformly continuous, for the sequences $x_n = 2n\pi + 1/n$ and $y_n = 2n\pi \Rightarrow\ x_n - y_n \to 0$ but

$$\begin{aligned} h(x_n) - h(y_n) &= (2n\pi + 1/n) \sin 1/n \\ &= 2n\pi \sin 1/n + 1/n \sin 1/n \to 2\pi. \quad \square \end{aligned}$$

6.11.150 Problem. The uniformly continuous functions can be extended by continuity at every finite point of accumulation of their domain of definition.

6.11.150.1 Solution. In fact, let $f : A \in \mathbb{R}$ be a uniformly continuous function and let $a \in \mathbb{R}\setminus A$ be a point of accumulation of A. If (x_n) is a sequence of elements of A, convergent to a, it is a Cauchy sequence and the sequence $(f(x_n))$ must be Cauchy in $\mathbb{R}$, so convergent, according to the property of completeness of $\mathbb{R}$. The proof ends by noticing that the value ℓ of the limit $\lim_{n\to\infty} f(x_n)$ does not depend on the particular sequence (x_n) such $x_n \to a$. For this, we use the method of interlacing. If $x_n \to a$ and $y_n \to a$, then the sequence

$$(x_0, y_0, x_1, y_1, ...)$$

obtained by interlacing the terms of the two sequences, is also convergent to a. The above reasoning shows that the sequence of their images

$$f(x_0), f(y_0), f(x_1), f(y_1), ...$$

is convergent, so $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(y_n)$. According to criterion on the existence of limits, this fact will yield $\lim_{x\to a} f(x)$. $\square$

6.11.151 Problem. Let $D \subseteq \mathbb{R}$ be bounded and $f : D \to \mathbb{R}$ a uniformly continuous function. Suppose $a \in D'$, then prove that

1. f has a finite limit at a.
2. f can be extended to a continuous function on the closure of D.
3. $f(D)$ is bounded.

6.11.151.1 Solution.

1. Let (a_n) be a sequence in $D - \{a\}$ converging to a. We show that $(f(a_n))$ converges. Since f is u-continuous, so for $\epsilon > 0$, $\exists \delta > 0$ such that $x, y \in D$ and $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$. Again, since (a_n) is cauchy, we can take $N \in \mathbb{N}$ such that $m, n \geq N$ implies $|a_m - a_n| < \delta$, and then $m, n \geq N$ implies $|f(a_m) - f(a_n)| < \epsilon$. Thus $(f(a_n))$ converges.
2. Define $g : \overline{D} \to \mathbb{R}$ by
$$g(x) = \begin{cases} f(x), & x \in D \\ \lim_{x\to a} f((y)), & x \in \overline{D} - D. \end{cases}$$
Show that g is well-defined and is continuous by construction.
3. Note that $\overline{D} \subseteq \mathbb{R}$ is closed by definition. Since D is bounded, so $\overline{D}$ is compact. Now g is continuous implies that $g(\overline{D})$ is compact hence bounded. As $f(D) = g(D) \subseteq g(\overline{D})$, we conclude that $f(D)$ is bounded. □

6.11.152 Problem. Prove that if f is uniformly continuous on a bounded set S, then f is a bounded function on S.

6.11.152.1 Solution. If S is bounded, then $\overline{S}$ is also bounded. Since f is uniformly continuous on S, its extension $\hat{f}$ must be continuous on $\overline{S}$. But $\overline{S}$ is also a compact set, so $\hat{f}$ is continuous on a compact set. Hence its image is a compact set. By the Heine-Borel Theorem, its image is bounded. Therefore f is bounded on S.

6.11.153 Problem. Suppose $f : D \to \mathbb{R}$ is uniformly continuous and D is a bounded set of real numbers, then $f(D)$ is bounded.

6.11.153.1 Solution. Here we use the property that uniform continuous function carries Cauchy sequences to Cauchy sequences. Suppose $f(D)$ is not bounded, then $\exists\ d_n \in D$ such that $f(d_n) > n, \forall n \in \mathbb{N}$, and the sequence (d_n) is bounded as D is bounded. So by B-W theorem the sequence (d_n) has a convergent subsequence (d_{n_k}). Now $f(d_{n_k}) > n_k$ shows that $(f(d_{n_k}))$ is unbounded. Which contradicts that $(f(d_{n_k}))$ is a Cauchy sequence. □

6.11.154 Problem. Let A and B be disjoint subsets of $\mathbb{R}$ and $f : A \cup B \to \mathbb{R}$ a continouus function.Assume that f is uniformly continuous on A and on B. Is it true that f is uniformly continuous on $A \cup B$?

6.11.154.1 Solution. Let $A = (0, 1)$ and $B = (1, 2)$, and define $f : A \cup B \to \mathbb{R}$ by
$$f(x) = \begin{cases} 0, & x \in A \\ 1, & x \in B. \end{cases}$$
It is easy to show that f is a continuous function.To show that f is uniformly continuous on A, let $\epsilon > 0$ and take any $\delta > 0$. Then $x, y \in A$ and $|x - y| < \delta$ implies $|f(x) - f(y)| = 0 < \epsilon$. Similarly f is continuous on B. Now we show that f is not uniformly continuous function on $A \cup B$, let $\epsilon = 1$ and take any $\delta > 0$. Let $x = 1 - \min\{1/2, \delta/4\}$ and $y = 1 + \min\{1/2, \delta/4\}$.Then $x, y \in A \cup B$ and $|x - y| < \delta$ but $|f(x) - f(y)| = 1 \geq \epsilon$. □

6.11.155 Problem. (Volterra's Theorem). Let $f, g : [a, b] \to \mathbb{R}$ and let C_f and C_g denote the continuity sets of f and g; respectively. Thus $C_f = \{x; x \in [a, b], f$ is continuous at $x\}$ and C_g defined similarly. Show that if C_f and C_g are both dense, then so is $C_f \cap C_g$. Deduce that there is no function that is continuous on $\mathbb{Q}$ and discontinuous on $\mathbb{Q}^C$.

6.11.155.1 Solution. Let $I_0 = [a, b]$ and choose a point $p_0 \in C_f \cap I_0^\circ$. The existence of p_0 is assured by the fact that C_f is dense in I_0. Using the definition of continuity at p, we infer the existence of a non-degenerate compact subinterval $J_0 \subseteq I_0$, centered at p_0, such that

$$x, y \in J_0 \text{ implies } |f(x) - f(y)| < 1.$$

Similarly, we can choose a point $q_0 \in C_g \cap J^\circ$. By the continuity of g at q_0, we infer the existence of a non-degenerate compact subinterval $I_1 \subseteq J_0$, centered at q_0, such that

$$x, y \in I_1 \text{ implies } |g(x) - g(y)| < 1/2.$$

By mathematical induction, we can choose a nested sequence (I_n) of compact intervals such that

$$|f(x) - f(y)| < 1/2^{n-1} \text{ and } |g(x) - g(y)| < 1/2^n \tag{6.1}$$

for all $x, y \in I_n$ and all $n \in \mathbb{N}$. By the Nested Intervals Lemma, there exists a point z in the intersection of all these intervals. Because of relations (6.1), z is necessarily a point of continuity both for f and g. □

6.11.156 Problem. Let $f, g \in C[a, b]$ satisfy $f(x) < g(x)\ \forall\, x \in [a, b]$. Show that there is a constant $c < 1$ such that $f(x) \le g(x)\ \forall\, x \in [a, b]$.

6.11.156.1 Solution. Hint: Pick $N \ge 0$ such that $f + N > 0$ on $[a, b]$ and look at $(f + N)/(g + N)$. □

6.11.157 Problem. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function which has a local minimum at every point. Prove that f must be constant.

6.11.157.1 Solution. For $t \ge 0$ define $M(t) = \sup\{f(x); x \in [0, t]\}$. Then M is continuous, non-decreasing, $M(0) = f(0)$, and $M(t) \ge f(t)$ for all $t \ge 0$. Since f has a local minimum at every point, it is easy to see that if $M(t) = f(t)$ for some $t > 0$, then $f(x) = f(0)$ for $0 \le x \le t$. Let $\alpha = \sup\{t \ge 0; M(t) = f(t)\}$ and assume that $\alpha < \infty$. Then $M(\alpha) = f(\alpha)$ since both M and f are continuous, $f(x) = f(0)$ for $0 \le x \le \alpha$ and $M(t) > f(t)$ for $t > \alpha$. It follows that the maximum of f on $[\alpha, \alpha + 1]$ occurs at some $\beta \in (\alpha, \alpha + 1)$. But this implies $M(\beta) = f(\beta)$, which contradicts the definition of α. Therefore,$\alpha = \infty$ and f is constant for $x \ge 0$. A similar argument proves that f is constant for $x \le 0$. □

6.11.158 Problem. Let $f, g : [0, 1] \to [0, 1]$ be two continuous functions. If g is non-decreasing and $f \circ g = g \circ f$, prove that f and g have a common fixed point in [0,1].

6.11.158.1 Solution. Let $Fix(f)$ denote the set of fixed points of f, and similarly define $Fix(g)$. An easy calculus exercise shows that these are closed non-empty sets. Since $f \circ g = g \circ f$, we have $g(Fix(f)) \subseteq Fix(f)$. Let $\alpha = \sup Fix(f)$. If $g(\alpha) = \alpha$, we are done. If not, $g(\alpha) < \alpha$ and hence $(g^n(\alpha))$ is a decreasing sequence in $Fix(f)$. Hence $\beta = \lim_{n\to\infty} g^n(\alpha)$ is a common fixed point of f and g. □

6.11.159 Problem. Show that the following statements do not imply each other

1. f is continuous a.e. on $[0,1]$.
2. $\exists\, g$ continuous on $[0,1]$ such that $g = f$ a.e.

6.11.159.1 Solution. Consider $f(x) = \frac{1}{x}$ then (1) does not imply (2), and $g(x) = 1, f(x) = 1\ \forall\, x \in \mathbb{Q}$ and $f(x) = 0$ for $x \in \mathbb{Q}^C$, hence (2) does not imply (1). □

6.11.160 Problem. If f is absolute continuous in $[0,1]$, then f^2 is absolute continuous in $[0,1]$.

6.11.160.1 Solution. $[f(x)]^2 - [f(y)]^2 = ([f(x)] + [f(y)])([f(x)] - [f(y)])$ and f is bounded. □

6.11.161 Problem. Show that there does not exist a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that f is constant on a bounded open interval.

6.11.161.1 Solution. Let $U \subseteq \mathbb{R}$ such that $f(x) = c$ on U, then $f^{-1}\{c\} = U$ is both open and closed. Since, non-empty open and closed set is only $\mathbb{R}$ hence $U = \mathbb{R}$, which is impossible. □

6.11.162 Problem. Show that if there exist a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that f is constant on an open interval U, then $U = \mathbb{R}$.

6.11.162.1 Solution. Hin: Previous solution. □

6.11.163 Problem. If $f : \mathbb{R} \to \mathbb{R}$ is continuous $\lim_{x\to\infty} f(x) = b$ and $\lim_{x\to-\infty} f(x) = a$, let $a < y < b$. Prove that $\exists\, x \in \mathbb{R}$ such that $f(x) = y$.

6.11.163.1 Solution. Let $b - y = \epsilon > 0$, so $\exists\, M > 0$ such that

$$\begin{aligned} x > M \Rightarrow& |f(x) - b| < \epsilon = b - y \\ \Rightarrow& -b + y < f(x) - b < b - y \\ \Rightarrow& y < f(x), \\ \Rightarrow& f(M+1) > y. \end{aligned}$$

Again, if $y - a = \epsilon_1 > 0$, so $\exists\, M_1 > 0$ such that

$$\begin{aligned} x < -M_1 \Rightarrow& |f(x) - a| < \epsilon_1 = y - a \\ \Rightarrow& a - y < f(x) - a < y - a \\ \Rightarrow& f(x) < y \\ \Rightarrow& f(-M_1 - 1) < y. \end{aligned}$$

Therefore, $f(-M_1 - 1) < y < f(M+1)$. Since f is continuous on $\mathbb{R}$, so f is continuous on $[-M_1 - 1, M+1]$, by IVP $\exists\, x \in [-M_1 - 1, M+1]$ such that $f(x) = y$. □

6.11.164 Problem. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x \in \mathbb{R}$, $f(x) + f(2x) = 0$.

6.11.164.1 Solution. Hint: Show that $f(x) = -f\left(\dfrac{x}{2}\right) = f\left(\dfrac{x}{4}\right) = -f\left(\dfrac{x}{8}\right) = ...$ □

6.11.165 Problem. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(0) = 1$ and

$$f(2x) - f(x) = x,\ \forall\, x \in \mathbb{R}.$$

6.11.165.1 Solution. We write the above functional equation as

$$f(x) - f\left(\frac{x}{2}\right) = \frac{x}{2}$$
$$f\left(\frac{x}{2}\right) - f\left(\frac{x}{4}\right) = \frac{x}{4}$$
$$f\left(\frac{x}{4}\right) - f\left(\frac{x}{8}\right) = \frac{x}{4}$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
$$f\left(\frac{x}{2^{n-1}}\right) - f\left(\frac{x}{2^n}\right) = \frac{x}{2^n}$$

Adding up, we obtain

$$f(x) - f\left(\frac{x}{2^n}\right) = x\left(\frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^n}\right)$$

which, when $n \to \infty$, becomes $f(x) - 1 = x$. Hence $f(x) = x + 1$ is the (unique) solution. □

6.11.166 Problem. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying $f(x) = f(x^2)\ \forall\ x \in \mathbb{R}$. Prove that f is constant.

6.11.166.1 Solution. The condition from the statement implies that $f(x) = f(-x)$, so it suffices to check that f is constant on $[0, \infty)$. For $x \geq 0$, define the recursive sequence $(x_n)\ x_n \geq 0$, by $x_0 = x$, and $x_{n+1} = \sqrt{x_n}, ;\forall\ n \geq 0$. Then $f(x_0) = f(x_1) = f(x_2) = ... = f(\lim_{n\to\infty} x_n)$. And $\lim_{n\to\infty} x_n = 1$ if $x > 0$. It follows that f is constant. □

6.11.167 Problem. (Kozepiskolai Matematikai Lapok (Mathematics Gazette for High Schools, Budapest)) Does there exist a continuous function $f : [0, 1] \to \mathbb{R}$ that assumes every element of its range an even (finite) number of times?

6.11.167.1 Solution. The answer is yes, there is a tooth function with this property. We construct f to have local maxima at $\dfrac{1}{2^{2n+1}}$ and local minima at 0 and $\dfrac{1}{2^{2n}}$, $n \geq 0$. The values of the function at the extrema are chosen to be $f(0) = f(1) = 0, f\left(\dfrac{1}{2}\right) = \dfrac{1}{2}, f\left(\dfrac{1}{2^{2n+1}}\right) = \dfrac{1}{2^n}, f\left(\dfrac{1}{2^{2n}}\right) = \dfrac{1}{2^{n+1}}$ and for $n \geq 1$. These are connected through segments. The reader is requested to draw the graph. □

6.11.168 Problem. (Vietnamese Mathematical Olympiad, 1999) Let $f(x)$ be a continuous function defined on [0,1] such that

1. $f(0) = f(1) = 0$;
2. $2f(x) + f(y) = 3f\left(\dfrac{2x + y}{3}\right)\ \forall\ x, y \in [0, 1]$.

Prove that $f(x) = 0\ \forall\ x \in [0, 1]$.

6.11.168.1 Solution. We prove by induction on n that $f(m/3^n) = 0$ for all integers $n \geq 0$ and all integers $0 \leq m \leq 3^n$. The given conditions show that this is true for $n = 0$. Assuming that it is true for $n - 1 \geq 0$, we prove it for n. If $n \equiv 0 \pmod 3$

$$f\left(\frac{m}{3^n}\right) = f\left(\frac{\frac{m}{3}}{3^{n-1}}\right) = 0$$

by the induction hypothesis. If $m \equiv 1 \pmod 3$, then $1 \leq m \leq 3n-2$ and

$$3f\left(\frac{m}{3^n}\right) = 2f\left(\frac{\frac{m-1}{3}}{3^{n-1}}\right) + f\left(\frac{\frac{m+2}{3}}{3^{n-1}}\right) = 0 + 0 = 0$$

Thus $f\left(\frac{m}{3^n}\right) = 0$. Since the set $D = \left\{\frac{m}{3^n}; m \in \mathbb{Z}, n \in \mathbb{N}\right\}$ is dense in [0,1] and $f(x) = 0$ on D, so, f is identically 0 on [0,1].

6.11.169 Problem. Give an example of a monotonic function whose points of discontinuity form an arbitrary countable set.

6.11.169.1 Solution. Let A be any arbitrary nonempty countable set of real numbers $a_1, a_2, ...$ and let $\sum p_n$ be a finite or convergent infinite series of positive numbers with sum p. If A is bounded below and x is less than every point of A, let $f(x) = 0$. Otherwise, define $f(x)$ to be the sum of all terms p_m of $\sum p_n$ such that $a_m < x$. The function f is increasing on $\mathbb{R}$, continuous at every point not in A, and discontinuous with a jump equal to p_n at each point a_n, i.e.,

$$\lim_{p \to a_n+} p_n - \lim_{p \to a_n-} p_n = a_n. \quad \square$$

6.11.170 Problem. Give an example of an one-one onto function $f : \mathbb{R} \to \mathbb{R}$ which is continuous at some $a \in \mathbb{R}$, but f^{-1} is not continuous at $f(a)$.

6.11.170.1 Solution. Let

$$f(x) = \begin{cases} \frac{1}{2n} & \text{if } x = \frac{1}{n}, n \in \mathbb{N} \\ \frac{1}{x} & \text{if } x = 3, 5, 7, .. \\ \frac{x}{2} & \text{if } x = 2, 4, 6, .. \\ x & \text{otherwise.} \end{cases}$$

i.e., $f(x) = x$, except when $x = 1, 2, 3, ...; 1/2, 1/3, 1/4, ...$; but for 1,1/2,1/3,1/4,..., the image under f is 1/2,1/4,1/6,1/8,...,respectively; and for 3,5, 7,..., the image under f is 1/3,1/5,1/7,... respectively; and for 2,4,6,..., the image is 1,2,3,... respectively. So the set of values 1,2,3,...; 1/2,1/3,1/4,... are just reshuffled by f and so, since everywhere else f(x)= x, we.see that f is onto. So f^{-1} exists and $f/$ is continuous at 0 (check!). However, f^{-1} is not continuous at $f(0) = 0$, since $f^{-1}(1/n) = n$ for $n = 3, 5, 7, ...$ $\square$

6.11.171 Problem. (45thW.L. Putnam Mathematical Competition, 2002, proposed by T. Andreescu) Let a and b be real numbers in the interval (0,1/2) and let f be a continuous real-valued function such that

$$f(f(x)) = af(x) + bx, \ \forall\, x \in \mathbb{R}.$$

Prove that $f(0) = 0$.

6.11.171.1 Solution. From the given condition, it follows that f is one-to-one. Indeed, if $f(x) = f(y)$, then $f(f(x)) = f(f(y))$, so $bx = by$, which implies $x = y$. Because f is continuous and one-to-one, it is strictly monotonic. We will show that f has a fixed point. Assume by way of contradiction that this is not the case. So either $f(x) > x$ for all x, or $f(x) < x$ for all x. In the first case f must be strictly increasing, and then we have the chain of implications

$$f(x) > x \Rightarrow f(f(x)) > f(x) \Rightarrow af(x) + bx > f(x) \Rightarrow f(x) < \frac{bx}{1-a}, \ \forall\, x \in \mathbb{R}.$$

In particular, $f(1) < \dfrac{b}{1-a} < 1$, contradicting our assumption. In the second case the simultaneous inequalities $f(x) < x$ and $f(f(x)) < f(x)$ show that f must be strictly increasing again. Again we have a chain of implications

$$f(x) < x \Rightarrow f(f(x)) < f(x) \Rightarrow af(x) + bx < f(x) \Rightarrow f(x) > \frac{bx}{1-a},\ \forall\, x \in \mathbb{R}.$$

In particular, $f(-1) > \dfrac{-b}{1-a} > -1$,, again a contradiction. In conclusion, there exists a real number c such that $f(c) = c$. The condition $f(f(c)) = af(c) + bc$ implies $c = ac + bc$; thus $c(a + b - 1) = 0$. It follows that $c = 0$, and we obtain $f(0) = 0$. Remark. This argument can be simplified if we use the fact that a decreasing monotonic function on $\mathbb{R}$ always has a unique fixed point. (Prove it!) □

6.11.172 Problem. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous decreasing function. Prove that the system $x = f(y), y = f(z), z = f(x)$ has a unique solution.

6.11.172.1 Solution. The fact that f is decreasing implies immediately that

$$\lim_{x\to-\infty}(f(x) - x) = \infty \text{ and } \lim_{x\to\infty}(f(x) - x) = -\infty.$$

By the intermediate value property, there is a such that $f(a) - a = 0$, that is, $f(a) = a$. The function cannot have another fixed point because if x and y are fixed points, with $x < y$, then $x = f(x) \geq f(y) = y$, impossible. The triple (a, a, a) is a solution to the system. And if (x, y, z) is a solution then $f(f(f(x))) = x$. The function $f \circ f \circ f$ is also continuous and decreasing, so it has a unique fixed point. And this fixed point can only be a. Therefore, $x = y = z = a$, proving that the solution is unique. □

6.11.173 Problem. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that

$$|f(x) - f(y)| \geq |x - y|,\ \forall\, x, y \in \mathbb{R}.$$

Prove that the range of f is all of $\mathbb{R}$.

6.11.173.1 Solution. The inequality from the statement implies right away that f is injective, and also that f transforms unbounded intervals into unbounded intervals. The sets $f((-\infty, 0])$ and $f([0, \infty))$ are unbounded intervals that intersect at one point. They must be two intervals that cover the entire real axis. □

6.11.174 Problem. A runner runs a six-mile course in 30 minutes. Prove that somewhere along the course the runner ran a mile in exactly 5 minutes.

6.11.174.1 Solution. Let x denote the distance along the course, measured in miles from the starting line. For each $x \in [0, 5]$, let $f(x)$ denote the time that elapses for the mile from the point x to the point $x + 1$. Note that f depends continuously on x. We are given that

$$f(0) + f(1) + f(2) + f(3) + f(4) + f(5) = 30.$$

It follows that not all of $f(0), f(1), ..., f(5)$ are smaller than 5, and not all of them are larger than 5. Choose $a, b \in \{0, 1, ..., 5\}$ such that $f(a) \leq 5 \leq f(b)$. By the intermediate value property, there exists c between a and b such that $f(c) = 5$. The mile between c and $c + 1$ was run in exactly 5 minutes. □

6.11.175 Problem. (Soviet Union University Student Mathematical Olympiad, 1975) Given a sequence (a_n) such that for any $\alpha > 1$ the subsequence $(a_{[\alpha^n]})$ converges to zero, does it follow that the sequence (a_n) itself converges to zero? $[x]$ is the greatest integer function.

6.11.175.1 Solution. The answer to the question is yes. We claim that for any sequence of positive integers n_k, there exists a number $\alpha > 1$ such that $([\alpha^k])$ and (n_k) have infinitely many terms in common. We need the following lemma.

6.11.7 Lemma. For any $\alpha, \beta, 1 < \alpha < \beta$, the set $\bigcup_{k=1}^{\infty}[\alpha^k, \beta^k - 1]$ contains some interval of the form (a, ∞).

Proof. Observe that $(\beta/\alpha)^k \to \infty$ as $k \to \infty$. Hence for large k, $\alpha^{k+1} < \beta^k - 1$, and the lemma follows. □

Let us return to the problem and prove the claim. Fix the numbers α_1 and β_1 $1 < \alpha_1 < \beta_1$. Using the lemma we can find some k_1 such that the interval $[\alpha^{k_1}, \beta^{k_1} - 1]$ contains some terms of the sequence (n_k). Choose one of these terms and call it t_1. Define

$$\alpha_2 = t_1^{\frac{1}{k_1}}, \;\; \beta_2 = \left(t_1 + \frac{1}{2}\right)^{\frac{1}{k_1}}$$

Then $[\alpha_2, \beta_2] \subset [\alpha_1, \beta_1]$, and for any $x \in [\alpha_2, \beta_2], [x^{k_1}] = t_1$. Again by the lemma, there exists k_2 such that $[\alpha^{k_2}, \beta^{k_2} - 1]$ contains a term of (n_k) different from n_1. Call this term t_2. Let

$$\alpha_3 = t_2^{\frac{1}{k_2}}, \;\; \beta_3 = \left(t_2 + \frac{1}{2}\right)^{\frac{1}{k_2}}$$

As before, $[\alpha_3, \beta_3] \subset [\alpha_2, \beta_2]$, and for any $x \in [\alpha_3, \beta_3], [x^{k_2}] = t_2$. Repeat the construction infinitely many times. By Cantor's nested intervals theorem, the intersection of the decreasing sequence of intervals $[\alpha_j, \beta_j], j = 1, 2, ..$ is nonempty. Let γ be an element of this intersection. Then $[\gamma^{k_j}] = t_j, j = 1, 2, ..$ which shows that the sequence (γ^j) contains a subset of the sequence (n_k). This proves the claim. □

6.11.176 Problem. Let $f : (0, \infty) \to \mathbb{R}$ be a continuous function with the property that for any $x > 0$, $\lim_{n\to\infty} f(nx) = 0$. Prove that $\lim_{x\to\infty} f(x) = 0$.

6.11.176.1 Solution. The solution follows closely that of the previous problem. Replacing f by $|f|$ we may assume that $f \geq 0$. We argue by contradiction. Suppose that there exists $a > 0$ such that the set

$$A = f^{-1}((a, \infty)) = \{x \in (0, \infty); f(x) > a\}$$

is unbounded. We want to show that there exists $x_0 \in (0, \infty)$ such that the sequence (nx_0) has infinitely many terms in A. The idea is to construct a sequence of closed intervals $I_1, I_2, I_3, ...$ with lengths converging to zero and a sequence of positive integers $n_1 < n_2 < n_3 < ....$ such that $n_k I_k \subseteq A \; \forall \; k \geq 1$. Let I_1 be any closed interval in A of length less than 1 and let $n_1 = 1$. Exactly as in the case of the previous problem, we can show that there exists a positive number m_1 such that $\cup_{m\geq m_1} mI_1$ is a half-line. Thus there exists $n_2 > n_1$ such that $n_2 I_1$ intersects A. Let J_2 be a closed interval of length less than 1 in this intersection. Let $I_2 = \frac{1}{n_1} J_2$. Clearly, $I_2 \subseteq I_1$, and the length of I_2 is less than $1/n_2$. Also, $n_2 I_2 \subseteq A$. Inductively, let $n_k > n_{k-1}$ be such that $n_k I_{k-1}$ intersects A, and let J_k be a closed interval of length less than 1 in this intersection. Define $I_k = \frac{1}{n_k} J_k$.

We found the decreasing sequence of intervals $I_1 \supseteq I_2 \supseteq I_3 \supseteq ....$ and positive integers $n_1 < n_2 < n_3 < ...$ such that $n_k I_k \subseteq A$. Cantor's nested intervals theorem implies the existence of a number x_0 in the intersection of these intervals. The subsequence $(n_k x_0)$ lies in A, which means that $(n_k x_0)$ has infinitely many terms in A. This implies that the sequence $f(nx_0)$ does not converge to 0, since it has a subsequence bounded away from zero. But this contradicts the hypothesis. Hence our assumption was false, and therefore $\lim_{x\to\infty} f(x) = 0$. □

Remark: This result is known as Croft's lemma. It has an elegant proof using the Baire category theorem.

6.11.177 Problem. Suppose that $f : [0,1] \to \mathbb{R}$ is continuous and has a local maximum at each point in [0,1]. Prove that f is constant.

6.11.177.1 Solution. Since [0,1] is compact and f is continuous, so by the extreme value theorem, f attains its infimum at some point $x_0 \in [0,1]$. Fix some $a \in [0,1]$ such that there exists some $b \in [0,1], x_0 \in (a,b)$ and $f(x_0) \geq f(x)\ \forall\ x \in (a,b)$. The existence of such a, b is guaranteed by assumption that f has local maximum at every point. Define the following set

$$B_a = \{b; b \in [0,1], b > a, x_0 \in (a;b) \text{ and } \forall\ x \in (a,b); f(x_0) \geq f(x)\}$$

Let $b \in B_a$. Observe that for each $x \in (a,b)$, we have $f(x_0) \geq f(x)$ by construction and that $f(x_0) \leq f(x)$ by the fact that $f(x_0)$ is the infimum of f on [0,1]. Hence, f is constant on (a,b). Moreover, since this gives us that the left-hand side limit of f at b is $f(x_0)$, then by continuity of f, we have $f(b) = f(x_0)$. This holds for each $b \in B_a$. Let $\beta = \sup B_a$. Since B_a is bounded, then β is finite and $\beta \leq 1$. We claim that belongs to B_a. Suppose not. Then, there exists $x \in (a,\beta)$ such that $f(x) > f(x_0)$. But, then for all $b \in (x,\beta)$, we have that $b \notin B_a$. Hence, for all $b \in (x,\beta)$, we have $b > \sup B_a = \beta$, a contradiction. Now, suppose that $\beta < 1$. Then, by the observation that $f(b) = f(x_0)$ for each $b \in Ba$, we have $f(b) = f(x_0)$. But, f has a local maximum at β by assumption, so we can find some $b \leq 1$ such that $\beta \in (a,b)$ with $f(x_0) = f(\beta) \geq x$ for all $x \in (a,b)$. But, then $b \in B_a$ and $b \gneq \beta$, which is a contradiction. Therefore, $b = 1$.

Now, if we fix $b = 1$ and consider the set

$$A_1 = \{a; a \in [0,1]; a < 1; x_0 \in (a,1] \text{ and } \forall\ x \in (a,1]; f(x_0) \geq f(x)\}$$

Following the same argument as above, we find that $\inf A_1 = 0$ and $0 \in A_1$. Thus, $f(x_0)$ is both the supremum and infimum on [0,1] and therefore f is constant.

6.11.177.2 Solution. 2. Since f is continuous on the compact set [0,1], by extreme value theorem it attains absolute minimum, say, m, at some $c \in [0,1]$. Let $K = f^{-1}[\{m\}]$. Since $c \in K, K$ is nonempty. It suffices to show that $K = [0,1]$. Since f is continuous and K is the pull-back of a singleton, which is closed in $\mathbb{R}$, K is closed in [0,1]. Now since [0,1] is connected, any nonempty subset of [0,1] which is both open and closed must be the whole space [0,1]. As we know K is closed in [0,1], it suffices to show that K is also open in [0,1]. To this end, let $y \in K$. Since f has a local minimum at y, there is $\epsilon > 0$ such that f has absolute maximum at y on $U = (y-\epsilon, y+\epsilon) \cap [0,1]$. But since $y \in K, f$ has absolute minimum on [0,1] at y. Thus f must be constant on the ϵ- ball U of y. But since $f(y) = m, f$ must be identically m on U. Hence $U \subseteq K$. So y is an interior point of K. Since $y \in K$ was arbitrary, K is open in [0,1]. Thus $K = [0,1]$. This shows the assertion. □

6.11.178 Problem. Suppose that $f : \mathbb{R} \to \mathbb{R}$ has a closed graph $gr(f)$ (closed as a subset of $\mathbb{R}^2$). Then f is locally bounded implies that f is continuous.

6.11.178.1 Solution. f is locally bounded $\Rightarrow$ f is continuous: Suppose f is not continuous at the point $x = \xi$. Then there exists a sequence $(a_n) \to \xi$ such that $f(a_n) \to \eta \neq f(\xi)$ with $\xi \in \mathbb{R}$, as f is locally bounded. Thus, the sequence of points $\{(a_n, f(a_n))\} \to (\xi, \eta)$ in $\mathbb{R}^2$, contradicting the fact that $gr(f)$ is closed, since the point $(\xi, \eta) \notin gr(f)$ and is a limit point of $gr(f)$. $\square$

6.11.179 Problem. Let t and ϵ be real numbers with $|\epsilon| < 1$. Prove that the equation $x - \epsilon \sin x = t$ has a unique real solution.

6.11.179.1 Solution. Define the function $f(x) = \epsilon \sin x + t$. Then for any real numbers x_1 and x_2,

$$|f(x_1) - f(x_2)| = |\epsilon||\sin x_1 - \sin x_2| \leq 2|\epsilon|.\left|\sin \frac{x_1 - x_2}{2} \cos \frac{x_1 + x_2}{2}\right|$$
$$\leq 2|\epsilon|.\left|\sin \frac{x_1 - x_2}{2}\right| \leq |\epsilon||x_1 - x_2|.$$

Hence f is a contraction, and there exists a unique x such that $f(x) = \epsilon \sin x + t = x$. This x is the unique solution to the equation. (J. Kepler) $\square$

6.11.180 Problem. If $f : [a, b] \to \mathbb{R}$ be continuous, then for each $\epsilon > 0$ $\exists$ a step-function g with domain $[a, b]$ such that $|f(x) - g(x)| < \epsilon \; \forall \; x \in [a, b]$.

6.11.180.1 Solution. By previous result, for each $\epsilon > 0 \; \exists \; \delta > 0$ such that $|f(x) - f(y)| < \epsilon \; \forall \; x, y \in [a, b]$ for which $|x - y| < \delta$. Now, we choose $n \in \mathbb{N}$ such that $(b - a)/n < \delta$, and divide the interval $[a, b]$ into n subintervals of equal lengths, by points $a = c_0, c_1, ..., c_r, ...c_n = b$. Then if g is the step-function given by

$$g(a) = f(a), \;\; g(x) = f(c_r) \;\; x \in (c_{r-1}, c_r) \; r = 1, 2, ...n,$$

we have that $|f(a) - g(a)| = 0$ and that $|f(x) - g(x)| = |f(x) - g(c_r)|$ for $x \in (c_{r-1}, c_r)$ $r = 1, 2, .., n$. Since $|x - c_r| < (b - a)/n < \delta$ whenever $x \in (c_{r-1}, c_r)$, it follows that $|f(x) - g(x)| < \epsilon \; \forall \; x \in [a, b]$. $\square$

6.11.181 Problem. Let $f, g : [a, \infty)$ be continuous functions on $[a, \infty)$. If

$$\lim_{x \to \infty} (f(x) - g(x)) = 0$$

then then f is uniformly continuous on $[a, \infty)$ if and only if g is uniformly continuous on $[a, \infty)$.

6.11.181.1 Solution. Let us suppose that g is uniformly continuous on $[a, \infty)$ and let us prove that so f is. Let $\epsilon > 0$ be fixed. The following claims hold:

1. There exists $b > a$ such that $|f(z) - g(z)| < \epsilon/6 \; \forall \; z \in [a, \infty)$ (because $\lim_{x \to \infty}(f(x) - g(x)) = 0$;

2. There exists $\delta_1 > 0$ such that $|g(x) - g(y)| < \epsilon/6 \; \forall \; x, y \in [a, \infty)$ such that $|x - y| < \delta_1$ (because g is uniformly continuous on $[a, \infty)$.

3. There exists $\delta_2 > 0$ such that $|f(x) - f(y)| < \epsilon/2 \; \forall \; x, y \in [a, b]$ such that $|x - y| < \delta_2$ (because g is uniformly continuous on $[a, b]$ (by Heine-Borel theorem).

Next we prove that for $x, y \in [a, \infty)$ the relation $|x - y| < \min\{\delta_1, \delta_2\}$ implies $|f(x) - f(y)| < \epsilon$. By virtue of (iii), we just have to consider the following cases:

Case 1. $x, y \in [b, \infty)$ and $|x - y| < \min\{\delta_1, \delta_2\}$. The triangle inequality and properties (i) and (ii) yield

$$|f(x) - f(y)| \le |f(x) - g(x)| + |g(x) - g(y)| + |g(y) - f(y)| \le \epsilon/2.$$

Case 2. $a < x < b < y$ and $|x - y| < \min\{\delta_1, \delta_2\}$. We deduce from (iii) and Case 1 that

$$|f(x) - f(y)| \le |f(x) - f(b)| + |f(b) - f(y)| < \epsilon. \quad \square$$

6.11.182 Problem. Let $f : [a, \infty) \to \mathbb{R}$ be continuous on $[a, \infty)$. The function f is uniformly continuous on $[a, \infty)$ provided that one of the following conditions is satisfied:

1. $\lim_{x\to\infty} f(x)$ exists (equivalently, the graph of f has an horizontal asymptote).
2. There exist $m, n \in \mathbb{R}, m \neq 0$, such that $\lim_{x\to\infty}(f(x) - mx - n) = 0$ (equivalently, the line $y = mx + n$ is the oblique asymptote for the graph of f).

6.11.182.1 Solution. Hint: The function $g(x) = mx + n$ is uniformly continuous on for every $m, n \in \mathbb{R}$.

6.11.183 Problem. If $f : [a, \infty) \to \mathbb{R}$ is uniformly continuous on $[a, \infty)$ then there exist $A > 0$ and $B > 0$ such that

$$|f(x)| < Ax \;\forall\, x \in [B, \infty).$$

6.11.183.1 Solution. Let $\delta > 0$ be such that

$$[x, y \in [a, \infty), |x - y| < \delta] \Rightarrow |f(x) - f(y)| < 1, \tag{A}$$

and let $x > a + 2^{-1}\delta$ be fixed. Now let us consider points

$$a = x_0 < x_1 < ... < x_n = x,$$

such that $x_i - x_{i-1} = 2^{-1}\delta \;\forall\, i \in \{2, .., n\}$ and $0 < x_1 - x_0 < 2^{-1}\delta$. Note that we have

$$x - a = \sum_{i=1}^{n}(x_i - x_{i-1}) > \sum_{i=2}^{n}(x_i - x_{i-1}) = (n-1)2^{-1}\delta. \tag{B}$$

and

$$f(x) - f(a) = \sum_{i=1}^{n}(f(x_i) - f(x_{i-1}))$$

Hence we deduce, using the triangle inequality, (A), and (B), that

$$|f(x)| \le \sum_{i=1}^{n}|f(x_i) - f(x_{i-1})| + |f(a)| < n + |f(a)| < 2\delta^{-1}(x - a) + 1 + |f(a)|,$$

and the result follows with $A = 2\delta^{-1} + 1$ and $B = 2\delta^{-1}|a| + 1 + |f(a)|$. $\square$

The previous problem is particularly useful to determine functions that are not uniformly continuous.

6.11.184 Problem. If $f : [a, \infty) \to \mathbb{R}$ satisfies

$$\lim_{x\to\infty} \frac{f(x)}{x} = \infty,$$

then f is not uniformly continuous on $[b, \infty)$ for any $b \geq a$.

6.11.184.1 Solution. Assume, reasoning by contradiction, that f is uniformly continuous on $[b, \infty)$ for some $b \geq a$. Previous problem guarantees the existence of $A > 0$ and $B > 0$ such that

$$\frac{|f(x)|}{x} < A \,\forall\, x \in [B, \infty),$$

and then the limit $\lim_{x\to\infty} \frac{|f(x)|}{x}$ could be, at most, A, a contradiction.

There are lots of functions that are not uniformly continuous. We point out the most important ones in the following problem:

6.11.185 Problem.

1. The function $g(x) = x^\alpha$ with $\alpha \in \mathbb{R}, \alpha > 1$, is not uniformly continuous on $[a, \infty)$ for any $a \geq 0$.
2. The function $g(x) = e^x$ is not uniformly continuous on $[a, \infty)$ for any $a \in \mathbb{R}$.

6.11.185.1 Solution. In all cases we have

$$\lim_{x\to\infty} \frac{g(x)}{x} = \infty,$$

so the conclusions follow by virtue of (6.4). □

Next we present a family of uniformly continuous functions on $[0, \infty)$:

6.11.186 Problem.

1. The function $g(x) = x^\alpha$ with $\alpha \in \mathbb{R}$, with $\alpha \leq 1$, is uniformly continuous on $[a, \infty)$ for every $a > 0$; moreover, g is uniformly continuous on $(0, \infty)$ if and only if $0 \leq \alpha \leq 1$.
2. The function $g(x) = \ln x$ is uniformly continuous on $[a, \infty)$ for every $a > 0$, and it is not uniformly continuous on $(0, \infty)$.

6.11.186.1 Solution.

1. The cases $\alpha = 1$ and $\alpha = 0$ correspond to affine functions. For $\alpha < 0$ we can prove that g is uniformly continuous on $[a, \infty)$ for every $a > 0$; on the other hand, $\alpha < 0$ implies $\lim_{x\to 0+} g(x) = \infty$, hence g is not uniformly continuous on $(0, \infty)$ in these cases.
   For $\alpha \in (0, 1)$ it suffices to prove that g is Lipschitzian on $[1, \infty)$, because g is continuous on [0,1], thus uniformly continuous on [0,1] by virtue of Heine-Borel theorem.
   Let $1 \leq x \leq y$, then

$$\begin{aligned} |x^\alpha - y^\alpha| \left(x^{1-\alpha} + y^{1-\alpha}\right) &= |x^\alpha y^{1-\alpha} - y^\alpha y^{1-\alpha} - y^\alpha x^{1-\alpha} + x^\alpha x^{1-\alpha}| \\ &\leq |x - y| + |x^\alpha y^{1-\alpha} - y^\alpha x^{1-\alpha}| \end{aligned} \tag{A}$$

Note that $\alpha > 0$ and $1 - \alpha > 0$, so $1 \le x \le y$ implies that

$$x^\alpha y^{1-\alpha} - y^\alpha x^{1-\alpha} \le y^\alpha y^{1-\alpha} - x^\alpha x^{1-\alpha} = y - x,$$

and, analogously, $x^\alpha y^{1-\alpha} - y^\alpha x^{1-\alpha} \ge y - x$, thus (A) gives

$$|x^\alpha - y^\alpha| \le \frac{2}{y^{1-\alpha} + x^{1-\alpha}} |x - y|.$$

Finally, note that $y^{1-\alpha} + x^{1-\alpha} \ge 2$, if $x, y \in [1, \infty)$ and then the claim follows.

2. Let $a > 0$ be fixed and let and let $(x_n), (y_n)$ be a pair of sequences in $[a, \infty)$ such that $(x_n - y_n) \to 0$. We have to prove that $\ln(x_n) - \ln(y_n) = \ln(x_n/y_n)$ tends to 0. Note that $0 < y_n^{-1} \le a^{-1}\ \forall\, n \in \mathbb{N}$, thus

$$\left| \frac{x_n}{y_n} - 1 \right| = \frac{|x_n - y_n|}{y_n} \le a^{-1} |x_n - y_n|\ \forall\, n \in \mathbb{N},$$

which implies that

$$\lim_{n\to\infty} \frac{x_n}{y_n} = 1,$$

and therefore

$$\lim_{n\to\infty} \left(\frac{x_n}{y_n} \right) = \ln 1 = 0.$$

To prove that g is not uniformly continuous on $(0, \infty)$ simply note that $\lim_{x\to 0+} \ln(x) = -\infty$. □

6.11.187 Problem. Let $f : [0, 1) \to \mathbb{R}$ be continuous, and let $\lim_{x\to\infty} f(x) = 0$. Then f is uniformly continuous on $[0, \infty)$.

6.11.187.1 Solution. Let $\epsilon > 0$ be fixed. Since $\lim_{x\to\infty} f(x) = 0$, then there exists $M \in [0, \infty)$ such that for all $x > M$, we have that $f(x) < \epsilon/2$. Moreover, we have that since f is continuous on $[0, M + 1]$ which is compact, then it is uniformly continuous. Hence, there exists $\delta > 0$ such that for all $x, y \in [0, M + 1]$, if $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$. We may assume that $\delta < 1$.
Now, let $x, y \in [0, 1)$ with $|x-y| < \delta$. If $x, y > M$, then we have that $|f(x)-f(y)| < |f(x)|+|f(y)| < \epsilon/2 + \epsilon/2 = \epsilon$. If both $x, y < M$, then uniform continuity of f on $[0, M + 1]$ applies. If $y > M$ and $x \le M$, then since $\delta < 1$ and $|x - y| < \delta$, then $y < M + 1$ and again uniform continuity of f on $[0, M + 1]$ applies. Thus, f is uniformly continuous on all $[0, \infty)$ as desired. □

6.11.188 Problem.

Give examples of each of the following:

1. Composition of two functions, one uniformly continuous other is not uniformly continuous.
2. Composition of two functions, both are not uniformly continuous.

6.11.188.1 Solution.

1. Let $f, g : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x,\ g(x) = x^2; (f \circ g)(x) = x^2$ and $f \circ g$ is not uniformly continuous. Again, if $f, g : [0, \infty) \to \mathbb{R}$ defined by $f(x) = \sqrt{x},\ g(x) = x^2;\ ; (f \circ g)(x) = x$ is uniformly continuous.

2. Let $f, g : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2,\ g(x) = x^3; (f \circ g)(x) = x^6$ and $f \circ g$ is not uniformly continuous. Again, if $f, g : (0,1) \to \mathbb{R}$ defined by $f(x) = \frac{1}{x} = g(x)$; $(f \circ g)(x) = x$ is uniformly continuous. $\square$

6.11.189 Problem. Prove or disprove that the function $f(x) = \sin x^3/x, x > 0$ is uniformly continuous on $(0, \infty)$.

6.11.189.1 Solution. We wish to show that $f(x)$ is uniformly continuous on (0,1). First, define g on [0,1) as follows:

$$g(x) = \begin{cases} f(x) & \text{if } x > 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Now, by L'Hospital's rule, we find that

$$\lim_{x \to 0+} f(x) = 0.$$

Thus, since f is continuous on (0,1) and $\lim_{x \to 0+} g(x) = g(0)$ then g is continuous on $[0, \infty)$. Therefore, g is uniformly continuous on any compact subset of [0,1). Moreover, for all $x > y > 0$, observe that

$$|g(x) - g(y)| \leq |g(x)| + |g(y)| \leq \frac{1}{x} + \frac{1}{y} \leq \frac{2}{y}.$$

Now, let $\epsilon > 0$ be fixed. Let $x_0 > 0$ be so that $\frac{2}{x_0} < \epsilon$. Hence, we have that g is uniformly continuous on $[0, x_0]$ and that for all $x, y > x_0$, we have

$$|g(x) - g(y)| = |f(x) - f(y)| \leq \frac{2}{x_0} < \epsilon.$$

Thus, g is uniformly continuous on [0,1) and so f is uniformly continuous on (0,1) as desired. $\square$

6.11.190 Problem. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies $f(f(x)) = x$. Let $a \in \mathbb{R}$ and suppose $a < f(a)$. Show that $f[a, f(a)] = [a, f(a)]$.

6.11.190.1 Solution. Let $y \in f[a, f(a)]$, there exists $x \in [a, f(a)]$ such that $y = f(x)$ then $f(y) = f(f(x)) = x$. Hence $a \leq f(y) \leq f(a)$ implies $f[a, f(a)] \subseteq [a, f(a)]$. Again, let $v \in [a, f(a)]$ then by IVP $\exists\ u \in [a, f(a)]$ such that $f(u) = v \Rightarrow v \in f([a, f(a)])$. Thus $f[a, f(a)] = [a, f(a)]$. $\square$

6.11.191 Problem. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and increasing and $E \subset \mathbb{R}$ be bounded. Prove that $f(\sup E) = \sup f(E)$.

6.11.191.1 Solution. E is bounded implies $\sup E$ exists, let $\sup E = \alpha$, two cases can arise, (1) $\alpha \in E$ and (2) $\alpha \notin E$. Let $\sup f(E) = \beta$, so

$$\begin{aligned} & x \leq \alpha \ \forall\ x \in E \\ \Rightarrow & f(x) \leq f(\alpha) \ \forall\ x \in E \text{ as } f \text{ is increasing} \\ \Rightarrow & \sup_{x \in E} f(x) \leq f(\alpha) \\ \Rightarrow & \beta \leq f(\alpha). \end{aligned}$$

If possible, suppose $\beta < f(\alpha)$ and let $\epsilon < f(\alpha) - \beta$. Since f is continuous at α so $\exists\, \delta > 0$ such that $|x - \alpha| < \delta$ and $x \in E$ implies

$$\begin{aligned} &|f(x) - f(\alpha)| < \epsilon < f(\alpha) - \beta \\ \Rightarrow &f(\alpha) - f(x) < \epsilon < f(\alpha) - \beta \\ \Rightarrow &f(x) > \beta = \sup f(E), \text{ a contradiction.} \end{aligned}$$

Hence $f(\alpha) = \beta$ i.e. $f(\sup E) = \sup f(E)$. □

6.11.192 Problem. Let $f : [a, b] \to \mathbb{R}$ be increasing and continuous and $A \subseteq \mathbb{R}$; $g : A \to \mathbb{R}$, where $g(A) \subseteq [a, b]$. Prove that

$$\sup_{x\in A}(f \circ g)(x) = f(\sup_{x\in A} g(x))$$

and

$$\inf_{x\in A}(f \circ g)(x) = f(\inf_{x\in A} g(x)).$$

6.11.192.1 Solution. Let $\sup_{x\in A} g(x) = \alpha$, then $g(x) \le \alpha \in [a, b]$ for all $x \in A$. As f is increasing, so $f(g(x)) \le f(\alpha)\ \forall\, x \in A \Rightarrow f(\alpha)$ is an upper bound of $\{(f \circ g)(x); x \in A\}$. Since f is continuous at α, for $\epsilon > 0\ \exists\, \delta > 0$ such that $x \in (\alpha - \delta, \alpha + \delta) \Rightarrow f(x) \in (f(\alpha) - \epsilon, f(\alpha) + \epsilon)$. Now, α being a supremum, so $\exists\, y \in A$ such that $g(y) > \alpha - \delta$ and this implies $f(g(y)) > f(\alpha) - \epsilon$ which shows that $f(\alpha)$ is the supremum of $\{(f \circ g)(x); x \in A\}$. Hence

$$\sup_{x\in A}(f \circ g)(x) = f(\sup_{x\in A} g(x)).$$

Similiarly for infimum. □

6.11.193 Problem. Suppose $f : \mathbb{R} \to \mathbb{R}$ be continuous, such that $\lim_{x\to-\infty} f(x) = 0$ and $\lim_{x\to\infty} f(x) = 0$, prove that f is bounded and attains either a maximum or a minimum on $\mathbb{R}$.

6.11.193.1 Solution. Let $\epsilon > 0$, then $\exists\ M_1, M_2 > 0$ such that $|f(x)| < \epsilon,\ \forall\ x < -M_1$ and $|f(x)| < \epsilon,\ \forall\ x > M_2$. Since f is continuous on $[M_1, M_2]$, thus f is bounded on $\mathbb{R}$. □

6.11.8 Note. If $f(x) > 0,\ \forall\ x \in \mathbb{R}$, then f does not attain a minimum, and if $f(x) < 0, \forall\ x \in \mathbb{R}$, then f does not attain a maximum,and if $\exists\ x \in \mathbb{R}$ such that$f(x) = 0$, then f attains both maximum and minimum.
Consider$f(x) = e^{-x^2}, g(x) = -e^{-x^2}$,

and $h(x) = \begin{cases} = e^x, \text{if } x \le \text{-log2}, \\ \dfrac{1}{2} - \dfrac{1}{2\text{log2}}(x + \text{log2}), \text{if -log2} < x < \text{log2} \\ -e^{-x}, \text{if } x > \text{log2}. \end{cases}$

6.11.194 Problem. Let $f : [a, b] \to \mathbb{R}$, such that $\forall\ x \in [a, b]\ \exists\ \delta_x > 0$ and f is bounded on $N(x; \delta_x) \cap [a, b]$. Prove that f is bounded on $[a, b]$.

6.11.194.1 Solution. Let $\mathcal{A} = \{N(x; \delta_x); x \in [a, b]$ and f is bounded on $[a, b] \cap N(x; \delta_x)\}$. Now $\mathcal{A}$ becomes an open cover of $[a, b]$. Since $[a, b]$ is compact, then $\exists\ x_1, x_2, .., x_n$ such that $[a, b] \subset \bigcup_{i=1}^{n} N(x_i; \delta_{x_i})$. Suppose $|f(x_i)| \le\ M_i; i = 1, ..n.$.
Then $|f(x)| \le\ M = \max\{M_i; i = 1, ..n.\}\forall\ x \in [a, b]$. Thus f is bounded on $[a, b]$. □

6.11.194.2 Solution. Let f be unbounded on $[a,b]$. So for each $n \in \mathbb{N}, \exists\ x_n \in [a,b]$ such that $f(x_n) > n$. Since (x_n) is a bounded sequence on $[a,b]$, by B-W theorem there exists a subsequence (x_{n_k}) converges to c for some $c \in [a,b]$. By the condition $\exists\ \delta_c > 0$ such that f is bounded on $N(c;\delta_c)$. So let $|f(x)| \leq\ M_c,\ \forall\ x \in\ N(c;\delta_c)$. Now $x_{n_k} \to c \Rightarrow \exists\ m \in \mathbb{N}$ such that for $k \geq m, x_{n_k} \in N(c;\delta_c)$, then $\exists\ p \in \mathbb{N}$ such that $n_p \geq M_c$. If $t > \max\{p,m\}$, then $x_{n_t} \in\in N(c,\delta_c)$ and $f(x_{n_t}) > n_t \geq M_c$ contradicts $f(x_{n_t}) \geq M_c$. Hence f is bounded. □

6.11.195 Problem. Suppose that f is a real-valued function of a real variable, and that $f(x+y) = f(x)f(y)$ for all x and $y, f(1) \neq 0$, and $\lim_{x\to 0} f(x)$ exists. Prove that $\lim_{x\to 0} f(x) = 1$.

6.11.195.1 Solution. Since for all $x \in \mathbb{R}, x = 2y$ for some y, we have $f(x) = f(2y) = f(y)^2 > 0$. Further, $f(1) = f(0+1) = f(0)f(1)$, and since $f(1) \neq 0, f(0) = 1$. Now let $L = \lim_{x\to 0} f(x)$. Then

$$1 = f(0) = f(x-x) = f(x)f(-x) \to L^2.$$

Since L cannot equal -1 since this would imply $f(x) \leq 0$ for some x, we have $L = 1$. □

6.11.196 Problem. Let f be a continuous function that maps the closed unit interval $J = [0,1]$ into itself. Show that if $f(f(x)) = x$ for all $x \in J$, then either f is strictly increasing on J or f is strictly decreasing.

6.11.196.1 Solution. $f(a) = f(b) \Rightarrow f(f(a)) = f(f(b)) \Rightarrow a = b$; therefore f is injective on J. Since f is continuous, it assumes all intermediate values. It follows that if f were not strictly monotone then it could not be injective. □

6.11.197 Problem. Let $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0. & \text{if } x \in \mathbb{Q}^C \\ \frac{1}{q}, & \text{if } x = \frac{p}{q}, (p,q) = 1. \end{cases}$$

Show that f is continuous at every irrational number and discontinuous at every rational number.

6.11.197.1 Solution. If $a = p/q$ is rational, then if we take $\epsilon = 1/2q$, for every irrational x,

$$|f(x) - f(a)| = |0 - 1/q| = 1/q > 1/2q = \epsilon.$$

Since every interval $|x-a| < \delta$ contains irrational points, there is no $\delta > 0$ such that

$$|f(x) - f(a)| < \epsilon\ \forall\ |x-a| < \delta$$

Therefore f is discontinuous at every rational value of x. On the other hand, given any $\epsilon > 0$, we can find an integer Q such that $\epsilon > 1/Q$. For any irrational value a, consider the interval $|x-a| < 1/2$. In this interval there are at most q rational numbers of the form p/q, so that there are at most $Q(Q-1)/2$ such numbers for which $1/q > 1/Q$. If we take

$$\delta = \min_{q<Q} \left|\frac{p}{q} - a\right|$$

then $\delta \neq 0$. since a is irrational, and if $|x-a| < \min(1/2,\delta)$ $|f(x) - f(a)| = |0-0| < \epsilon$ when x is irrational; while if $x = p/q; q \geq Q$, so that

$$|f(x) - f(a)| = 1/q \leq 1/Q < \epsilon.$$

Therefore this function is continuous for every irrational value of x. □

6.11.197.2 Solution. Let $x_0 \in \mathbb{R}\backslash\mathbb{Q}$. Let us observe a sequence of rational numbers (r_k) converging to x_0, where $r_k = \frac{m_k}{n_k}$, and assume that $lcd(m_k, n_k) = 1$ for every $k \in \mathbb{N}$. But then $\lim_{k\to\infty} n_k = \infty$, hence

$$\lim_{k\to\infty} f(r_k) = \lim_{k\to\infty} f\left(\frac{m_k}{n_k}\right) = \lim_{k\to\infty} \frac{1}{n_k} = 0 = f(x_0).$$

It is trivial that for every sequence of irrational numbers (i_k) converging to (x_0), and we get

$$\lim_{k\to\infty} f(r_k) = 0 = f(x_0).$$

Thus f is continuous at every irrational number. Assume now that $x_0 \in \mathbb{Q}$. Then x_0 is a rational number of the form $x_0 = \frac{m}{n}$, where $lcd(m, n) = 1$. By the definition $f(x_0) = \frac{1}{n}$, Let us consider

$$r_k = \frac{mk+1}{nk},\ k \in \mathbb{N}$$

This sequence of rational numbers converges to x_0 as $k \to \infty$.

$$\lim_{k\to\infty} f(r_k) = \lim_{k\to\infty} \frac{1}{nk} = 0 \neq \frac{1}{n} = f(x_0).$$

Hence the function f has a discontinuity at every rational point. □

6.11.198 Problem. Let $\{r_n; n \in \mathbb{N}\}$ be an enumeration of the rational numbers in $\mathbb{R}$. For each $x \in \mathbb{R}$, we consider

$$L_x = \{n \in \mathbb{N}; r_n \le x\}.$$

Now define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \sum_{n\in L_x} \frac{1}{2^n}.$$

Show that f is strictly increasing, and has a jump discontinuity at every rational number and

$$f(x) \to 0 \text{ as } x \to -\infty \text{ and } f(x) \to 1 \text{ as } x \to \infty.$$

Also show that f is right continuous and $f|_{\mathbb{R}\backslash\mathbb{Q}}$ is continuous. (This gives, for example, a monotone function that is discontinuous at rational numbers and continuous at irrational numbers.)

6.11.198.1 Solution. We will first show that $f|_{\mathbb{R}\backslash\mathbb{Q}}$ is continuous. Let us fix an irrational number $e \in \mathbb{R} \setminus \mathbb{Q}$. Let $\epsilon > 0$ then we can find an $m \in \mathbb{N}$ such that

$$\sum_{n\ge m} \frac{1}{2^n} < \epsilon.$$

Let

$$\delta = \min\{|r_k - e|; r_k < e, k = 1, 2, .., m-1\}.$$

Suppose $x \in \mathbb{R}$ be such that $|x - e| < \delta$. If $x < e$, then $L_e \setminus L_x \subseteq \{m, m+1, ...\}$ and so

$$f(x) - f(e) \le \sum_{n\in L_e\setminus L_x} \frac{1}{2^n} \le \sum_{n\ge m} \frac{1}{2^n} < \epsilon.$$

If $x > e$, then $L_x \setminus L_e \subseteq \{m, m+1, ...\}$ and so

$$f(e) - f(x) \leq \sum_{n \in L_e \setminus L_x} \frac{1}{2^n} \leq \sum_{n \geq m} \frac{1}{2^n} < \epsilon.$$

Thus f is continuous at e. Thus $f|_{\mathbb{R}\setminus\mathbb{Q}}$ is continuous. Now we prove that f is right hand continuous at rational numbers. Let $\epsilon > 0$ and a rational number x, let $(x_n) \in \mathbb{R}^{\mathbb{N}}$ such that

$$x_n \to x \text{ as } n \to \infty.$$

Since f is strictly increasing, we have

$$f(r) < f(x_n) \; \forall \; n \in \mathbb{N}.$$

Then

$$|f(x_n) - f(x)| = f(x_n) - f(x) = \sum_{n \in S_{kx}} \frac{1}{2^n},$$

where

$$S_{kx} = \{n \in \mathbb{N}; x < r_k < x_k\}$$

Now choose an integer $m_0 \in \mathbb{N}$ such that

$$\sum_{n \geq m_0} \frac{1}{2^n} < \epsilon.$$

and consider the set $A \subseteq \mathbb{R}$, defined by

$$A = \bigcap_{\{n \in \mathbb{N}, n < m_0, r_n > x\}} (x, r_n)$$

Evidently $A = (x, r)$ for some $r \in \mathbb{Q}$. Moreover, every rational number in A is of the form r_n, with $n \geq m_0$. Because $x_k \to x+$ we can find an integer k_0 such that

$$x_k \in A \; \forall \; k \geq k_0.$$

Therefore

$$|f(x_k) - f(x)| < \epsilon \; \forall \; k \geq k_0.$$

and so f is right hand continuous at x. □

6.11.199 Problem. Construct a strictly increasing function that is continuous at each irrational and discontinuous at each rational.

6.11.199.1 Solution. Let $\{r_n; n \in \mathbb{N}\}$ be an enumeration of the rationals and define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \sum_{r_n < x} 1/2^n$. Note that $f(r_n-) = f(r_n) = f(r_n+) - 1/2^n$ and $f(x-) = f(x) = f(x+)$ for all $x \in \mathbb{R} \setminus \mathbb{Q}$. □

6.11.200 Problem. Let (r_n) be a sequence of rational numbers and a function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \sum_{\{n; r_n < x\}} \frac{1}{2^n}$$

is lower semicontinuous.

6.11.200.1 Solution. Suppose $y = r_m \in \mathbb{Q}$. Then $f(x) \geq \dfrac{1}{2^m} + f(y)\ \forall\ x > y$, so

$$\inf_{y<x<y+\delta} f(x) \geq \frac{1}{2^m} + f(y) > f(y)$$

for any $\delta > 0$. We now estimate $\inf_{y-\delta<x\leq y} f(x)$. If $x \leq y$, then

$$f(x) = f(y) - \sum_{\{n;x\leq r_n<y\}} \frac{1}{2^n}.$$

Suppose $y - \delta < x \leq y$, then $\{n; x \leq r_n < y\} \subset \{n; y - \delta < r_n < y\}$ and so

$$f(x) \geq f(y) - \sum_{\{n;y-\delta<r_n<y\}} \frac{1}{2^n}.$$

We claim that for each $N \geq 1\ \exists\ \delta > 0$ such that if $r_j \in (r_m - \delta, r_m)$ then $j \geq N$. Suppose by contradiction that this is false. Then $\exists\ N_0 \geq 1$ such that for each $\delta = 1/k\ \exists\ j(k)$ such that $r_{j(k)} \in (r_m - 1/k, r_m)$ and $j(k) < N_0$. Hence $j(k)$ takes only a finite umber of values between 1 and N_0, therefore there exists a subsequence (k_i) such that $k_i \to \infty$ as $i \to \infty$ with $j(k_i) = s$ and $r_s \in (r_m - 1/k_i, r_m)\ \forall\ i$. Letting $i \to \infty$ we get $r_m \leq r_s < r_m$, a contradiction.
Now given $\epsilon > 0, \exists\ N \geq 1$ such that $\sum_{n\geq N} \frac{1}{2^n} < \epsilon$, and by the claim there exists $\delta_0 > 0$ such that $\{n; y - \delta_0 < r_n < y\} \subset \{n; n \geq N\}$. consequently

$$\sum_{\{n;y-\delta_0<r_n<y\}} \frac{1}{2^n} \leq \sum_{n\geq N} \frac{1}{2^n} < \epsilon,$$

and so $f(x) \geq f(y) - \epsilon$ for $y - \delta < x \leq y$ and $0 < \delta \leq \delta_0$. Thus,

$$\inf_{y-\delta<x\leq y} f(x) \geq f(y) - \epsilon\ \forall\ 0 < \delta \leq \delta_0,$$

and so

$$\lim_{\delta\to 0} \inf_{y-\delta<x\leq y} f(x) \geq f(y) - \epsilon\ \forall\ \epsilon > 0.$$

6.11.201 Problem. Prove that, $\lim_{h\to 0} \omega_f(N(p;h) \cap [a,b])$ always exists. Explain why the limit in question is guaranteed to exist.

6.11.201.1 Solution. Hint. Since $\omega_f(N(p;h) \cap [a,b])$ is a decreasing function of h. In fact, $A \subseteq B$ implies $\omega_f(A) \leq \omega_f(B)$.

6.11.202 Problem. Let f be a uniformly continuous mapping from $\mathbb{R}$ into itself. Show that if $f(0) = 0$, there exists a positive real number $B > 0$ such that $|f(x)| \leq 1 + B|x|$ for every $x \in \mathbb{R}$.

6.11.202.1 Solution. There exists $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < 1$. Let $B = 1/\delta$. Take any $x > 0$ and let $n_x = [x/\delta]$ the greatest integer not exceeding x/δ. Then

$$\begin{aligned}|f(x)| = |f(x) - f(0)| \leq& |f(x) - f(n_x\delta)| + \sum_{i=1}^{n_x} |f(i) - f((i-1)\delta)| \\ \leq& 1 + n_x \leq 1 + B|x|.\end{aligned}$$

The proof for $x < 0$ is similar. $\square$

6.11.203 Problem. Let the real-valued function f on [0,1] have the following two properties:

1. If $[a,b] \subseteq [0,1]$, then $f([a,b])$ contains the interval with endpoints $f(a)$ and $f(b)$ (i.e., f has the Intermediate Value Property).

2. For each $c \in \mathbb{R}$, the set $f^{-1}(\{c\})$ is closed.

Prove that f is continuous.

6.11.203.1 Solution. Suppose that f is not continuous at $c \in [0,1]$. Then, $\exists\ \epsilon > 0$, such that there is a sequence (x_n) converging to $c \in [0,1]$ with $|f(x_n) - (f(c)| \geq \epsilon\ \forall\ n \in \mathbb{N}$, i.e. Hence $\forall\ n \in \mathbb{N}$, either $f(x_n) \geq f(c) + \epsilon > f(c)$ or $f(x_n) \leq f(c) - \epsilon < f(c)$. By the first condition, $\exists\ y_n$ such that y_n lies between c and x_n, i.e. $|y_n - c| < |x_n - c|$ and $f(y_n) = f(c) - \epsilon$ or $f(y_n) = f(c) + \epsilon$. Then, we can show that $y_n \to c$ and the set $\{y_n; i = 1, 2, ...\}$ is not closed, for, we observe that

$$c \notin \{y_n; i = 1, 2, ...\} = f^{-1}(f(c) - \epsilon) \cup f^{-1}(f(c) + \epsilon)$$

and this contradicts the second condition. □

6.11.204 Problem. Prove that a function f defined on an interval $\langle a, b\rangle$ is continuous if and only if:

1. f has the Cauchy property (that is, the image of each interval $\langle p, q\rangle \subseteq \langle a, b\rangle$ is an interval).

2. For each $c \in \mathbb{R}$, the set $f^{-1}(\{c\})$ is closed.

$\langle a, b\rangle$ stands for any one of the four kinds of intervals $[a, b], (a, b), [a, b)$, or $(a, b]$.

6.11.204.1 Solution. Hint: Prove that if $\underline{\lim}_{x\to x_0} f(x) < c < \overline{\lim}_{x\to x_0} f(x)$ and $c \neq f(x_0)$, then the set $f^{-1}(\{c\})$ is not closed. □

6.11.205 Problem. If a function f is defined on (a, b) and takes the same two distinct values in every arbitrary subinterval of (a, b) is totally discontinuous in (a, b).

6.11.205.1 Solution. Let A and B be the two constant values which f takes in every subinterval of (a, b). If $x_0 \in (a, b)$ then $\exists$ a nbhd. N of x_0 and a sequence (x_n) in N such that $x_n \to x$ and $\lim_{x_n\to x_0} f(x_n) = A$ and similarly $\exists$ a sequence (x'_n) in N such that $x'_n \to x_0$ and $\lim_{x'_n\to x_0} f(x'_n) = B$. But as $A \neq B$ it follows that $\lim_{x\to x_0} f(x)$ does not exist. Hence f is totally discontinuous in (a, b). □

6.11.206 Problem. Show that the function $f : (0, 1) \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0, & \text{if } x = .x_1x_2x_3.....\text{and the digits } x_1, x_3, x_5.. \\ & \text{in the odd places form a non-periodic sequence} \\ \frac{x_{2n}}{10} + \frac{x_{2n+2}}{10^2} + \frac{x_{2n+4}}{10^3} + ... & \text{if the digits } x_1, x_3, x_5.. \\ & \text{in the odd places form a periodic sequence} \\ & \text{from some point on, say after } x_{2n-1} \end{cases}$$

is totally discontinuous although it takes all values intermediate between any two of its values.

6.11.206.1 Solution. Let (a, b) be arbitrary subinterval of $(0, 1)$. We show that f takes all the values between 0 to 1 as x varies from a to b. It will follow then that f is totally discontinuous in $(0, 1)$. Let $y_0 \in (0, 1)$. We show that $\exists\ x_0 \in (a, b)$ such that $y_0 = f(x_0)$. Since $y_0 \in (0, 1)$, we can write $y_0 = 0.t_1t_2t_3.....$ The values of x may be written as

$$\begin{aligned} x = &\frac{x_1}{10} + \frac{x_3}{10^3} + \frac{x_5}{10^5} + ... + \frac{x_{2n-1}}{10^{2n-1}} + ... \\ &+ \frac{x_2}{10^2} + \frac{x_4}{10^4} + \frac{x_6}{10^6} + ... + \frac{x_{2n}}{10^{2n}} + ... \end{aligned} \tag{A}$$

If $f(x)$ is different from 0, the numerators in the first row must, from some point on, becomes periodic. Let x_{2n-1} be the last numerator before the periodicity begins. Suppose a point x' of (a, b) is taking the first $2n - 1$ terms of the fraction; that is, let

$$x' = \frac{x_1}{10} + \frac{x_2}{10^2} + \frac{x_3}{10^3} + ... + \frac{x_{2n-1}}{10^{2n-1}}.$$

Whatever digits be chosen as numerators following x_{2n-1}, the value of x can differ from x' by at most $\frac{1}{10^{2n-1}}$. By the proper choice of n we can make

$$\frac{1}{10^{2n-1}} < |b - x'|,$$

and hence the value of x lies within (a, b). In (A) we shall now replace those numerators following x_{2n-1} having even subscripts by the digits used in defining y_0; that is, we define x_0 by

$$x_0 = 0.x_1x_2........x_{2n-1}t_1x_{2n+1}t_2x_{2n+2}t_3......$$

The point x_0 then a point in the subinterval (a, b) and $f(x_0) = y_0$. But $y_0 \in (0, 1)$ and hence $f(x)$ must take all values between 0 and 1 as x is given the values in (a, b). Thus by the previous problem, f is totally discontinuous in (a, b). □

6.11.207 Problem. Show that a function $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if

1. $f(I)$ is an interval for each interval $I \subseteq \mathbb{R}$ and
2. $f^{-1}(\{y\})$ is closed for each $y \in \mathbb{R}$:

In fact, show that (2) may be replaced by $f^{-1}(\{y\})$ is closed for each $y \in \mathbb{Q}$.

6.11.207.1 Solution. Left to the reader.

6.11.208 Problem. Give an example of a function which satisfies Intermediate Value Property without being continuous.

6.11.208.1 Solution. Example: Consider the function $f : [0, 1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1, & \text{if } x = 0 \\ x, & \text{if } 0 < x < 1 \\ 0, & \text{if } x = 1. \end{cases} \quad \square$$

6.11.209 Problem. Is the condition "for any $c \in \mathbb{R}$ the set $f^{-1}(\{c\})$ is closed" sufficient for the continuity of f?

6.11.209.1 Solution. For example, a discontinuous function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 1 + x, & \text{if } x \text{ is irrational.} \end{cases} \qquad \square$$

6.11.210 Problem. Suppose that f is a continuous function on $\mathbb{R}$ which is periodic with period 1, i.e., $f(x + 1) = f(x)$. Show:

1. The function f is bounded above and below and achieves its maximum and minimum.

2. The function f is uniformly continuous on $\mathbb{R}$.

3. There exists a real number x_0 such that $f(x_0 + \pi) = f(x_0)$.

6.11.210.1 Solution.

1. Let f_1 be the restriction of f to [0,2]. The ranges of f and f_1 are the same by periodicity so f attains its extrema.

2. Let $\delta > 0$. f_1 is uniformly continuous, being a continuous function defined on a compact set so there is $\epsilon > 0$ such that

$$|f_1(a) - f_1(b)| < \epsilon \text{ for } a, b \in [0, 2], |a - b| < \epsilon.$$

   Let $x, y \in \mathbb{R}$ with $|x - y| < \epsilon$. Then there are $x_1, x_2 = x_1 + 1, y_1, y_2 = y_1 + 1 \in [0, 2]$ with $f(x_1) = f(x_2) = f(x), f(y_1) = f(y_2) = f(y)$ and $|f(x_i) - f(y_i)| < \epsilon$ for some choice of $i, j \in \{1, 2\}$, and the conclusion follows.

3. Let f attain its maximum and minimum at ξ_1 and ξ_2, respectively. Then

$$f(\xi_1 + \pi) - f(\xi_1) < 0 \text{ and } f(\xi_2 + \pi) - f(\xi_2) > 0;$$

   as f is continuous, the conclusion follows from the Intermediate Value Theorem. $\square$

6.11.211 Problem. Let f be a continuous real valued function defined on $[0, 1] \times [0, 1]$. Let the function g on [0,1] be defined by

$$g(x) = \max\{f(x, y); y \in [0, 1]\}.$$

Prove that g is continuous.

6.11.211.1 Solution. For each $y \in [0, 1]$, consider the function $g_y(x) = f(x, y)$. Then $g(x) = \sup g_y(x)$. The family $\{g_y\}$ is equicontinuous because f is uniformly continuous. It suffices then to show that the pointwise supremum of an equicontinuous family of functions is continuous. Let $\epsilon > 0$, $x_0 \in [0, 1]$. There is y_0 such that

$$g_{y_0}(x_0) \leq g(x_0) < g_{y_0}(x_0) + \epsilon.$$

Let $\delta > 0$ such that if $|r - s| < \delta$, then $|g_y(r) - g_y(s)| < \delta \ \forall \ y$, and $|x_0 - x_1| < \delta$. For some y_1, we have that

$$g_{y_1}(x_1) \leq g(x_1) < g_{y_1}(x_1) + \epsilon$$

Further, by equicontinuity of $\{g_y\}$, we have the two inequalities $|g_{y_0}(x_0)-g_{y_0}(x_1)|<\epsilon$ and $|g_{y_1}(x_0)-g_{y_1}(x_1)|<\epsilon$. By combining them we get

$$g_{y_0}(x_0)<g_{y_0}(x_1)+\epsilon<g(x_1)+\epsilon<g_{y_1}(x_1)+2\epsilon$$

and

$$g_{y_1}(x_1)<g_{y_1}(x_0)+\epsilon<g(x_0)+\epsilon<g_{y_0}(x_0)+2\epsilon.$$

These two inequalities imply $|g_{y_1}(x_1)-g_{y_0}(x_0)|<2\epsilon$. This, combined with the first two inequalities, shows that $|g(x_0)-g(x_1)|<3\epsilon$. Since this holds for all ϵ and x_0 and all x_1 close to x_0, thus g is continuous. □

6.11.212 Problem. Let f be a continuous function from $\mathbb{R}$ to $\mathbb{R}$ such that $|f(x)-f(y)|\geq|x-y|$ for all x and y. Prove that the range of f is all of $\mathbb{R}$.

6.11.212.1 Solution. The inequality given implies that f is one-to-one, so f is strictly monotone and maps open intervals onto open intervals, so $f(\mathbb{R})$ is open. Let $z_n=f(x_n)$ be a sequence in $f(\mathbb{R})$ converging to $z\in\mathbb{R}$. Then (z_n) is Cauchy, and, by the stated inequality, so is x_n. Let $x=\lim x_n$. By continuity we have $f(x)=f(\lim x_n)=\lim f(x_n)=z$, so $f(\mathbb{R})$ is also closed. Thus, $f(\mathbb{R})=\mathbb{R}$. □

6.11.213 Problem. Let $A=[a,b]$ and let $f:A\to A$ be a continuous function with the property that for all $x,y\in A, |f(x)-f(y)|\geq|x-y|$, ($f$ is expanding).

1. Prove that f is one-to-one and the inverse map $f^{-1}:f(A)\to A$ is continuous.
2. Show that $f(A)=A$.

6.11.213.1 Solution.

1. Suppose $x\neq y$ then $|f(x)-f(y)|\geq|x-y|>0$ implies $f(x)\neq f(y)$, i.e. f is one-one. Again, let $y_n\to b\in f(A)$. Since f is one-one, so, there exist $x_n\in A$ and $a\in A$ such that $y_n=f(x_n)$ and $b=f(a)$ which means $f(x_n)\to f(a)$, since f is continuous, hence $x_n\to a\Rightarrow f^{-1}(y_n)\to f^{-1}(b)$. Thus $f^{-1}:f(A)\to A$ is continuous.
2. Let $f(A)=[c,d]$ with $f(a)=c$ and $f(b)=d$ such that $a<c<d<b$ then $|f(b)-f(a)|=d-c<b-a$, a contradiction. □

6.11.214 Problem. Prove that a continuous function from $\mathbb{R}\to\mathbb{R}$ which maps open sets to open sets must be monotonic.

6.11.214.1 Solution. Suppose $f:\mathbb{R}\to\mathbb{R}$ is continuous, maps open sets to open sets but is not monotonic. Without loss of generality assume there are three real numbers $a<b<c$ such that $f(a)<f(b)>f(c)$. By Weierstrass Theorem f has a maximum, M, in $[a,c]$, which cannot occur at a or b. Then $f((a,c))$ cannot be open, since it contains M but does not contain $M+\epsilon$ for any positive ϵ. We conclude then that f must be monotonic. □

6.11.215 Problem. Either prove or disprove (by a counterexample) each of the following statements:

1. Let $f:\mathbb{R}\to\mathbb{R}, g:\mathbb{R}\to\mathbb{R}$ be such that

   $$\lim_{x\to a}g(x)=b \text{ and } \lim_{x\to b}f(x)=c$$

   Then $\lim_{x\to a}f(g(x))=c$.

2. If $f : \mathbb{R} \to \mathbb{R}$ is continuous and U is an open set in $\mathbb{R}$, then $f(U)$ is an open set in $\mathbb{R}$.

6.11.215.1 Solution.

1. This is false. For $f(t) = g(t) = \begin{cases} 0 & \text{if } t \neq 0 \\ 1 & \text{if } t = 0 \end{cases}$
   we have $\lim_{t\to 0} g(t) = \lim_{t\to 0} f(t) = 0$ but $\lim_{t\to 0} f(g(t)) = 1$.

2. This is false. $f(t) = t^2$ maps the open interval $(-1, 1)$ onto [0,1), which is not open. □

6.11.216 Problem. Let $f : [a, b] \to \mathbb{R}$ be bounded. Prove that the limit

$$\omega_f(p) = \lim_{h\to 0+} \omega_f(N(p; h) \cap [a, b])$$

always exists and that $\omega_f(p) = 0 \Leftrightarrow f$ is continuous at p.

6.11.216.1 Solution. Suppose $\omega_f(p) = 0$ and let $\epsilon > 0$. Then there exists $\delta(\epsilon) > 0$ such that if $0 < h < \delta$, then

$$\sup \{|f(x') - f(y')|; x', y' \in N(p; h) \cap [a, b]\} < \epsilon.$$

So, if $|p - x| < \delta$, and $x \in N(p; h) \cap [a, b]$ then $|f(p) - f(x)| < \epsilon$ implies that f is continuous at p. Conversely, suppose f is continuous at p and let $\epsilon > 0$. Then there exists $\delta > 0$ such that $|p - x| < \delta$ implies $|f(p) - f(x)| < \epsilon/2$, so for

$$\begin{aligned} &x, y \in N(p; h) \cap [a, b] \\ &|f(x) - f(y)| = |f(x) - f(p) + f(p) - f(y)| \\ &\leq |f(x) - f(p)| + |f(p) - f(y)| < \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

Therefore $\sup \{|f(x) - f(y)|; x, y \in N(p; \delta) \cap [a, b]\} < \epsilon$. Thus $\omega_f(p) = 0$. □

6.11.217 Problem. Let $f : [a, b] \to \mathbb{R}$ be bounded, and let $\epsilon > 0$ be given. Assume that $\omega_f(x) < \epsilon$ for every $x \in [a, b]$. Then there exists a $\delta > 0$ (depending only on ϵ) such that for every closed subinterval $T \subseteq [a, b]$, we have $\omega_f(T) < \epsilon$ whenever the length of T is less than δ.

6.11.217.1 Solution. For each $x \in [a, b]$ there exists $N_x = N(x; \delta_x)$ such that

$$\omega_f(N_x \cap [a, b]) < \omega_f(x) + [\epsilon - \omega_f(x)] = \epsilon.$$

The set of all balls $N(x; \delta_x/2)$ forms an open cover of $[a, b]$. By compactness, a finite number (say k) of these will cover $[a, b]$. Let their radii be $\delta_1/2, \delta_2/2, .., \delta_k/2$ and let $\delta = \min\{\delta_1/2, \delta_2/2, .., \delta_k/2\}$. When the interval T has length $< \delta$, then T is partly covered by at least one of these balls, say by $N(x_p; \delta_p/2)$. However, the ball $N(x_p; \delta_p)$ completely covers T (since $\delta_p \geq 2\delta$). Moreover, in $N(x_p; \delta_p) \cap [a, b]$ the oscillation of f is less than ϵ. This implies that $\omega_f(T) < \epsilon$ and the theorem is proved. □

6.11.218 Problem. Let f be defined and bounded on $[a, b]$. For each $\epsilon > 0$, define the set A_ϵ as $A_\epsilon = \{x; x \in [a, b],\ \omega_f(x) \geq \epsilon\}$. Then A_ϵ is a closed set.

6.11.218.1 Solution. Let x be an accumulation point of A_ϵ. If $x \notin A_\epsilon$, we have $\omega_f(x) < \epsilon$. Hence there exists $B_x = B(x; \delta_x)$, such that $\omega_f(B_x \cap [a, b]) < \epsilon$. Thus no points of B_x can belong to A_ϵ, contradicting the statement that x is an accumulation point of A_ϵ. Therefore, $x \in A_\epsilon$ and A_ϵ is closed. □

6.11.219 Problem.

1. Prove that every continuous function $f : [a, b] \to [a, b]$ has a fixed point.

2. Prove that if the function $f : [-a, a] \to [-a, a]$ is continuous, then the equation $f(x) = -x$ has at least one solution.

6.11.219.1 Solution.

1. Consider the function F defined by $F(x) = f(x) - x$ then $F(a) = f(a) - a, F(b) = f(b) - b$, so $F(a) > 0, F(b) < 0$ thus $\exists\, c \in (a, b)$ such that $F(c) = 0$, hence $f(c) = c$.

2. By the problem, $f(-a) = a$ and $f(a) = -a$ thus $f(-a)f(a) = -a^2 < 0$, hence $\exists\, c \in [-a, a]$ such that $f(c) = 0$. □

6.11.220 Problem. Prove that there exists no continuous function $f : \mathbb{R} \to \mathbb{R}$ such that

$$(f \circ f)(x) = -x^3 \ \forall\, x \in \mathbb{R}.$$

6.11.220.1 Solution (Hint: If a function g is strictly monotone, then $g \circ g$ is strictly increasing.)**.** □

6.11.221 Problem. Find all continuous functions $f(x)$ whose graph G (of $y = f(x)$) has the following property: For each chord C of G, if C's projection onto the x-axis has length d^2, then C's midpoint lies d units above G.

6.11.221.1 Solution. Let $(a, f(a))$ and $(b, f(b))$ be arbitrary points on the graph with $b > a$ and let $c = \frac{a+b}{2}$. We are given that

$$\frac{f(a) + f(b)}{2} - f(c) = (b - a)^2$$

or equivalently,

$$\frac{\dfrac{f(b) - f(c)}{b - c} - \dfrac{f(c) - f(a)}{c - a}}{b - a} = 8.$$

Take the limit as $b \to a$ (and $b \to c$, $c \to a$) to see that $f''(a) = 8$. Since a is arbitrary, we have $y = 4x^2 + Ax + B$. □

6.11.222 Problem. Let A and B be disjoint subsets of $\mathbb{R}$ and $f : A \cup B \to \mathbb{R}$ a continouus function. Assume that f is uniformly continuous on A and on B. Is it true that f is uniformly continuous on $A \cup B$?

6.11.222.1 Solution. Let $A = (0, 1)$ and $B = (1, 2)$, and define $f : A \cup B \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0, & x \in A \\ 1, & x \in B \end{cases}$$

It is easy to show that f is a continuous function. To show that f is uniformly continuous on A, let $\epsilon > 0$ and take any $\delta > 0$. Then $x, y \in A$ and $|x - y| < \delta$ implies $|f(x) - f(y)| = 0 < \epsilon$. Similarly f is continuous on B. Now we show that f is not uniformly continuous function on $A \cup B$, let $\epsilon = 1$ and take any $\delta > 0$. Let $x = 1 - \min\{1/2, \delta/4\}$ and $y = 1 + \min\{1/2, \delta/4\}$. Then $x, y \in A \cup B$ and $|x - y| < \delta$ but $|f(x) - f(y)| = 1 \geq \epsilon$. □

6.11.223 Problem. (Knaster's Fixed Point Theorem). Prove that every increasing function $f : [a, b] \to [a, b]$ admits at least a fixed point. Indicate an example where the set of fixed points is countable.

6.11.223.1 Solution. Hint: Consider the point $c = \sup\{x \in [a, b]; x \leq f(x)\}$. □

6.11.224 Problem. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz. Show that f extends to a continuous function $h : \mathbb{R} \to \mathbb{R}$. Is h unique? Explain.

6.11.224.1 Solution. Hint: Given $x \in \mathbb{R}$, choose a sequence of rationals (r_n) converging to x and argue that $h(x) = \lim_{n\to\infty} f(r_n)$ exists and is actually independent of the sequence (r_n).

6.11.225 Problem. Does there exist a function $f : \mathbb{R} \to \mathbb{R}$, such that f is continuous at

1. a point.
2. finitely many points.
3. countably many points with no limit point.
4. countably many points with limit point(s).
5. countably many points dense in $\mathbb{R}$.
6. irrationals.
7. uncountably many points dense in $\mathbb{R}$.

6.11.225.1 Solution.

1. Function defined by $f(x) = \begin{cases} x & \text{if } x \text{ is rational,} \\ 1 - x & \text{if } x \text{ is irrational} \end{cases}$
2. Let $S = \{x_1, x_2, ..., x_n\}$ be any finite subset of $\mathbb{R}$. Define a function f by
$$f(x) = \begin{cases} 1 - (x - x_1)(x - x_2)....(x - x_n) & \text{if } x \text{ is rational,} \\ 1 + (x - x_1)(x - x_2)....(x - x_n) & \text{if } x \text{ is irrational.} \end{cases}$$
3. Consider $f(x) = [x]$.
4. Let f be defined on $[0, 1]$ by
$$f(x) = \begin{cases} 0, & \text{if } x = 0 \\ (-1)^{r-1}, & \text{if } \frac{1}{r+1} < x \leq \frac{1}{r} \text{ for } r = 1, 2, ... \end{cases}$$
f is continuous on $[0, 1]$ except at the points $0, 1, \frac{1}{2}, \frac{1}{3}, ...$ The set of points of discontinuity of f has only one limit point 0. Also f is bounded on [0,1].
or
$$f(x) = \begin{cases} 0, & \text{if } x = 0 \\ \frac{1}{2^{n-1}}, & \text{if } \frac{1}{2^n} < x \leq \frac{1}{2^{n-1}} \text{ for } n = 1, 2, ... \end{cases}$$
f is continuous on $[0, 1]$ except at the points $0, \frac{1}{2}, \frac{1}{2^2}, ...$ The set of points of discontinuity of f has only one limit point 0. Also f is bounded on [0,1].

5. Left to the reader.

6. Thomae's function;

7. Thomae's function; □

6.11.226 Problem. Let $I = [a, b]$, and $f : I \to \mathbb{R}$ be strictly monotone and continuous on I. Then there exists an inverse function $g : J \to \mathbb{R}$ where $J = f(I)$ such that

1. g is strictly monotone in J,

2. g is continuous on J.

6.11.226.1 Solution. Suppose that f is strictly monotone on I. Let $x, y \in I$ with $x < y$, then $f(x) < f(y)$ which shows that $x \neq y \Rightarrow f(x) \neq f(y)$ i.e f is one-one, and $f : I \to f(I)$ is surjective. Hence f is bijective so there exists an inverse function $g : f(I) \to I$ such that $y \in J = f(I)\ \exists\ x$ such that $y = f(x) \Leftrightarrow g(y) = x$. We prove that g is increasing (strictly). Let $p, q \in f(I)$ then $\exists\ x, y \in I$, such that $f(x) = p, f(y) = q$ with $g(p) = x$ and $g(q) = y$. So $p < q \Rightarrow f(x) < f(y) \Rightarrow x < y \Rightarrow g(p) < g(q)$ implies g is strictly increasing. Since J is bounded and closed, let (y_n) be a sequence in J converging to $y \in J$ and let $y_n = f(x_n)$,then $g(y_n) = x_n$ is also a sequence in I. Since x_n is bounded and infinite in I, x_n is convergent in I. Suppose $x_n \to x \in I$, by continuity of $f, f(x_n) \to f(x)$ i.e. $y_n \to f(x)$ since $y_n \to y$ then $y = f(x)$ so $g(y) = x$. Hence $x_n \to x$ means $g(y_n) \to g(y)$. This shows that g is continuous on J. □

6.11.227 Problem. Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \lim_{n\to\infty} \left[\lim_{t\to 0} \frac{\sin^2(n!\pi x)}{\sin^2(n!\pi x) + t^2} \right]$$

is equal to 0 when x is rational and equal to 1 when x is irrational. Hence show that the f is totally discontinuous.

6.11.227.1 Solution. Let $x = p/q$ be a rational number. Then by taking n sufficiently large $n!\pi x$ can be made an integral multiple of π so that $\sin(n!\pi x) = 0$. Hence

$$f(x) = \lim_{t\to 0} \frac{0}{0 + t^2} = 0, \text{ when } x \text{ is rational.}$$

If x is irrational, then $0 < \sin^2 n!\pi x < 1$.

$$f(x) = \lim_{n\to\infty} \left[\lim_{t\to 0} \frac{1}{1 + \frac{t^2}{\sin^2(n!\pi x)}} \right] = \frac{1}{1+0} = 1.$$

6.11.228 Problem. Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \lim_{t\to\infty} \frac{(1 + \sin \pi x)^t - 1}{(1 + \sin \pi x)^t + 1}$$

is discontinuous at the points $x = 0, 1, 2, ..n, ..$

6.11.228.1 Solution. At $x = 0, 1, 2, ..n, ..,$ we have $\sin \pi x = 0$, so that

$$f(x) = \lim_{t\to\infty} \frac{(1+0)^t - 1}{(1+0)^t + 1} = 0$$

at these values. Now, if $2m < x < 2m+1$ (m being an integer), then $\sin \pi x$ is positive. Hence for such values of x, we have

$$f(x) = \lim_{t\to\infty} \frac{1 - \frac{1}{(1+\sin \pi x)^t}}{1 + \frac{1}{(1+\sin \pi x)^t}} = 1,$$

and if $2m+1 < x < 2m+2$, $\sin \pi x$ is negative and so $\lim_{t\to\infty}(1+\sin \pi x)^t = 0$. Therefore $f(x) = -1$ for these values of x. Hence if x is an even integer, then $f(x) = 0, f(x+0) = 1$ and $f(x-0) = -1$, and if x is an odd integer, then $f(x) = 0, f(x+0) = -1$ and $f(x-0) = 1$. Hence f has the discontinuities of the first kind at $x = 0, 1, 2, 3, ..n, ...$ □

6.11.229 Problem. If a function f is uniformly continuous on a bounded interval I, then f is bounded on I.

6.11.229.1 Solution. Here we suppose that I is one of $(a, b), [a, b], [a, b)$, or $(a, b]$. To check that f is bounded, choose $\delta > 0$ so that $|f(x) - f(y)| < 1$ whenever $x, y \in I$ and $|x - y| < \delta$. There is a finite set $a = x_0 < x_1 < ... < x_n = b$ such that $|x_i - x_{i-1}| < \delta$ for $i = 1, 2, .., n$. Our definition of δ implies that f is bounded on each of the intervals $[x_{i-1}, x_i] \cap I$. Let

$$\begin{aligned}
m_i &= \inf\{f(x); x_{i-1} \le x \le x_i, x \in I\}\\
M_i &= \sup\{f(x); x_{i-1} \le x \le x_i, x \in I\}\\
m &= \min\{m_1, ..., m_n\}\\
M &= \max\{M_1, ..., M_n\}.
\end{aligned}$$

Then, for every $x \in I, m \le f(x) \le M$, so f is bounded on I. □

6.11.230 Problem. Any function that is continuous on (0,1) but unbounded cannot be uniformly continuous there. Give an example of a continuous function on (0,1) that is bounded, but not uniformly continuous.

6.11.230.1 Solution. $f(x) = \sin\left(\frac{1}{x}\right)$. □

6.11.231 Problem. Use Baire's Category Theorem to show that the set of all rationals Q is not the intersection of a countable collection of open sets. Use this result to show that the set of irrationals is not the union of a countable collection of closet sets.

6.11.231.1 Solution. Assume not. Then there exists (O_n) a sequence of open sets such that $\mathbb{Q} = \bigcap_{n=1}^{\infty} O_n$. Since $\mathbb{Q}$ is countable, then we may write $\mathbb{Q} = \{r_n; n \in \mathbb{N}\}$. Let $\hat{O_n} = O_n \setminus \{r_n\}$, for $n \in \mathbb{N}$. It is clear that $\hat{O_n}$ is open as an intersection of two open sets. Since $\mathbb{Q} \subseteq O_n$, then O_n is dense in $\mathbb{R}$ and consequently $\hat{O_n}$ is also dense in $\mathbb{R}$, for any $n \in \mathbb{N}$. Baire's Category Theorem implies that $\bigcap_{n=1}^{\infty} \hat{O_n}$ is not empty and is dense in $\mathbb{R}$. But this contradicts

$$\bigcap_{n=1}^{\infty} \hat{O_n} = \bigcap_{n=1}^{\infty} O_n \setminus \mathbb{Q} = \emptyset$$

Finally assume that the set of irrationals is the union of a countable collection of closed sets. Then by taking the complement one can easily prove that $\mathbb{Q}$ is the intersection of a countable collection of open sets. Contradiction.
This conclusion means that $\mathbb{Q}$ is not G_δ-set and $\mathbb{R}\setminus\mathbb{Q}$ not an F_σ.

6.11.232 Problem. Let C be a subset of $\mathbb{R}$. Define the **Cantor-Bendixson derivative** of C, denoted C', by

$$C' = \{x \in \mathbb{R};\ x \text{ is an accumulation point of } C\}.$$

Show that C' is closed, and if $C' \neq \emptyset$, then C is infinite.

6.11.232.1 Solution. If $C' = \emptyset$, we have nothing to prove. So assume $C' \neq \emptyset$. Let us prove that C' is closed. Let $a \notin C'$,then a is not a limit point of C. Hence there exists $\epsilon > 0$ such that $(a-\epsilon, a+\epsilon)\cap C$ does not contain a point different from a. Since $a \notin C$, we have $(a-\epsilon, a+\epsilon)\cap C = \emptyset$. In fact, we have $(a-\epsilon, a+\epsilon)\cap C' = \emptyset$. Indeed assume not, i.e., $(a-\epsilon, a+\epsilon)\cap C' \neq \emptyset$. Let $a^* \in (a-\epsilon, a+\epsilon)\cap C'$. Since $(a-\epsilon, a+\epsilon)$ is open, there exists $\delta > 0$ such that $(a^*-\delta, a^*+\delta) \subseteq (a-\epsilon, a+\epsilon)$. Since a^* is a limit point of C, $(a^*-\delta, a^*+\delta)\cap C \neq \emptyset$, which in turns implies $(a-\epsilon, a+\epsilon)\cap C \neq \emptyset$, a contradiction. Hence $\mathbb{R}\setminus C'$ is open or equivalently C' is closed. Finally let us prove that C is infinite. Assume not. So $C = \{c_1, ..., c_n\}$. Since C' is not empty, let $c^* \in C'$. Let $\epsilon = \min\{|c^* - c_i|; c_i \neq c^*, i = 1, ..., n\}$. It is clear that $\epsilon > 0$ and $(c^*-\epsilon, c^*+\epsilon)\cap C = \emptyset$, a contradiction. Hence C is not finite. □

6.11.233 Problem. A subset P of $\mathbb{R}$ is said to be **perfect** if and only if $P' = P$, where P' is the Cantor–Bendixson derivative of P. Show that any nonempty perfect subset of $\mathbb{R}$ is not countable.

6.11.233.1 Solution. Let $P \subseteq \mathbb{R}$ be a nonempty perfect set. Since P has a limit point (being not empty), P is infinite and is closed. Assume P is countable. Write $P = \{p_n; n \in \mathbb{N}\}$. Since p_0 is a limit point of $P, P\cap(p_0-1, p_0+1)$ is not empty and is infinite. Take a $p \in P\cap(p_0-1, p_0+1)$, with $p \neq p_0$. Then there exists an open interval I_1 which contains p, such that $\overline{I_0}$ does not contain p_0 and $\overline{I_0} \subseteq (p_0-1, p_0+1)$, where $\overline{I_0}$ denotes the closure of I_0. Since I_1 contains a limit point of P, it contains infinitely many points of P. In particular, it contains a point different from p_1. Since I_1 is open, it will contain an open interval I_2, which contains a point from P such that $\overline{I_2}$ does not contain p_1 and $\overline{I_2} \subseteq I_1$. By induction, we will construct a sequence of open intervals $I_n \subseteq (p_0-1, p_0+1)$ such that

1. $p_n \notin \overline{I_n}$.
2. $\overline{I_{n+1} \subseteq I_n}$
3. $I_n \cap P \neq \emptyset$.

for any $n \in \mathbb{N}$. The sequence $(\overline{I_n}\cap P)$ is a decreasing sequence of bounded closed nonempty sets. Hence $I = \cap_{n\in\mathbb{N}}\overline{I_n}\cap P \neq \emptyset$. Let $p^* \in I$ then $p^* \in \overline{I_n}\cap P$ for any $n \in \mathbb{N}$. Hence $p^* \neq p_n$, for any $n \in \mathbb{N}$. So $P\setminus\{p_n; n \in \mathbb{N}\} \neq \emptyset$. Contradiction. □

6.11.234 Problem. A point a is a condensation point of $A \subseteq \mathbb{R}$ if for any $\epsilon > 0$, the set $(a-\epsilon, a+\epsilon)\cap A$ is infinite not countable. Let $P = \{x \in \mathbb{R}; x \text{ is a condensation point of } A\}$. Show that P is either empty or perfect.

6.11.234.1 Solution. Assume P not empty. Before we prove that P is perfect, let us prove that $C = A \setminus P$ is countable. Indeed, let $a \in C$, then there exists $r_1 < a < r_2$ with $r_1, r_2 \in \mathbb{Q}$ such that $(r_1, r_2) \cap A$ is countable. Hence

$$C \subseteq \{(q_1, q_2) \cap A; q_1, q_2 \in \mathbb{Q} \text{ such that } (q_1, q_2) \cap A \text{ is countable.}\}$$

Since a countable union of countable sets is countable, we conclude that C is countable. Next let us prove that P is perfect. Clearly any condensation point of A is also a limit point of A. The converse is not true in general. Indeed, if we take $A = \{1/n; n \geq 1\}$, then 0 is a limit of A but it is not a condensation point since A is countable. Let $a \in P'$, i.e., a is a limit point of P. Let $\epsilon > 0$, then there exists $p \in (a - \epsilon, a + \epsilon) \cap P$, such that $p \neq a$. Since $(a - \epsilon, a + \epsilon)$ is open, there exists $\delta > 0$ such that $(p - \delta, p + \delta) \subseteq (a - \epsilon, a + \epsilon)$. Since $(p - \delta, p + \delta) \cap A$ is infinite not countable, then $(a - \epsilon, a + \epsilon) \cap A$ is infinite not countable, i.e., $a \in P$. So $P' \subseteq P$. Let $p \in P$. Then for any $\epsilon > 0$, $(p - \epsilon, p + \epsilon) \cap A$ is infinite and not countable. Also $(p - \epsilon, p + \epsilon) \cap C$ is countable. Since

$$(p - \epsilon, p + \epsilon) \cap A = ((p - \epsilon, p + \epsilon) \cap P) \cup ((p - \epsilon, p + \epsilon) \cap C)$$

we conclude that $(p - \epsilon, p + \epsilon) \cap P$ is infinite and not countable. Hence p is a limit point of P, i.e., $p \in P'$. Therefore we have $P = P'$, or P is perfect. □

6.11.235 Problem. Let C be a closed subset of $\mathbb{R}$. Show that $C = P \cup F$, where P is perfect, F is countable, and $P \cap F = \emptyset$. This is known as the **Cantor-Bendixson** Theorem.

6.11.235.1 Solution. Let P be the set of all condensation points of C. Let $F = C \setminus P$. Then $C = P \cup F$. We have $P \cap F = \emptyset$. According to previous problem, P is perfect, and F is countable. □

6.11.236 Problem. Suppose that $f : [0, 1] \to \mathbb{R}$ is continuous and has a local maximum at each point in [0,1]. Prove that f is constant.

6.11.236.1 Solution. Since f is continuous on the compact set [0,1], by extreme value theorem it attains absolute minimum, say, m, at some $c \in [0, 1]$. Let $K = f^{-1}[\{m\}]$. Since $c \in K$, K is nonempty. It suffices to show that $K = [0, 1]$. Since f is continuous and K is the pull-back of a singleton, which is closed in R, K is closed in [0,1]. Now since [0,1] is connected, any nonempty subset of [0,1] which is both open and closed must be the whole space [0,1]. As we know K is closed in [0,1], it suffices to show that K is also open in [0,1]. To this end, let $y \in K$. Since f has a local minimum at y, there is $\epsilon > 0$ such that f has absolute maximum at y on $U = (y - \epsilon, y + \epsilon) \setminus [0, 1]$. But since $y \in K$, f has absolute minimum on [0,1] at y. Thus f must be constant on the relative ϵ-ball U of y. But since $f(y) = m$, f must be identically m on U. Hence $U \subseteq K$. So y is an interior point of K. Since $y \in K$ was arbitrary, K is open in [0,1]. Thus $K = [0, 1]$. This shows the assertion. □

6.11.237 Problem. Let f and g be continuous on [0,1], and suppose that $f(0) < g(0), f(1) > g(1)$. Show that there exists x in (0,1) such that $f(x) = g(x)$. Deduce that the equation

$$\frac{x + 1}{3} = \sin\left(\frac{\pi x}{2}\right)$$

has a solution in (0,1).

6.11.237.1 Solution. The function $f - g$ is continuous, and $(f - g)(0) < 0, (f - g)(l) > 0$. Hence, by the intermediate value theorem, there exists $c \in (a, b)$ such that $(f - g)(c) = 0$. □

6.11.238 Problem. Let $a, b > 1$, and let f be a bounded function on [0,1] such that

$$f(ax) = bf(x) \quad 0 \leq x \leq 1/a.$$

Show that f is continuous at 0.

6.11.238.1 Solution. Putting $x = 0$ in the functional equation, we find that $f(0) = f(a.0) = bf(0)$, where $b > 1$, and this gives a contradiction unless $f(0) = 0$. Let $|f(x)| < M \ \forall\ x \in [0,1]$. Let $f > 0$ be given, and let $n \in \mathbb{N}$. Then $f(a^n x) = b^n f(x) \ \forall\ x \in [0, a^{-n}]$, and so there exists $N \in \mathbb{N}$ such that

$$|f(x) - f(0)| = |f(x)| = b^{-n}|f(a^n x)| < Mb^{-n} \leq \epsilon$$

if $n > N$. Hence, taking $\delta = a^{-N}$, we see that $|f(x) - f(0)| < \epsilon \ \forall\ x \in (-\delta, \delta) \cap dom f$. Thus f is continuous at 0. □

6.11.239 Problem. Let $f : (-1,1) \to \mathbb{R}$ be continuous at 0, and suppose that $f(x) = f(x^2) \ \forall\ x \in (-1,1)$. Show that $f(x) = f(0) \ \forall\ x \in (-1,1)$.

6.11.239.1 Solution. Consider the sequence (x_n) defined by

$$x_n = x^{2^{n-1}}$$
$$\left(x, x^2, x^4, x^8, ...\right)$$

where $x \in (-1,1)$. Then $x^{2^{n-1}} \to 0$, and so, by continuity at 0, the sequence $\left(f\left(x^{2^n}\right)\right) \to f(0)$. But $\left(f\left(x^{2^n}\right)\right)$ is the constant sequence $(f(x))$, with limit $f(x)$, and so we conclude that $f(x) = f(0) \ \forall\ x \in (-1,1)$. □

6.11.240 Problem. Show that $\mathbb{R}$ cannot be covered by a countable family consisting of more than one closed disjoint sets.

6.11.240.1 Solution. This is Sierpinski's result. Note its relationship to the connectedness property of $\mathbb{R}$. Assume that $\mathbb{R}$ is the union of disjoint closed sets $\{C_i; i = 0, 1, 2, ...\}$. Choose $x < y \in \mathbb{R}$ so that $x \in C_0, y \notin C_0$. Put $a_0 = \sup(C_0 \cap (-\infty, y))$. Then $a_0 \in C_0$ and $C_0 \cap (a_0, y) = \emptyset$. Let i_1 be the smallest positive integer such that $C_{i_1} \cap (a_0, y) = \emptyset$, and put $b_0 = \inf(C_{i_1} \cap (a_0, y))$. Then $b_0 \in C_{i_1}, a_0 < b_0$, and the sets $C_i, i \leq i_1$ do not meet (a_0, b_0). Let i_2 be the smallest positive integer such that $C_{i_2} \cap (a_0, b_0) \neq \emptyset$, and put $a_1 = \sup(C_{i_2} \cap (a_0, b_0))$. Then $a_1 \in C_{i_2}$, and $a_0 < a_1 < b_0$. Moreover, the sets $C_i, i \leq i_2$ do not meet (a_1, b_0). Let i_3 be the smallest positive integer such that $C_{i_3} \cap (a_1, b_0) \neq \emptyset$, and put $b_1 = \inf(C_{i_3} \cap (a_1, b_0))$. Then $b_1 \in C_{i_3}$, and $a_0 < a_1 < b_1 < b_0$. Moreover, the sets C_i for $i \leq i_3$ do not meet (a_1, b_1). If we keep on going, we get a sequence of intervals $\{[a_i, b_i]; i = 0, 1, 2, ..., \}$ that is nested, and thus there is z such that $z \in [a_i, b_i]$ for all $i \geq 0$. It follows from the construction that then $z \notin C_i \ \forall\ i \geq 0$. This is a contradiction. □

6.11.241 Problem. If θ is irrational, show that every point of the interval $[-1, 1]$ is a limit point of the sequence $(\sin n\pi\theta)$.

6.11.241.1 Solution. For $n \in \mathbb{N}$, let $\{\frac{n\theta}{2}\} = \frac{n\theta}{2} - \left[\frac{n\theta}{2}\right]$ is the fractional part of $n\frac{\theta}{2}$. Multiply by 2π to get $2\pi\{\frac{n\theta}{2}\} = \pi n\theta - \left[\frac{n\theta}{2}\right]$, hence $\sin(2\pi(n\theta/2)) = \sin(n\pi\theta)$. Now, the sequence $(n\theta/2)$ is dense in [0,1], hence the sequence $(2\pi(n\theta/2))$ is dense in $[0, 2\pi]$. The continuity of the function "sin" shows the result. □

6.11.242 Problem. Prove that the set $\{(\cos n\theta, \sin n\theta); n \in \mathbb{N}\}$ is dense in the unit circle ($\mathbb{T}$ in $\mathbb{C}$) endowed with the Euclidean metric, whenever θ is an irrational multiple of π.

6.11.242.1 Solution. Put $\theta = \alpha\pi$, for a fixed irrational number α. We know that $2\pi(n\alpha/2)$ is dense in $[0, 2\pi]$. The mapping $x \mapsto \exp i(nx)$ maps continuously $[0, 2\pi]$ onto the unit circle $\mathbb{T}$ in $\mathbb{C}$, hence the set

$$\{\exp i(2\pi n\alpha/2) = \exp i(\pi n\alpha) = \exp i(n\theta) = (\cos n\theta, \sin n\theta) : n \in \mathbb{N}\}$$

is dense in $\mathbb{T}$. □

6.11.243 Problem. Construct a real-valued function on [0,1] that has a limit at each point but is not continuous at infinite countably many points.

6.11.243.1 Solution. Let $f(x) = \begin{cases} \frac{1}{n}, & \text{if } x = 1 - \frac{1}{n+1} \\ 0, & \text{otherwise.} \end{cases}$

Another example is the Thomae function. Indeed, it has a limit 0 at each point $x \in (0, 1)$, and so it is discontinuous precisely at points $\mathbb{Q} \cap (0, 1)$. □

6.11.244 Problem.

1. Construct a function f on $\mathbb{R}$ that is continuous at all integers and discontinuous at all numbers that are not integers.

2. Construct a function on $\mathbb{R}$ such that f is continuous at all numbers that are not integers and f is at the same time discontinuous at all integers.

6.11.244.1 Solution.

1. $f(x) = (\sin \pi x)D(x)$, where D is the Dirichlet function, $D = \chi_{\mathbb{Q}^C}$.

2. Let $f(x) = \begin{cases} 0 & \text{if } x \text{ is not an integer and} \\ 1 & \text{if } x \text{ is an integer.} \end{cases}$ □

6.11.245 Problem. Assume that f is a continuous function on [0,1], and that it is strictly increasing on (0,1). Is f strictly increasing on [0,1]?

6.11.245.1 Solution. Hint. Yes. By continuity, $f(0) \le f(x)$ for all $x \in (0, 1)$. Assume that $f(0) = f(x)$ for some $x \in (0, 1)$. Then, for $0 < y < x$ we get $f(x) = f(0) \le f(y) < f(x)$, a contradiction. The proof for $x = 1$ is similar. □

6.11.246 Problem. Construct a continuous function f on $\mathbb{R}$ such that

1. $f(x) = 0$ for all $x \in \mathbb{R}$ with $|x| \ge 1$, and

2. $f(0) = 2$.

6.11.246.1 Solution. Let $f(x) = \begin{cases} 2\cos \frac{\pi}{2}x, & \text{if } |x| \le 1 \\ 0, & \text{if } |x| \ge 1. \end{cases}$ □

6.11.247 Problem. If $M \subseteq \mathbb{R}$, then the characteristic function χ_M of M is discontinuous exactly at the points of the boundary ∂M of M.

6.11.247.1 Solution. Assume that $x \notin \partial M$. Then, if $x \in M (x \in M^C)$, there exists a neighborhood $U(x)$ of x such that $U(x) \subseteq M$, hence $\chi_M = 1$ on $U(x)$ (respectively, $U(x) \subseteq M^C$, hence $\chi_M = 0$ on $U(x)$). This shows that χ_M is continuous at x. Assume now that $x \in \partial M$. Then, every neighborhood of x intersects both M and M^C. It follows that χ_M is not continuous at x. $\square$

6.11.248 Problem. (Tietze's extension theorem) Show that, if f is a continuous function on a closed subset C in $\mathbb{R}$, then f can be continuously extended to $\mathbb{R}$.

6.11.248.1 Solution. Left to the reader.

6.11.249 Problem. Assume that f is uniformly continuous on $[a, b]$ and also on $[b, c]$. Show that f is uniformly continuous on $[a, c]$.

6.11.249.1 Solution. Hint. If $a \leq x \leq b \leq y \leq c$, then $|f(y) - f(x)| \leq |f(y) - f(b)| + |f(b) - f(x)|$. $\square$

6.11.250 Problem. Show that the function defined for $x \in \mathbb{R}$ by $f(x) = x \sin x$ is not uniformly continuous on $\mathbb{R}$.

6.11.250.1 Solution. Consider the sequences $(x_n), (y_n)$ defined by $x_n = 2n\pi$ and $y_n = 2n\pi + \frac{1}{n}$, for $n \in \mathbb{N}$, we get $x_n - y_n = \frac{1}{n} \to 0$, but $f(x_n) - f(y_n) = \left(2n\pi + \frac{1}{n}\right) \sin \frac{1}{n} \to 2\pi \neq 0$. This shows that the product of two uniformly continuous functions is not uniformly continuous. $\square$

6.11.251 Problem. Let f be a real-valued uniformly continuous function on a bounded open interval (a, b). Show that f is bounded on (a, b).

6.11.251.1 Solution. For $\epsilon = 1$ get δ from the uniform continuity of f. Let the points $a = a_1 < a_2 < a_3 < ... < a_n = b$ be chosen so that the distance of a_i to the next a_{i+1} is less than $\delta/2$. Let $M = \max\{f(a_2), ..., f(a_{n-1})\}$. If $x \in (a, b)$, there is $i \in \{2, 3, ..., (n-1)\}$ such that $|x - a_i| < \delta$. Then $|f(x) - f(a_i)| < \epsilon = 1$. Hence $|f(x)| < |f(a_i)| + 1 \leq M + 1$. $\square$

6.11.252 Problem. Show that a real-valued function f defined on a subset D of $\mathbb{R}$ is uniformly continuous on D, if and only if, $\delta(\epsilon) \to 0$ as $\epsilon \to 0+$, where δ is the modulus of continuity of f on D.

6.11.252.1 Solution. Assume first that f is uniformly continuous on D. Then, given $\alpha > 0$ we can find $\epsilon > 0$ such that for every $x, y \in D$ with $|x - y| \leq \epsilon$ we have $|f(x) - f(y)| \leq \alpha$. It follows from the definition of δ that $\delta(\epsilon) \leq \alpha$. Since δ is increasing on $[0, \infty)$ we obtain $\lim_{\epsilon \to 0+} \delta(\epsilon) = 0$. Conversely, assume that $\lim_{\epsilon \to 0+} \delta(\epsilon) = 0$.. Then, given $\alpha > 0$ we can find $\epsilon > 0$ such that $\delta(\epsilon) \leq \alpha$. This shows that $|f(x) - f(y)| \leq \alpha \; \forall \, x, y \in D$ with $|x - y| \leq \epsilon$. Therefore, f is uniformly continuous on D. $\square$

6.11.253 Problem. Let P be a polynomial of an odd order. Show that the intermediate value property implies that P has at least one real root, i.e., $P(x) = 0$ for some real number x.

6.11.253.1 Solution. Let $P(x) = a_n x^n + ... + a_0$, where n is odd. Assume without loss of generality that $a_n = 1$. Then, $P(x) = x^n \left(1 + a_{n-1}\frac{1}{x} + ... + a_0 \frac{1}{x^n}\right)$. Thus, $\lim_{x \to \infty} P(x) = \infty$ and $\lim_{x \to -\infty} P(x) = -\infty$. This shows that there exist $a, b \in \mathbb{R}$ such that $a < b, P(a) < 0$, and $P(b) > 0$. By the intermediate value property, there is $x \in (a, b)$ such that $P(x) = 0$. $\square$

6.11.254 Problem. There exists a function ϕ defined on [0,1] such that $\phi(J) = [0, 1]$ for every nondegenerate interval $J \subseteq [0, 1]$. Observe that this gives an example of a discontinuous function satisfying the intermediate value property.

6.11.254.1 Solution. Hint. (B. Knaster, K. Kuratowski) For $x \in [0,1]$, put $x = 0.a_1a_2a_3.......$(base 2) (if x has two such expansions, choose one with finitely many nonzero digits). For $n \in \mathbb{N}$, put $p_x(n) = a_1 + a_2 + ... + a_n$. Then define

$$\phi(x) = \limsup_{n\to\infty} \frac{p_x(n)}{n}.$$

The discontinuity of the function ϕ at each point in [0,1] is a straightforward consequence of its behavior on each open subinterval. □

6.11.255 Problem. If the functions f and g have the IVP, it does not follow that $f + g$ has the IVP.

6.11.255.1 Solution. Hint. Let

$$\phi(t) = \begin{cases} t^2 \sin \frac{1}{t}, & \text{if } t \neq 0 \\ 0, & \text{if } t = 0. \end{cases} ;\ \psi(t) = \begin{cases} t^2 \cos \frac{1}{t}, & \text{if } t \neq 0 \\ 0, & \text{if } t = 0. \end{cases}$$

Then, let $f(t) = (\phi'(t))^2$ and $g(t) = (\psi'(t))^2$, and then show that $f + g$ does not have the IVP on any interval containing 0. □

6.11.256 Problem. Prove that it is not possible to partition $\mathbb{R}$ into a countably infinite union of nonempty closed subsets.

6.11.256.1 Solution. Hint: Assuming that $\mathbb{R} = \cup_{n=1}^{\infty} C(n)$ is such a partition, construct a sequence of nested intervals $[a_n, b_n]$ such that $C(n) \cap [a_n, b_n] = \emptyset$ for all $n \in \mathbb{N}$. □

6.11.257 Problem. Let I be an interval in $\mathbb{R}$ and $f : I \to \mathbb{R}$. Show that f is uniformly continuous on I iff for every $\epsilon > 0$, there exists $M > 0$ such that

$$\left|\frac{f(x) - f(y)}{x - y}\right| > M \Rightarrow |f(x) - f(y)| < \epsilon.$$

D. Paine, "Visualizing Uniform Continuity,"
American Mathematical Monthly 75 (1968), 44—45.

6.11.257.1 Solution. Suppose the condition holds and $\epsilon > 0$ is given, let $\delta = \epsilon/M$ then we get

$$\begin{aligned} \left|\frac{f(x) - f(y)}{x - y}\right| &> M \Rightarrow |f(x) - f(y)| < \epsilon. \\ \text{i.e. } \epsilon > |f(x) - f(y)| &> M|x - y| \Rightarrow |f(x) - f(y)| < \epsilon, \\ \text{by the condition } |x - y| &< \delta \Rightarrow |f(x) - f(y)| < \epsilon, \\ \text{i.e. } |f(x) - f(y)| &\geq \epsilon \Rightarrow |x - y| \geq \delta. \end{aligned}$$

If f is uniformly continuous and $\epsilon > 0$ is given, choose $\delta > 0$ so that $|f(x) - f(y)| \geq \epsilon \Rightarrow |x - y| \geq \delta$ and let $M = 2\delta/\epsilon$. Assume (without loss of generality) that $f(x) < f(y)$. and choose $n \in \mathbb{N}$ such that $\eta = (f(x) - f(y))/n \in [\epsilon, 2\epsilon]$. Divide $[f(x), f(y)]$ into n equal parts using the partition $\{f(x) + k\eta; 0 \leq k \leq n\}$ Using the Intermediate Value Theorem, choose $x_0 = x, x_1, x_2, ..., x_n = y$ such that $f(x_k) = f(x) + k\eta$. Now, we can deduce that $|x - y| \geq n\delta$ and hence $|f(x) - f(y)|/|x - y| \leq M$. □

6.11.258 Problem. Find a subset S of the real numbers $\mathbb{R}$ such that both (1) and (2) hold for S:

1. S is not the countable union of closed sets.
2. S is not the countable intersection of open sets.

6.11.258.1 Solution. Let A be a subset of [0,1] that is not a countable union of closed sets, and let B be a subset of [2,3] that is not a countable intersection of open sets. (Irrationals and rationals for instance, respectively.) We show that $S = A \cup B$ satisfies (1) and (2). Suppose for the sake of contradiction that S is the countable union of closed sets $\{F_n\}$. Then

$$A = S \cap [0,1] = \left(\bigcup_{n=1}^{\infty} F_n \right) \cap [0,1] = \bigcup_{n=1}^{\infty} \left(F_n \cap [0,1] \right).$$

Note that $F_n \cap [0,1]$ is closed for each n, so A is a countable union of closed sets, a contradiction. Likewise, if S is the countable intersection of open sets, it follows that B is the countable intersection of open sets, a contradiction. Hence S satisfies (1) and (2). □

6.11.259 Problem. Suppose that $f : [a,b] \to \mathbb{R}$ is a continuous and non-constant function. Prove that the function f cannot have any small periods.

6.11.259.1 Solution. Proof: Let f is continuous at $q \in [a,b]$, and by hypothesis that f is non-constant, there is a point $p \in [a,b]$, such that $|f(p) - f(q)| = M > 0$. Since f is continuous at q, then given $\epsilon < M$, there is a $\delta > 0$ such that for $x \in (q-\delta, q+\delta) \cap [a,b]$, we have

$$|f(x) - f(q)| < M. \tag{A}$$

If f has any small periods, then there is a point r in the set $(q-\delta, q+\delta) \cap [a,b]$ such that $f(r) = f(p)$. It contradicts to (A). Hence, the function f cannot have any small periods. □

6.11.9 Remark.

1. There is a function with any small periods.

   6.11.10 Example. Consier the function $f : \mathbb{R} \to \mathbb{R}$ defined by

   $$f(x) = \begin{cases} 0, & \text{if } x \in \mathbb{Q}^C \\ 1, & \text{if } x \in \mathbb{Q}. \end{cases}$$

   Since $f(x+q) = f(x)$ for any rational q, we see that f has any small periods.

2. Prove that there cannot have a non-constant continuous function which has two periods p, and q such that q/p is irrational.
   Since q/p is irrational, there is a sequence (q_n/p_n) with $q_n/p_n \in \mathbb{Q}$ such that

   $$\left| \frac{q_n}{p_n} - \frac{q}{p} \right| < \frac{1}{p_n^2} \Rightarrow |pq_n - p_n q| < \left| \frac{p}{p_n} \right| \to 0 \text{ as } n \to \infty.$$

   So, f has any small periods, by this exercise, we see that this f cannot a non-constant continuous function.

6.11.260 Problem. Show that the composition of

1. an uniformly continuous function and another uniformly continuous function is again an uniformly continuous function.
2. an uniformly continuous function and a non-uniformly continuous function may be uniformly continuous function or not an uniformly continuous function.
3. a non-uniformly continuous function and an uniformly continuous function may be uniformly continuous function or not an uniformly continuous function.
4. a non-uniformly continuous function and another non-uniformly continuous function may be an uniformly continuous function or not an uniformly continuous function. $\square$

6.11.260.1 Solution.

1. Left to the reader.
2. (a) Let $f, g : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x$ and $g(x) = x^2$, then $f(g(x)) = x^2$ is not uniformly continuous.
   (b) Let $f, g : (0, \infty) \to \mathbb{R}$ be defined by $f(x) = \sqrt{x}$ and $g(x) = x^2$, then $f(g(x)) = x$ is uniformly continuous.
3. (a) Let $f, g : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$ and $g(x) = x$, then $f(g(x)) = x^2$ is not uniformly continuous.
   (b) Let $f, g : [0, \infty) \to \mathbb{R}$ be defined by $f(x) = x^2$ and $g(x) = \sqrt{x}$, then $f(g(x)) = x$ is uniformly continuous.
4. (a) $f, g : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$ and $g(x) = x^3$, then $f(g(x)) = x^6$ is not uniformly continuous.
   (b) $f, g : (0, 1) \to \mathbb{R}$ be defined by $f(x) = 1/x$ and $g(x) = 1/\sqrt{x}$, then $f(g(x)) = \sqrt{x}$ is uniformly continuous.

6.11.261 Problem. Let $f : \mathbb{R} \to \mathbb{R}$ be uniformly continuous, then there exist $A, B > 0$ such that $|f(x)| \leq A|x| + B$.

6.11.261.1 Solution. Since f is uniformly continuous on $\mathbb{R}$, given $\epsilon > 0$, there is a $\delta > 0$ such that

$$|x - y| \leq \delta \Rightarrow |f(x) - f(y)| < 1$$

Given any $x \in \mathbb{R}$, then there is a positive integer N such that $(N-1)\delta < |x| < N\delta$. If $x > 0$, we consider

$$y_0 = 0, y_1 = \delta/2, y_2 = \delta, ..., y_{2N-1} = N\delta - \delta/2, y_{2N} = x.$$

Then we have

$$\begin{aligned} |f(x) - f(0)| &\leq \sum_{k=1}^{N} |f(y_{2k}) - f(y_{2k-1})| + |f(y_{2k-1}) - f(y_{2k-2})| \\ &\leq 2N \end{aligned}$$

which implies that

$$\begin{aligned}|f(x)| &\leq 2N + |f(0)| \\ &\leq 2\left(1 + \frac{|x_0|}{\delta}\right) + |f(0)| \text{ since } |x| > (N-1)\delta \\ &\leq \frac{2}{\delta}|x| + (2 + |f(0)|).\end{aligned}$$

Similarly for $x < 0$. So, we have proved that $|f(x)| \leq A|x| + B$ for all x. □

6.11.262 Problem. Let f be a continuous mapping of $\mathbb{R}$ into itself. Show that if f is monotone and bounded in $\mathbb{R}$, f is uniformly continuous in $\mathbb{R}$.

6.11.262.1 Solution. Left to the reader. □

6.11.263 Problem. Let f be a continuous function on $[a, b]$. Prove that if s, t are distinct values taken by f in $[a, b]$, then

$$\delta(s, t) = \inf\{|x - y|; f(x) = s, f(y) = t\}$$

is positive. Prove that if r, s, t are values taken by f in $[a, b]$, where $r < s < t$, then $\delta(r, s) \leq \delta(r, t)$. Prove further that $\delta(r, s) < \delta(r, t)$.

6.11.263.1 Solution. $\delta(s, t)$ implies that there exists $x_n, y_n, n = 1, 2, ...$, with $|x_n - y_n| < \frac{1}{n}$ such that $f(x_n) = s, f(y_n) = t$. But $|x_n - y_n| < 1/n < \delta$ (if $n > N$) implies that $|f(x_n) - f(y_n) < \epsilon$ (since f is continuous in $[a, b]$ and so uniformly continuous), i.e., $|s - t| \leq \epsilon$, which is false since $s \neq t$.
Next, choose $x_n < y_n$ with $f(x_n) = r, f(y_n) = t$, such that $|x_n - y_n| = \delta(r, t) + 1/n$. Since $r < s < t$,, f takes the value s between x_n, y_n, say at w_n, i.e., $x_n < w_n < y_n$, such that $f(w_n) = s$. Then $|x_n - w_n| < |x_n - y_n| = \delta(r, t) + l/n$ (for all n) and so is less than or equal to $\delta(r, t)$, i.e., $\delta(r, s) \leq \delta(r, t)$. Finally, if $\delta(r, s) = \delta(r, t)$. for some $r < s < t$, then there exist $x_n < w_n < y_n$, with $f(x_n) = r, f(w_n) = s, f(y_n) = t$ and then $(w_n - x_n) - (y_n - x_n) \to 0$ as $n \to \infty$, i.e., $y_n - w_n \to 0$ as $n \to \infty$, and so $f(y_n) - f(w_n) \to 0$, by the continuity of f, i.e., $|s - t| < \epsilon$, which is false since $t \neq s$. □

6.11.264 Problem. Let $f : \mathbb{R} \to \mathbb{R}$ and for $\delta > 0$, define

$$\begin{aligned}&\phi(x, \delta) = \sup\{|f(x) - f(y)|; y \in B(x; \delta)\} \text{ and} \\ &\psi(\delta) = \sup\{\phi(x; \delta); x \in \mathbb{R}\}.\end{aligned}$$

Prove that

1. ψ is nondecreasing,
2. ψ is bounded,
3. $\psi(\delta) \to 0$ as $\delta \to 0$ if and only if f is uniformly continuous on $\mathbb{R}$.

6.11.264.1 Solution.

1. Let $\delta_1 > \delta_2$. Then $\phi(x, \delta_1) \geq \phi(x, \delta_2)$, $\forall\ x$, since the left side is the oscillation of f in $(x - \delta_1, x + \delta_1)$ while the right side is the oscillation of f in $(x - \delta_2, x + \delta_2)$. Therefore, $\sup_x \phi(x, \delta_1) \geq \sup_x \phi(x, \delta_2)$ i.e. $\psi(\delta_1) \geq \psi(\delta_1)$.

2. First, let f be bounded: $m \leq f(x) \leq M$. Then for all $x, y \in \mathbb{R}, |f(x) - f(y)| \leq M - m$, so $\phi(x, \delta) < M - m$ for all $x, \delta(> 0)$ and hence $\psi(\delta) \leq M - m$ for all $\delta > 0$, i.e., ψ is bounded. Conversely, let ψ be bounded, say $\sup \psi(\delta) = M$ (inf is greater than or equal to 0, of course). Then for all x and all $\delta > 0, \phi(x, \delta) \leq \psi(\delta) \leq M$. In this case, take $x = 0$ (say) to get $\phi(0, \delta) \leq M$. for all $\delta > 0$ and so $|f(0) - f(y)| < M$, if $-\delta < y < \delta$ for all $\delta > 0$, i.e., $M - f(0) < f(y) < M + f(0)$ for all y (since δ can be any positive real number), i.e., f is bounded, as required.

3. First, let f be uniformly continuous on $\mathbb{R}$. Then given $\epsilon > 0$, there exists an $\eta > 0$, such that $|f(x) - f(y)| < \epsilon$ if $|x - y| < \eta$ and so $\phi(x, \delta) < \epsilon$ if $\delta < \eta$ (for all $x \in \mathbb{R}$), and so $\psi(\delta) \leq \epsilon$ if $\delta < \eta$, i.e., $\psi(\delta) \to 0$ as $\delta \to 0$. Conversely, let $\psi(\delta) \to 0$ as $\delta \to 0$, and we just work backwards to get f is uniformly continuous on $\mathbb{R}$. □

6.11.265 Problem. Show that a continuous, rational-valued function must be a constant.

6.11.265.1 Solution. Apply IVP.

6.11.266 Problem. Which one of the following statements is sufficient to determine the value $f(0)$ of $f(x)$? Give the values when determined:

1. f is continuous at $x = 0$ and takes both positive and negative values in any neighbourhood of $x = 0$.

2. Given $\epsilon > 0$, there exists a $\delta > 0$ such that $|f(x)| < \epsilon$ for $0 < |x| < \delta$.

3. $(f(h) + f(-h) - 2f(0))/h \to 1$ and $f(h) \to a$ as $h \to 0$.

6.11.266.1 Solution.

1. $f(0)$ is determined and equals 0. Indeed, being continuous at $x = 0$, we have $f(0) = \lim_{x \to 0+} f(x) = \lim_{x \to 0-} f(x)$, so $f(0)$ is determined. If $f(0)$ were positive, then, being continuous at $x = 0, f(x) > 0$ in a sufficiently small neighbourhood of 0, a contradiction to the hypothesis. Thus $f(x)$ is not greater than 0. Similarly, $f(0)$ is not less than 0. So $f(0) = 0$.

2. $f(0)$ is not determined by this condition; for example, let $f(x) = x \sin(1/x)$, for $x \neq 0$. Then $|f(x)| = |x||\sin(1/x)| < |x| < \epsilon$, if $0 < |x| < \delta$; so the condition is satisfied but $f(0)$ is not defined.

3. Write $f(h) = a + \epsilon_h$, where $\epsilon_h \to 0$ as $h \to 0$. Then

$$\begin{aligned}\frac{(f(h) + f(-h) - 2f(0))}{h} &= \frac{(a + \epsilon_h + a + \epsilon_{-h} - 2f(0))}{h} \\ &= \frac{(2a + \epsilon_h + \epsilon_{-h} - 2f(0))}{h} \\ &\to 1 \text{ as } h \to 0 \text{ (given)}.\end{aligned}$$

   It follows that $2a = 2f(0)$, giving $f(0) = a$. □

6.11.267 Problem. Let $f(x + y) = f(x).f(y)$.

1. Prove that if f is not identically equal to 0, then $f(x) > 0$.

2. Prove also that if f is bounded in some interval $[-X, X]$, then $\inf_{x \in [-X, X]} f(x) > 0$.

6.11.267.1 Solution.

1. Take $y = x$ to get $f(2x) = (f(x))^2 > 0$ for all x. Suppose $f(a) = 0$ for some a. Then $f(a+x) = f(a)f(x) = 0$ for all x, i.e., $f(x) = 0$, a contradiction.

2. Let $A = \sup f$ in $[-X, X]$ attained at ξ, i.e., $f(\xi) = A$. Suppose, to the contrary, that $\inf f = 0$. Then, given $\epsilon > 0$, there exists η; such that $0 < f(\eta) < \epsilon$. So $f(\eta+x) = f(\eta).f(x)$, i.e., $f(x) = (1/f(\eta)).f(\eta+x) > (1/\epsilon).f(\eta+x)$. Now take $x = \xi - \eta$; so $f(\xi-\eta) > (1/\epsilon)f(\xi) = (1/\epsilon)A > A$, if $\epsilon < 1$, a contradiction. □

6.11.268 Problem. Let f be defined in [0,1], $f(0) = 0$. For $x \in [0,1]$, there exists an $h(x) > 0$ such that $f(x) - f(x') \in \mathbb{Q}$ if $x' \in (x - h(x), x + h(x))$, for all $0 < x' < 1$. Prove that $f(x) \in \mathbb{Q}$ for all $x \in [0,1]$.

6.11.268.1 Solution. The open intervals $B(x;\delta_x)$ form an open cover of [0,1]. By the Heine-Borel theorem, there exists a finite subcover, say, $\{A_i = B(x_i;\delta_{x_i}); i = 1,2,..,n\}$ where, without loss of generality, assume that $A_i \cap A_{i+1} \neq \emptyset$ for each $i = 1,2,...,n-1$. Choose $p_i \in A_i \cap A_{i+1}$; then

$$\begin{aligned}
&(1)\ f(0) - f(p_1) = \rho_1 \in \mathbb{Q},\\
&(2)\ f(x_2) - f(p_1) = \rho_2 \in \mathbb{Q},\\
&(3)\ f(x_2) - f(p_2) = \rho_3 \in \mathbb{Q},\\
&(4)\ f(x_3) - f(p_2) = \rho_4 \in \mathbb{Q}.
\end{aligned}$$

Now (1)$\Rightarrow$ $f(p_1) \in \mathbb{Q}$, since $f(0) = 0$, and so (2)$\Rightarrow$ $f(x_2) = f(p_1) + \rho \in \mathbb{Q}$ Therefore, by (3), $f(p_2) \in \mathbb{Q}$, and so by (4), $f(x_3) \in \mathbb{Q}$, and so on. Thus $f(x_i) \in \mathbb{Q}$ for all i and so $f(x) \in \mathbb{Q}$, by the equation $f(x) - f(x') \in \mathbb{Q}$. The condition $f(0) = 0$ is essential, for, take $f(x) = \sqrt{2}$ for all $x \in [0,1]$. This function satisfies the conditions of the theorem (except that $f(0) = 0$) but fails to satisfy the conclusion. □

6.11.269 Problem. Does there exist a real nonconstant function f such that $f(x+p) = f(x)$ for all $p > 0$?

6.11.269.1 Solution. No: For otherwise there would exist $x_1 \neq x_2$ such that $f(x_1) \neq f(x_2)$. Suppose, without loss of generality, that $x_1 < x_2$, say $x_1 + p = x_2$. Then $f(x_1+p) = f(x_2) \Rightarrow f(x_1) = f(x_2)$, which is a contradiction. □

6.11.270 Problem. Give two examples to show that a function f can satisfy one of the following properties without satisfying the other:

1. Given $\epsilon > 0$, there exists a $\delta > 0$, such that $|x-1| < \delta$ implies that $|f(x) - 1| < \epsilon$.

2. Given $\delta > 0$, there exists a $\epsilon > 0$, such that $|x-1| < \delta$ implies that $|f(x) - 1| < \epsilon$.

6.11.270.1 Solution.

$$(1).\ f(x) = \begin{cases} 0, & \text{if } 1 - \frac{1}{10} \leq x \leq 1 + \frac{1}{10} \\ \frac{1}{x - \left(1+\frac{1}{10}\right)} & \text{if } 1 + \frac{1}{10} < x \\ \frac{-1}{x-\left(1-\frac{1}{10}\right)} & \text{if } x < 1 - \frac{1}{10} \end{cases} \qquad (2).\ f(x) = \begin{cases} 1 & \text{if } x \geq 1 \\ 0 & \text{if } x < 1. \end{cases}$$ □

6.11.271 Problem. Determine whether the given conditions are sufficient to ensure that $f(x)$ tends to a finite limit as $x \to \infty$

1. $f(x) > 0, g(x) < 0, f(x) - g(x) \to 0$ as $x \to \infty$.
2. $(f(x))^2 \to \lambda$ as $x \to \infty, \lambda < \infty$.

Distinguish between the cases $\lambda = 0, \lambda \neq 0$.

6.11.271.1 Solution.

1. The condition ensures that $f(x) \to 0$ as $x \to \infty$. Indeed, $|f(x) - g(x)| < \epsilon$ if $x > A$, i.e., $g(x) - \epsilon < f(x) < g(x) + \epsilon$ if $x > A$. Here, the right side is less than ϵ, since $g(x) < 0$ and $-\epsilon < f(x)$, since $f(x) > 0$. Thus $-\epsilon < f(x) < \epsilon$ if $x > A$, i.e., $|f(x)| < \epsilon$ if $x > A$, which means $f(x) \to 0$ as $x \to \infty$.
2. If $\lambda \neq 0$, this does not ensure that $f(x)$ tends to a finite limit, for example, let
$$f(x) = \begin{cases} 1, & \text{if } x \in [2n, 2n+1) \\ -1, & \text{if } x \in [2n+1, 2n+2). \end{cases}$$
Then $f^2 = 1$, which tends to 1, but f does not tend to any limit as $x \to \infty$. However, if $\lambda = 0$, then $f^2 \to 0$ as $x \to \infty$. □

6.11.272 Problem. Let $f : [0,1] \to \mathbb{R}$ be continuous, with $f(0) = f(1)$ and let $n \in \mathbb{N}$. Then the graph of f has at least n horizontal chords whose lengths are integral multiples of $1/n$.

6.11.272.1 Solution (Rosenbaum)**.** For each of $k = 1, 2, ..., n-1$, if f has no horizontal chord of length $a = k/n$, then it has two horizontal chords of length $1 - a = (n-k)/n$ and vice versa. Thus, there always are 2 horizontal chords, either of length k/n or of length $(n-k)/n$ or one each of length $k/n, (n-k)/n$ and this is the case for each of $k = 1, 2, ..., n-1$. For n odd, this, gives $2(n-1)/2 = n-1$ horizontal chords of the required type and for $k = n$, there always is a horizontal chord of length $1 = n(1/n)$ (an integral multiple of n). This gives the full set of n horizontal chords, as required. For n even, $n-1$ is odd and the counting is different, viz. either there are 2 horizontal chords of length k/n or two of length $(n-k)/n$ or one for each of $k = 1, 2, ..., (n-2)/2$; in all $2(n-2)/2 = n-2$ plus one of length $1 = n(1/n)$, giving the full set of n again. □

6.11.273 Problem. Construct an example of a two-to-one function $f : [0,1] \to \mathbb{R}$. Prove that no such f can be continuous on [0,1].

6.11.273.1 Solution. Let $\{r_n; n \in \mathbb{N}\}$ be an enumeration of the rationals in $[0,1]$ and define $f : [0,1] \to \mathbb{R}$ by
$$f(x) = \begin{cases} |2x-1| & \text{if } x \text{ is rational} \\ r_{2k-1} & \text{if } x = r_{2k-1} \\ r_{2k-1} & \text{if } x = r_{2k}. \end{cases}$$

Suppose now that f is a continuous two-to-one function. We can then assume that its, say, minimum is attained at the points $x_1 < x_2$, and x_2 is not an endpoint. Choose disjoint closed intervals $[a_1, b_1], [a_2, b_2]$ with $x_1 \in [a_1, b_1], x_1 \neq b_1$ and $x_2 \in (a_2, b_2)$. Then the intermediate value theorem implies that a value r with $\min\{f(b_1), f(a_2), f(b_2)\} > r > \min f$ is taken on at least three times. □

6.11.274 Problem. If $f : \mathbb{R} \to \mathbb{R}$ has a local maximum at each $x \in \mathbb{R}$, then $f(\mathbb{R})$ is countable.

6.11.274.1 Solution. For every $a \in f(\mathbb{R})$ choose $x_a \in \mathbb{R}$ with $f(x_a) = a$ and an open interval I_a with rational endpoints such that $x_a \in I_a$ and for each $x \in I_a, f(x) \geq f(x_a) = a$. Then for $a \in f(\mathbb{R})$ the function

$$a \mapsto I_a \in \{(p,q); p, q \in \mathbb{Q}\}$$

is one-to-one. □

6.11.275 Problem. Let $p : \mathbb{R} \to \mathbb{R}$ be an odd degree polynomial, that is $p(x) = \sum_{k=0}^{2m+1} a_k x^k$, where $a_{2m+1} \neq 0$ and $m \in \mathbb{N}$. Prove that p has a real root, i.e. there is an $x_0 \in \mathbb{R}$ such that $p(x_0) = 0$.

6.11.275.1 Solution. Assuming that $a_{2m+1} > 0$, we have

$$\lim_{x\to\infty} \frac{p(x)}{x^{2m+1}} = \sum_{k=0}^{2m+1} a_k \lim_{x\to\infty} x^{k-2m-1} = a_{2m+1} > 0$$

because all the limits are 0 except for the $k = 2m+1$ term where the limit is 1. Take $\epsilon = a_{2m+1}/2 > 0$, then there is an $R_0 > 0$ such that if $x > R_0$, then $p(x)/x^{2m+1} > a_{2m+1}/2$, which implies that $p(x) > 0$. Similarly we have

$$\lim_{x\to-\infty} \frac{p(x)}{x^{2m+1}} = \lim_{y\to\infty} \frac{p(-y)}{y^{2m+1}} = \sum_{k=0}^{2m+1} (-a_k) \lim_{y\to\infty} y^{k-2m-1} = -a_{2m+1} < 0$$

and so there is an $R_1 < 0$ such that for $x \leq R_1 < 0, p(x)/x^{2m+1} < -a^{2m+1}/2 < 0$ which implies that $p(x) < \frac{a_{2m+1}}{2} x^{2m+1} < 0$ as p is continuous since it is a polynomial, so as $p(R_1) < 0 < p(R_0)$, the Intermediate Value Theorem implies there is an $x_0 \in [R_1, R_0]$ such that $p(x_0) = 0$. □

6.11.276 Problem. For each $x \in [0,1]$, let

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 1-x & \text{if } x \text{ is irrational.} \end{cases}$$

Prove that:

1. $f(f(x)) = x \;\forall\, x \in [0,1]$.
2. $f(x) + f(1-x) = 1 \;\forall\, x \in [0,1]$.
3. f is continuous only at the point $x = 1/2$.
4. f assumes every value between 0 and 1.
5. $f(x+y) - f(x) - f(y)$ is rational $\forall\, x, y \in [0,1]$.

6.11.276.1 Solution.

1. If x is rational, then $f(f(x)) = f(x) = x$. If x is irrational, then $f(f(x)) = f(1-x) = 1-(1-x) = x$.
2. If x is rational, then $1-x$ is also rational, so $f(x) + f(1-x) = x + 1 - x = 1$.

3. Suppose that f is continuous at $x = c$. Let (r_n) be a sequence of rationals converges to c, then $f(r_n)$ converges to $f(c)$. Since $f(r_n) = r_n$, and (r_n) converges to c, so $f(c) = c$, Again, let (i_n) be a sequence of irrationals converges to c, then $f(i_n)$ converges to $f(c)$. Since $f(i_n) = 1 - i_n$, and (i_n) converges to c, so $f(c) = 1 - c$. Thus $c = 1 - c \Rightarrow c = \frac{1}{2}$.

4. Given $a \in [0, 1]$, we want to find $x \in [0, 1]$ such that $f(x) = a$. If $a \in \mathbb{Q}$, then choose $x = a$, we have $f(a) = a$. If $a \in \mathbb{Q}^C$, then choose $x = 1 - a \in \mathbb{Q}^C$, we have $f(1 - a) = 1 - (1 - a) = a$. Thus f assumes every value between 0 and 1.

5. Consider the cases

   (a) If $x, y \in \mathbb{Q}$ then $x + y \in \mathbb{Q}$. Hence $f(x + y) - f(x) - f(y) = x + y - x - y = 0 \in \mathbb{Q}$.

   (b) If $x \in \mathbb{Q}$ and $y \in \mathbb{Q}^C$ then $x+y \in \mathbb{Q}^C$. Hence $f(x+y)-f(x)-f(y) = 1-x-y-x-1+y = -2x \in \mathbb{Q}$

   (c) If $x \in \mathbb{Q}^C$ and $y \in \mathbb{Q}$ then $x+y \in \mathbb{Q}^C$. Hence $f(x+y)-f(x)-f(y) = 1-x-y-1+x-y = -2y \in \mathbb{Q}$

   (d) If $x, y \in \mathbb{Q}^C$ then either $x + y \in \mathbb{Q}$ or $x + y \in \mathbb{Q}^C$. Hence

$$\begin{aligned} & f(x+y) - f(x) - f(y) \\ = & \begin{cases} x + y - 1 + x - 1 + y = 2(x + y) - 2 \in \mathbb{Q} & \text{if } x + y \in \mathbb{Q} \\ 1 - x - y - 1 + x - 1 + y = -1 \in \mathbb{Q}, & \text{if } x + y \in \mathbb{Q}^C. \end{cases} \end{aligned}$$

   Thus $f(x + y) - f(x) - f(y)$ is rational $\forall\, x, y \in [0, 1]$. □

6.11.277 Problem. Let $U_n : \mathbb{R} \to \mathbb{R}$ denote the "ramp" function defined by

$$U_n(x) = \begin{cases} -n & \text{if } x \le -n \\ x & \text{if } -n < x \le n \\ n & \text{if } x > n \end{cases}$$

and let $F : \mathbb{R} \to \mathbb{R}$ denote a real function of a real variable. Show that F is continuous if and only if $U_n \circ F$ is continuous for all $n \in \mathbb{N}$.

6.11.277.1 Solution. Clearly U_n is continuous. So, if F is continuous, then $U_n \circ F$ is the composition of continuous functions and hence is continuous. Conversely, suppose that $U_n \circ F$ is continuous for all n. To prove F is continuous it is enough to show $F^{-1}(a, b)$ is open for every bounded interval (a, b). Let $n > \max\{|a|, |b|\}$. Then $U_n^{-1}(a, b) = (a, b)$ so

$$F^{-1}((a, b)) = F^{-1}\left(U_n^{-1}((a, b))\right) = \left(F^{-1} \circ U_n^{-1}\right)((a, b)) = (U_n \circ F)^{-1}((a, b))$$

which is an open set by the continuity of $U_n \circ F$. □

6.11.278 Problem. Suppose that a function f is defined and one-to-one on (a, b) and continuous at a point $c \in (a, b)$.

1. Is f^{-1} necessarily continuous at the point $f(c)$? Does f^{-1} always have one-sided limits at $f(c)$?

2. Is f^{-1} continuous at the point $f(c)$ if f is strictly monotone?

6.11.278.1 Solution.

1. Consider, for example, the function

$$f(x) = \begin{cases} 1-x & \text{for } -1 < x \leq 0 \\ \dfrac{x^{-1}+[x^{-1}]}{1+x^{-1}+[x^{-1}]} & \text{for } 0 < x < 1 \\ \dfrac{1+(2-x)^{-1}+[(2-x)^{-1}]}{2+(2-x)^{-1}+[(2-x)^{-1}]} & \text{for } 1 < x < 2 \end{cases}$$

   and the point $c = 0$.

2. Left to the reader.

6.11.279 Problem. Call a mapping of $X \subseteq \mathbb{R}$ into $Y \subseteq \mathbb{R}$ open if $f(V)$ is an open set in Y whenever V is an open set in X. Prove that every continuous open mapping $f : \mathbb{R} \to \mathbb{R}$ is monotonic.

6.11.279.1 Solution. Claim(1). $a \neq b \Rightarrow f(a) \neq f(b)$. Since f is a continuous function on a compact set, so it must attain its maximum and its minimum, so let

$$M = \sup_{x\in[a,b]} f(x) \text{ and } m = \inf_{x\in[a,b]} f(x),$$

and there must be some $z_1, z_2 \in [a, b]$ such that $f(z_1) = m$ and $f(z_2) = M$. If $M = m$, then f is constant on $[a, b]$, but then $f((a, b)) = \{M\} = \{m\}$ is just a point, which is not an open set, contradicting the openness of f. Therefore, we can assume $M > m$. There are four cases left:

1. $f(z_1) = m$ for some $z_1 \in (a, b)$, so $f((a, b))$ contains $z_1 \in \mathbb{R}$ but does not contain any real numbers less than z_1. Then any neighborhood N of z_1 will contain real numbers less than z_1, and N thus cannot be contained in $f((a, b))$, so it is not open, a contradiction.

2. $f(z_2) = M$ for some $z_2 \in (a, b)$, so $f((a, b))$ contains $z_2 \in \mathbb{R}$ but does not contain any real numbers greater than z_2. Then any neighborhood N of z_2 will contain real numbers greater than z_2, and N thus cannot be contained in $f((a, b))$, so it is not open, a contradiction.

3. $f(a) = m$ and $f(b) = M$. Then $f(a) \neq f(b)$.

4. $f(a) = M$ and $f(b) = m$. Then $f(a) \neq f(b)$.

Claim(2). If $a < b < c$ and $f(a) < f(b)$, then $f(b) < f(c)$.

1. $f(c) = f(a)$. Since $c \neq a$ this case contradicts claim 1.

2. $f(c) = f(b)$. Since $c \neq b$ this case contradicts claim 1.

3. $f(c) < f(a)$. Then $f(c) < f(a) < f(b)$, and by the Intermediate Value Theorem there is some $d \in (b, c)$ such that $f(d) = f(a)$, but $a \notin (b, c)$, so $a \neq d$, contradicting claim 1.

4. $f(a) < f(c) < f(b)$. Then $f(a) < f(c) < f(b)$, and by the Intermediate Value Theorem there is some $d \in (a, b)$ such that $f(d) = f(c)$, but $c \notin (a, b)$, so $c \neq d$, contradicting claim 1.

The only case left is $f(c) > f(b)$, and since all the other cases lead to contradictions, this is the only possible one.
Finally, if we assume f is not monotonic, then either there are some $a < b < c$ such that $f(a) < f(b)$ and $f(c) < f(b)$ or there are some $a < b < c$ such that $f(a) > f(b)$ and $f(c) > f(b)$. The first case contradicts claim (2). The second case turns into the first case if we replace $f(x)$ by $-f(x)$, which is still a continuous, open function. □

6.11.280 Problem. Assume that f is a continuous real valued function defined in (a, b) such that

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} \; \forall\, x, y \in (a, b).$$

Prove that f is convex.

6.11.280.1 Solution. To prove that f is convex on (a, b) we must show that

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \; \forall\, x, y \in (a, b), \lambda \in [0, 1] \tag{A}$$

Let Λ represent the set of values for λ for which (A) holds. It's immediately clear that $0, 1 \in \Lambda$. To prove that f is convex we must show that $[0, 1] \subseteq \Lambda$.
We first prove that, if $i, j \in \Lambda$ then $\frac{i+j}{2} \in \Lambda$, let $m = \frac{i+j}{2}$ then

$$\begin{aligned}
f(mx + (1-m)y) &= f\left(\frac{ix + (1-i)y}{2} + \frac{jx + (1-j)y}{2}\right) \\
&\leq \frac{f(ix + (1-i)y) + f(jx + (1-j)y)}{2} \\
&\leq \frac{if(x) + (1-i)f(y) + jf(x) + (1-j)f(y)}{2} \\
&\leq \frac{i+j}{2} f(x) + \frac{2-i-j}{2} f(y) \\
&\leq mf(x) + (1-m)f(y)
\end{aligned}$$

which shows that $m \in \Lambda$.
Next, we show that the set $D = \left\{\frac{m}{2^n}; m \in \mathbb{Z}, n \in \mathbb{N}, 0 \leq m \leq n\right\}$ is a subset of [0,1]. We prove this by induction. Let E be the set of all n for which the lemma is true. We know that $\{0, 1/2, 1\} \subset \Lambda$, so we have $0, 1 \in E$. Now assume that E contains 1, 2,...,k. Choose an arbitrary m such that $0 \leq m \leq 2^{k+1}$.

1. m is even: Let $m = 2a$ for some $a \in \mathbb{Z}$ then $0 \leq a \leq 2^k$ and therefore $m/2^{k+1} = a/2k$ By the hypothesis of induction $a/2k \in \Lambda$ therefore $m/2^{k+1} \in \Lambda$.

2. m is odd: Then let $\frac{m-1}{2} = a$ and $\frac{m+1}{2} = b$ for some $a, b \in \mathbb{Z}$ with $a, b \in [0, 2^k]$. From this, we have

$$\frac{m}{2^{k+1}} = \frac{m+1}{2^{k+2}} + \frac{m-1}{2^{k+2}} = \frac{1}{2}\left(\frac{a}{2^k} + \frac{b}{2^k}\right).$$

By the hypothesis of induction $\frac{a}{2^k}, \frac{b}{2^k} \in \Lambda$. Therefore, by the above, $m/2^{k+1} \in \Lambda$. This covers all possible cases for m, so $k+1 \in E$. This completes the inductive step, therefore $E = \mathbb{Z}$.
Lastly, every element of [0,1] is a member of Λ. Choose any $p \in [0, 1]$. We see that D is

dense in [0,1] so we can construct a sequence (p_n) in D such that $\lim_n p_n = p$. Each (p_n) is an element of D, so for each n we have

$$f(p_n x + (1-p_n)y) \leq p_n f(x) + (1-p_n)f(y)$$

The function f is continuous, so, taking the limit to get

$$f(px + (1-p)y) \leq pf(x) + (1-p)f(y) \;\forall\; x, y \in (a,b)$$

which means that $p \in \Lambda$. This shows that f is convex. □

6.11.281 Problem. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and that $f(x+y) = f(x) + f(y) \;\forall\; x, y \in \mathbb{R}$. Define $f(1) = a; g(x) = ax$. Show that $f(x) = g(x) \;\forall\; x \in \mathbb{R}$.

6.11.281.1 Solution.

1. We show that $f(n) = g(n), \forall\; n \in \mathbb{Z}$. Let $n \in \mathbb{N}$, then $f(n) = f(1+1+..+1) = nf(1) = na = g(n)$. Again, $f(1+0) = f(1) + f(0)$ implies $f(0) = 0$, so $0 = f(0) = f(1+(-1))$ implies $f(-1) = -f(1)$. Thus, $f(-n) = -f(n) = -na = g(-n)$. Hence $f(n) = g(n) \;\forall\; n \in \mathbb{Z}$.

2. Next, we show that $2f\left(\frac{x}{2}\right) = f(x)$. Now, $f(x) = f(x/2 + x/2) = f(x/2) + f(x/2) = 2f(x/2)$ shows that $f\left(\frac{x}{2}\right) = \frac{1}{2}f(x)$.

3. Define $D = \left\{\frac{m}{2^n}; m \in \mathbb{Z}, n \in \mathbb{N}\right\}$. Since

$$\begin{aligned} f\left(\frac{m}{2^n}\right) &= \frac{1}{2}f\left(\frac{m}{2^{n-1}}\right) \\ &= \frac{1}{2}f\left(\frac{m}{2^{n-1}}\right) \\ &= \frac{1}{2^2}f\left(\frac{m}{2^{n-2}}\right) = ... \\ &= \frac{1}{2^n}f(m) = \frac{1}{2^n}mf(1) = \frac{1}{2^n}ma = g\left(\frac{m}{2^n}\right). \end{aligned}$$

As f and g agree on a dense set D and f is continuous, Hence $f(x) = g(x) \;\forall\; x \in \mathbb{R}$. □

6.11.282 Problem. A function $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and

$$f\left(\frac{x+y}{2}\right) = \frac{1}{2}(f(x) + f(y))$$

for all $x, y \in \mathbb{R}$. Prove that $f(x) = ax + b, (a, b \in \mathbb{R})$ for all $x \in \mathbb{R}$.

6.11.282.1 Solution. From the relation $f\left(\frac{x+y}{2}\right) = \frac{1}{2}(f(x) + f(y))$, we get

$$\begin{aligned} f\left(\frac{2x+0}{2}\right) &= \frac{1}{2}(f(2x) + f(0)) \\ \Rightarrow f\,(x) &= \frac{1}{2}(f(2x) + f(0)) \\ \Rightarrow f(x) + f(y) &= \frac{1}{2}(f(2x) + f(0)) + \frac{1}{2}(f(2y) + f(0)) \\ &= f(x+y) + f(0). \end{aligned}$$

Let $\phi(x) = f(x) - f(0)$, then

$$\begin{aligned}\phi(x) + \phi(y) &= f(x) - f(0) + f(y) - f(0) \\ &= f(x+y) + f(0) - 2f(0) \\ &= f(x+y) - f(0) = \phi(x+y).\end{aligned}$$

Thus $\phi(x+y) = \phi(x) + \phi(y)$. Since ϕ is continuous, we can prove easily that $\phi(x) = ax$ for some $a \in \mathbb{R}$. Hence $\phi(x) = ax$ implies that $f(x) - f(0) = ax$, i.e., $f(x) = ax + b$. where $f(0) = b$. $\square$

6.11.283 Problem. Fix a set of distinct real numbers, indexed by the natural numbers, $A = \{a_n; n \in \mathbb{N}\}$. Define

$$f_A = \begin{cases} 0, & \text{if } x \notin A \\ \frac{1}{n} & \text{if } x = a_n. \end{cases}$$

Show that f_A is discontinuous at exactly the points of A.

6.11.283.1 Solution. We begin by showing that f_A is discontinuous at the points of A. Let (x_m) be a sequence in A^C converging to the point $a_n \in A$. (This sequence exists because A^C is dense.) Then $\lim f(x_m) = 0 \neq 1/n = f(a_n)$ and thus f_A is discontinuous at all points of A.
We now must show that f_A is continuous at all points of A^C. Fix one point $b \in A^C$. So $f_A(b) = 0$. Let $\epsilon > 0$. We need to find a $\delta > 0$ to guarantee that $|x - b| < \delta$ implies $|f_A(x) - f_A(b)| = |f_A(x)| < \epsilon$. So choose some $N \in \mathbb{N}$ where $n > N$ implies that $1/n < \epsilon$. Let $\delta = \frac{1}{2}\min\{|a_j - b|; j = 1, 2, .., N\}$. Then if $|x - b_j| < \epsilon$ there are two possibilities:

- $x \in A^C$ and $f_A(x) = 0 < \epsilon$ or
- $x = a_n$ for $n > N$ and so $f_A(a_n) = 1/n < \epsilon$.

In either case $|x - b| < \delta$ implies $|f_A(x)| < \epsilon$ and so f_A is continuous at b. We are done. $\square$

6.11.284 Problem. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. A point x is called a **shadow point** if there exists a point $y \in \mathbb{R}$ with $y > x$ such that $f(y) > f(x)$. Let $a < b$ be real numbers and suppose that

- all the points of the open interval $I = (a, b)$ are shadow points;
- a and b are not shadow points.

Prove that

1. $f(x) \leq f(b)\ \forall\ a < x < b$;
2. $f(a) = f(b)$.

6.11.284.1 Solution.

1. We prove by contradiction. Suppose that exists a point $cin(a, b)$ such that $f(c) > f(b)$. By Weierstrass theorem, f has a maximal value m on $[c, b]$; this value is attained at some point $d \in [c, b]$. Since $f(d) = \max_{x \in [c.b]} f(x) \geq f(c) > f(b)$, we have $d \neq b$, so $d \in [c, b) \subset (a, b)$. The point d, lying in (a,b), is a shadow point, therefore $f(y) > f(d)$ for some $y > d$. From combining our inequalities we get $f(y) > f(d) > f(b)$.
Case 1: $y > b$. Then $f(y) > f(b)$ contradicts the assumption that b is not a shadow point.
Case 2: $y \leq b$. Then $y \in (d, b] \subset [c, b]$, therefore $f(y) > f(d) = m = \max_{x \in [c,b]} f(x) \geq f(y)$, contradiction again. $\square$

2. Since $a < b$ and a is not a shadow point, we have $f(a) \geq f(b)$. By part (1), we already have $f(x) \leq f(b)\ \forall\ x \in (a, b)$. By the continuity at a we have

$$f(a) = \lim_{x\to a+} f(x) \leq \lim_{x\to a+} f(b) = f(b).$$

Hence we have both $f(a) \geq f(b)$ and $f(a) \leq f(b)$, so $f(a) = f(b)$. □

6.11.285 Problem. Prove that if $f : [0, 1] \to [0, 1]$ is a continuous function, then the sequence (x_n) of iterates $x_{n+1} = f(x_n)$ converges if and only if

$$\lim_{n\to\infty}(x_{n+1} - x_n) = 0.$$

6.11.285.1 Solution. The "only if" part is obvious. Now suppose that $\lim_{n\to\infty}(x_{n+1} - x_n) = 0$ and the sequence (x_n) does not converge. Then there are two cluster points a, b and $a < b$. There must be points from the interval (K, L) in the sequence. There is an $x \in (a, b)$ such that $f(x) \neq x$. Put $\epsilon = \frac{1}{2}|f(x) - x| > 0$. Then from the continuity of the function f we get that for some $\delta > 0$ for all $y \in (x-\delta, x+\delta)$ and $|f(y)-y| > \epsilon$. On the other hand for n large enough we get $|x_{n+1}-x_n| < 2\delta$ and $|f(x_n)-x_n| < |x_{n+1}-x_n| < \epsilon$. So the sequence cannot come into the interval $(x-\delta, x+\delta)$, but also cannot jump over this interval. Then all cluster points have to be at most $x-\delta$ (a contradiction with b being a cluster point), or at least $x + \delta$ (a contradiction with a being a cluster point). □

6.11.286 Problem. Let f be continuous and nowhere monotone on [0,1]. Show that the set of points on which f attains local minima is dense in [0,1]. (A function is **nowhere monotone** if there exists no interval where the function is monotone. A set is dense if each non-empty open interval contains at least one element of the set.)

6.11.286.1 Solution. Let $(x - \alpha, x + \alpha) \subset [0, 1]$ be an arbitrary non-empty open interval. The function f is not monotone in the intervals $[x - \alpha, x]$ and $[x, x + \alpha]$, thus there exist some real numbers $x - \alpha \leq\, < p < q \leq x, x \leq r < s \leq x + \alpha$ so that $f(p) > f(q)$ and $f(r) < f(s)$. By Weierstrass theorem, f has a global minimum in the interval $[p, s]$. The values $f(p)$ and $f(s)$ are not the minimum, because they are greater than $f(q)$ and $f(s)$, respectively. Thus the minimum is in the interior of the interval, it is a local minimum. So each non-empty interval $(x - \alpha, x + \alpha) \subset [0, 1]$ contains at least one local minimum. □

6.11.287 Problem. Suppose $m, n \in \mathbb{Z}$, and f is defined on $(0, \infty)$ by

$$f(x) = x^m \sin\left(\frac{1}{x^n}\right).$$

Discuss the uniform continuity of f for different values of m and n.

6.11.287.1 Solution. We shall discuss about the following cases.

i. If $m = n = 0$ then the function $f(x) = x^m \sin\left(\dfrac{1}{x^n}\right)$ becomes the constant function and so f is uniformly continuous.

ii. If $m = 0, n < 0$ then $f(x) = \sin x^s$ where $s = -n$ then for $n = -1$ being uniformly continuous but for $n < -1$ not.

iii. If $m = 0, n > 0$ then $f(x) = \sin\left(\frac{1}{x^n}\right)$ which is not uniformly continuous.

iv. If $m < 0, n = 0$ then $f(x) = \frac{1}{x^r}\sin 1$ where $r = -m$, then f is not uniformly continuous by 6.3.2(1).

v. If $m > 0, n = 0$ then $f(x) = x^m \sin 1$ then f is uniformly continuous on $(0, \infty)$ for $m = 1$ and not for $m > 1$ by using result 6.3.2(3)

vi. If $m < 0$ and $n > 0$ then $\lim_{x\to 0} f(x)$ does not exist then by 6.3.2(1) f is not uniformly continuous.

vii. If $m > 0$ and $n < 0$ then the function is of the form $f(x) = x^m \sin x^k$ where $k = -n$, for $m = 1$ the function is not uniformly continuous as can be proved by contradicting the definition of uniformly continuous by taking $x = \sqrt[k]{2n\pi}$ and $y = \sqrt[k]{2n\pi + \frac{\pi}{2}}$. For $m > 1$ the function is not uniformly continuous as the function $\frac{|f(x)|}{x}$ is not bounded in the interval $[1, \infty)$.

viii. If $m < 0$ and $n < 0$, then f is of the form $f(x) = \frac{1}{x^r}\sin x^s$ where $r = -m, s = -n$. If $r = s$ then clearly $\lim_{x\to 0} f(x) = 1$ and $\lim_{x\to\infty} f(x) = 0$, so the function is uniformly continuous in this case. If $r < s$ then $m > n$ and $\lim_{x\to 0} f(x) = \lim_{x\to\infty} f(x) = 0$, and again uniformly continuous. But if $r > s$ then $m < n$ and $\lim_{x\to 0} f(x)$ does not exist, therefore not uniformly continuous.

ix. for $m > 0$ and $n > 0$, we consider $m = n, m = n+1, m < n$ and $m > n+1$.
For $m > 0$ and $n > 0$, that is $m, n \in \mathbb{N}$. For $m = n$, the function

$$x^m \sin\left(\frac{1}{x^n}\right) = x^n \sin\left(\frac{1}{x^n}\right) = \frac{\sin\left(\dfrac{1}{x^n}\right)}{\dfrac{1}{x^n}} \text{ and } \lim_{x\to\infty} \frac{\sin\left(\dfrac{1}{x^n}\right)}{\dfrac{1}{x^n}} = 1,$$

so the function $x^n \sin\left(\frac{1}{x^n}\right)$ is uniformly continuous on the interval $[1, \infty)$ again since

$$\lim_{x\to 0} x^n \sin\left(\frac{1}{x^n}\right) = 0$$

implies the function can be extended continuously on the interval [0,1] and hence uniformly continuous on the the interval [0,1] as well as on the interval $[1, \infty)$. So the function $x^m \sin\left(\frac{1}{x^n}\right)$ is uniformly continuous on the interval $(0, \infty)$ by the esult 6.3.2(5).
If $m = n+1$ then the function $f(x) = x^m \sin\left(\frac{1}{x^n}\right)$ becomes $f(x) = x^{n+1} \sin\left(\frac{1}{x^n}\right)$ and

$$f'(x) = (n+1)x^n \sin\left(\frac{1}{x^n}\right) - \cos\left(\frac{1}{x^n}\right)$$

Again since $\lim_{x\to\infty} f'(x) = n$ so the derivative of the function f, that is $f'(x)$ is bounded on the interval $[1, \infty)$ [using the result 6.3.2(2) for f'] so the function is uniformly continuous on the interval $[1, \infty)$ by 6.3.2(5). Again function f is also uniformly continuous on (0,1] as can be extended continuously on the interval [0,1] and hence f is uniformly continuous on the interval $(0, \infty)$ if $m = n+1$.

If $m < n$ then the function $f(x) = x^m \sin\left(\dfrac{1}{x^n}\right) = \frac{1}{x^r}\frac{sin\left(\frac{1}{x^n}\right)}{\frac{1}{x^n}}$ where r is such that $m + r = n$. Now in this case $\lim_{x\to\infty} f(x) = \lim_{x\to 0} f(x) = 0$ and hence the function is uniformly continuous in this by 6.3.2(1).
If $m > n + 1$ then

$$\frac{|f(x)|}{x} = \left|\frac{x^m \sin\left(\dfrac{1}{x^n}\right)}{x}\right| = \left|x^{m-1} \sin\left(\frac{1}{x^n}\right)\right|$$

on the interval $[1, \infty)$. But since $\lim_{x\to\infty} x^{m-1} \sin\left(\frac{1}{x^n}\right) \lim_{x\to\infty} x^r \sin\left(\frac{1}{x^n}\right) \to \infty$ as $m > n+1$ and hence there does not exist any positive number such that $\frac{|f(x)|}{x} \le M$ for $x \ge 1$. Therefore by result 6.3.2(3) f is not uniformly continuous on the interval $(0, \infty)$. □

Thus we conclude that the function $f(x) = x^m \sin\left(\frac{1}{x^n}\right)$ is uniformly continuous only for $m = n$ for $m, n \in \mathbb{Z}$, for $m = 0$ and $n = -1$, for $m = 1$ and $n = 0$, for $m, n < 0$ with $m > n$, and for $m, n \in \mathbb{N}$ with $m = n+1$ and for $m, n \in \mathbb{N}$ with $m < n$. For the othere cases f is not uniformly continuous on $(0, \infty)$.

6.11.288 Problem. Let $f : \mathbb{R} \to \mathbb{R}$ be a real function. Prove or disprove each of the following statements.

1. If f is continuous and $range(f) = \mathbb{R}$ then f is monotonic.
2. If f is monotonic and $range(f) = \mathbb{R}$ then f is continuous.
3. If f is continuous and monotonic, then $range(f) = \mathbb{R}$.

6.11.288.1 Solution.

1. False. Consider function $f(x) = x^3 - x$.
2. True. Assume first that f is non-decreasing. For an arbitrary number a, the limits $\lim_{x\to a-}$ and $\lim_{x\to a+}$ exist and $\lim_{x\to a-} \le \lim_{x\to a+}$. If the two limits are equal, the function is continuous at a. Otherwise, if $\lim_{x\to a-} = b < \lim_{x\to a+} = c$, we have $f(x) \le b \;\forall\; x < a$ and $f(x) \ge c \;\forall\; x > a$; therefore $range(f) \subset (-\infty, b) \cup (c, \infty) \cup \{a\}$ cannot be the complete $\mathbb{R}$.
3. False. Consider he function $g(x) = \tan^{-1} x$.

6.11.289 Problem. Suppose that f and g are real-valued functions on the real line and $f(r) \le g(r)$ for every rational r. Does this imply that $f(x) \le g(x)$ for every real x if

1. f and g are non-decreasing?
2. f and g are continuous?

6.11.289.1 Solution.

1. No. Suppose that, f and g are chosen as the characteristic functions of $[\sqrt{3}, \infty)$ and $[\sqrt{3}, \infty)$ respectively and we are done.

2. Yes. By the assumptions $g - f$ is continuous on the whole real line and nonnegative on the rationals. Since any real number can be obtained as a limit of rational numbers we get that $g - f$ is nonnegative on the whole real line. $\square$

6.11.290 Problem (Continuity of Arithmetic operations in $\mathbb{R}$)**.** Addition is a mapping $+ : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ that assigns to (x, y) the real number $x + y$. Subtraction $- : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and multiplication $\cdot : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are also such mappings. Division is a mapping $\div : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ that assigns to (x, y) the number x/y. Prove that all the operations $+, -, \cdot, \div$ are continuous functions.

6.11.290.1 Solution. Sketch: Let $(a, b) \in \mathbb{R} \times \mathbb{R}$ be given and let $\epsilon > 0$ be given.
$(+)$: Let $\delta = \epsilon$. If $|x - a| + |y - b| < \delta$ then

$$|(x + y) - (a + b)| = |(x - a) + (y - b)| \leq |x - a| + |y - b| < \delta = \epsilon.$$

$(-)$: Let $\delta = \epsilon$. If $|x - a| + |y - b| < \delta$ then

$$|(x - y) - (a - b)| = |(x - a) - (y - b)| \leq |x - a| + |y - b| < \delta = \epsilon.$$

$(\cdot)$: Let $\delta = \min(1, \epsilon/(|a| + |b|))$. If $|x - a| + |y - b| < \delta$ then

$$|(x \cdot y) - (a \cdot b)| = |x||y - b| + |x - a||b| < \epsilon.$$

$(\div)$: Let $\delta = \min(|b|/2, 1, \epsilon b^2/(2(|a| + |b|) + 2)$. If $|x - a| + |y - b| < \delta$ then

$$\begin{aligned} |(x \div y) - (a \div b)| &= \left|\frac{bx - ay}{by}\right| \\ &= \frac{|xb - xy| + |xy - ay|}{|by|} < \epsilon. \quad \square \end{aligned}$$

6.11.291 Problem. Prove that any Lipschitz function $f : \mathbb{Q} \to \mathbb{R}$ extends uniquely to a continuous function $g : \mathbb{R} \to \mathbb{R}$.

6.11.291.1 Solution. Lipschitz functions are, in particular, uniformly continuous. Since $\mathbb{Q}$ is dense in $\mathbb{R}$ and $\mathbb{R}$ is complete, then by the extension theorem, we conclude that f extends uniquely to a continuous function $g : \mathbb{R} \to \mathbb{R}$ (in fact, we showed that g is uniformly continuous).

6.12 Additional Exercises on Chapter 6.

6.12.1 Exercise. Show that, if $f : \mathbb{R} \to \mathbb{R}$ satisfies the Intermediate Value Property and $f(x \pm 0)$ both exist at every $x \in \mathbb{R}$, then f is continuous.

6.12.2 Exercise.

1. Give an example of a function $f : (0, 1) \to \mathbb{R}$ which is continuous, but such that there is no continuous function $g : [0, 1] \to \mathbb{R}$ which agrees with f on (0,1).

2. Suppose $f : A \to \mathbb{R}$. Prove that if f is uniformly continuous then there is a unique continuous function $g : \overline{A} \to \mathbb{R}$ which agrees with f on A.

6.12.3 Exercise. Let $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. If c be an isolated point of D then f is continuous at c.

6.12.4 Exercise. Let $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. If $c \in D \cap D'$ then f is continuous at c if and only if $\lim_{x \to c} f(x) = f(c)$.

6.12.5 Exercise. Let f be a continuous function on $0 < x < \infty$ satisfying $f(1) = 5$ and $f\left(\frac{x}{x+1}\right) = f(x) + 2$ for $0 < x < \infty$.

1. Find $\lim_{x \to \infty} f(x)$.
2. Prove that $\lim_{x \to 0^+} f(x) = +\infty$.
3. Find all such functions f.

6.12.6 Exercise. Let $I = [a, b]$ and let $f : I \to \mathbb{R}$ be continuous on I. If f has an absolute maximum [respectively, minimum] at an interior point c of I, show that f is not injective on I.

6.12.7 Exercise. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function that does not take on any of its values twice and with $f(0) < f(1)$. Show that f is strictly increasing on [0,1].

6.12.8 Exercise. If $I = [a, b]$ is an interval and $f : I \to \mathbb{R}$ is an increasing function, then the point a [respectively, b] is an absolute minimum [respectively, maximum] point for f on I. If f is strictly increasing, then a is the only absolute minimum point for f on I.

6.12.9 Exercise. Show that if $I = [a, b]$ and $f : I \to \mathbb{R}$ is increasing on I, then f is continuous at a if and only if $f(a) = \inf\{f(x); x \in (a, b]\}$.

6.12.10 Exercise. Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be increasing on I. Suppose that $c \in I$ is not an endpoint of I. Show that f is continuous at c if and only if there exists a sequence (x_n) in I such that $x_{2n-1} < c$ and $x_{2n} > c$ for $n \in \mathbb{N}$ and such that $x_n \to c$ and $f(x_n) \to f(c)$.

6.12.11 Exercise. Consider the function g defined by $g(x + y) = g(x) + g(y)$ for all $x, y \in \mathbb{R}$ where $\mathbb{R}$ is a vector space over the field $\mathbb{Q}$ of rationals. Show that g is not continuous on $\mathbb{R}$.

6.12.12 Exercise. Is the inverse image of a convergent sequence under a continuous function necessarily a convergent sequence?

6.12.13 Exercise. Give an example of a continuous function with domain $\mathbb{R}$ such that the image of a closed set is not closed.

6.12.14 Exercise. Let $f = p + g$ where p is a polynomial of odd degree and g is a bounded continuous function on the real line. Show that there is at least one solution of $f(x) = 0$.

6.12.15 Exercise. If g is a continuous function on a compact set, show that either g has a zero or g is bounded away from zero ($|g(s)| > 1/n$ for all x in the domain, for some $1/n$).

6.12.16 Exercise. If f is continuous on $[a, b]$ and g is continuous on $[b, c]$, show that

$$h(x) = \begin{cases} f(x), & \text{if } x \in [a, b] \\ g(x), & \text{if } x \in [b, c] \end{cases}$$

is continuous if and only if $f(b) = g(b)$.

6.12.17 Exercise. Give an example of a function on $\mathbb{R}$ that assumes its sup and inf on every compact interval and yet is not continuous.

6.12.18 Exercise. Show that a continuous, rational-valued function must be a constant.

6.12.19 Exercise. Suppose that $f : [0, 1] \to \mathbb{R}$ is continunous with $f(0) = f(1) = 0$. Prove that

1. there exist two points x_1 and x_2 such that as $|x_1 - x_2| = 1/n$, we have $f(x_1) = f(x_2) \neq 0$ for all n. In this case, we call $1/n$ the length of **horizontal strings.**

2. Could you show that there exists $2/3$ as the length of horizontal strings?

6.12.20 Exercise. Show that every rational function is continuous at every point where it is neither infinite nor indeterminate. What is the most set of points of discontinuity which such a function may have?

6.12.21 Exercise. A function $f : [0, 1] \to [0, 1]$ is defined by

$$f(x) = \begin{cases} x, & \text{if } x \in \mathbb{Q} \\ 1 - x, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Prove that

1. f is bijective., and $f^{-1} = f$.

2. f is continuous only at $1/2$, but not continuous on $[0, 1]$

Hence f^{-1} exists without being continuity of f.

6.12.22 Exercise. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous at a. If $\forall\, \delta > 0,\ \exists\, x \in N(a; \delta)$ such that $f(x) = 0$, prove that $f(a) = 0$.

6.12.23 Exercise. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous, let $a \in \mathbb{R}$ such that $f(a) > \mu$. Prove that $\exists$ a nbhd. U of a such that $f(x) > \mu\ \forall\, x \in U$.

6.12.24 Exercise. Give an example of a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that

1. $Z(f) = \{x \in \mathbb{R}; f(x) = 0\}$ is a bounded enumerable set.

2. $Z(f) = \{x \in \mathbb{R}; f(x) = 0\}$ is an unbounded enumerable set.

6.12.25 Exercise. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous, a point $x \in \mathbb{R}$ is said to be a fixed point of f if $f(x) = x$. Prove that that the set $S = \{x \in \mathbb{R}; x \text{ is a fixed point of } f\}$ is a closed set.

6.12.26 Exercise. Give an example of a bounded continuous function $f : \mathbb{R} \to \mathbb{R}$ with both $\sup\{f(x); x \in \mathbb{R}\}$ and $\inf\{f(x); x \in \mathbb{R}\} \notin ran(f)$.

6.12.27 Exercise. Let f be a non-constant, periodic and continuous at least one point. Prove that this function cannot have arbitrarily small positive periods.

6.12.28 Exercise. Let $f : [0, 1] \to \mathbb{R}$ be continuoous with maximum and minimum values not occuring at either 0 or 1. Show that f is not injective.

6.12.29 Exercise. Suppose that f, g are uniformly continuous on an interval I.

1. Prove that $f+g$ is uniformly continuous on I.
2. Give an example to show that fg may not be uniformly continuous on I.
3. Find additional hypothesis on f and g that will guarantee that fg is uniformly continuous on I.
4. Under what conditions, $\dfrac{1}{f}$ is uniformly continuous on I?

6.12.30 Exercise. Suppose that f is uniformly continuous on (a,b). Prove that f is bounded on (a,b).

6.12.31 Exercise. Suppose that f is uniformly continuous on $[a,b]$. Given a positive integer n, let $x_i = a + i(b-a)/n$, for $0 \leq i \leq n$, and define a function s_n and l_n on $[a,b]$ by

$$s_n = \begin{cases} f(a), & \text{if } x = a; \\ f(x_i), & \text{if } x_{i-1} \leq x \leq x_i; \end{cases}$$

and

$$l_n = \begin{cases} f(a), & \text{if } x = a; \\ \dfrac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}(x - x_i) + f(x_i), & \text{if } x_{i-1} \leq x \leq x_i; \end{cases}$$

Begin by drawing a graph to indicate how the graphs are related to the graph of f. Then prove that $\forall\ \epsilon > 0, \exists$ a positive integer p such that $|s_p(x) - f(x)| < \epsilon$ and $|l_p(x) - f(x)| < \epsilon, \forall x \in [a,b]$.

6.12.32 Exercise. Suppose that a function $f : [a,b] \to [a,b]$ be continuous. Prove that there exists at least one point x such that $f(x) = x$.
A point with this property is known as a **fixed point** of f.

6.12.33 Exercise. A function $f : [a,b] \to [a,b]$ is said to be **contraction** on $[a,b]$ if $\exists\ \alpha \in (0,1)$ such that $|f(x) - f(y)| \leq \alpha|x-y|, \forall\ x, y \in [a,b]$.

1. Let f be a contraction on $[a,b]$. Prove that f is uniformly continuous on $[a,b]$.
2. The previous problem shows that a contraction on $[a,b]$ must have a fixed point on it. Prove that a fixed point of contraction is unique.
3. Suppose thaat f is a contraction on $[a,b]$. Let x_0 be any point on $[a,b]$ and define a sequence (x_n) recursively by $x_n = f(x_{n-1}), \forall\ n \geq 1$. Prove that$(f(x_n))$ converges to the fixed point of f.
4. Prove that $f(x) = \sqrt{3x+2}$ is a contraction on $[1,5]$ and find its fixed point.
5. Find an interval on which $f(x) = x^3 + \dfrac{x^2}{6}$ is a contraction.
6. Use the identity
$$\cos(a-b) - \cos(a+b) = 2\ \sin a \sin b$$
and some properties of sine function to prove that $\cos x$ is a contraction on $[0,1]$. What is the fixed point of this contraction?

6.12.34 Exercise. Give an example of a continuous function $f : [0,1] \to \mathbb{R}$ for which the set $\{x \in [0,1]; f(x) = v\}$ is infinite for at least one $v \in \mathbb{R}$.

6.12.35 Exercise.

1. Let $f : [0,1] \to [0,1]$ be monotone and surjective. Show that f is continuous,
2. Let $g : [0,1] \to [0,1]$be continuous and injective. Show that g is strictly monotone.

6.12.36 Exercise. Let f be a finite function on $\mathbb{R}$ and define

$$\omega_f(\delta) = \sup\{|f(x) - f(y)|; |x - y| < \delta\},$$

$\delta > 0$, to be the modulus of continuity of f. Show that $\omega_f(\delta)$ decreases 0 as δ decreases to 0 and that f is uniformly continuous if and only if $\omega_f(\delta) \to 0$ as $\delta \to 0$.

6.12.37 Exercise. If $f : [0,1] \to [0,1]$, defined by

$$f(x) = \begin{cases} 0, & \text{if } x = 0, \\ 1/n, & \text{if } x \in \left(\dfrac{1}{n+1}, \dfrac{1}{n}\right]. \end{cases}$$

Prove that

1. f is increasing,
2. f is continuous at 0,1 and on $\left(\dfrac{1}{n+1}, \dfrac{1}{n}\right]$
3. f is discontinuous at $1/n; \forall\, n \geq 2$.

6.12.38 Exercise. Let $f : [0,1] \to \mathbb{R}$ be continuoous with maximum and minimum values not occuring at either 0 or 1. Show that f is not injective.

6.12.39 Exercise.

1. The $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} x, & \text{when} |x| \leq 1 \\ 1/x, & \text{when} |x| > 1 \end{cases}$$

   Prove that $f \circ f = f$.

2. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 - x, & \text{if } x \text{ is a rational number,} \\ -x, & \text{if } x \text{ is an irrational number.} \end{cases}$$

   Show that $f(f(x)) = x, \forall\, x \in \mathbb{R}$ and that f is not continuous. Furthermore, show that there is no interval $[a,b]$ such that $f[a,b] \subset [a,b]$.

6.12.40 Exercise. Let (x_n) be a sequence in $\mathbb{R}$ with no convergent subsequence. Prove that $\{x_n; n \in \mathbb{N}\}$ is a closed subset of $\mathbb{R}$.

6.12.41 Exercise. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be **periodic** if $\exists$ a positive integer p such that $f(x+p) = f(x), \forall x \in \mathbb{R}$. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a periodic function. Prove that a continuous periodic function on $\mathbb{R}$ is bounded and uniformly continuous on $\mathbb{R}$.

6.12.42 Exercise. Let $f : [a, b] \to \mathbb{R}$ be continuous and (x_n) is Cauchy sequence in $[a, b]$. Prove that $f(x_n)$ is a Cauchy sequence.

6.12.43 Exercise. Let $f : [a, b] \to \mathbb{R}$ be monotone, then f can be uniformly approximated on $[a, b]$ by step functions.

6.12.44 Exercise. A $f : [a, b] \to \mathbb{R}$ is called a **polygonal function** if there exists points $c_0, c_1, .., c_m$ and numbers $p_1, p_2, ..., p_m$ such that $a = c_0 < c_2 < ... < c_m = b$ and that for $r = 1, 2, .., m$ we have

$$f(x) = \left(\frac{c_r - x}{c_r - c_{r-1}}\right) p_{r-1} + \left(\frac{x - c_{r-1}}{c_r - c_{r-1}}\right) p_r, \text{ for } c_{r-1} \le x \le c_r$$

(i.e. the graph of f consists of a finite number of segments joining the points (c_r, p_r) to form a polygon.)
Prove that if f is continuous on $[a, b]$, then f can be uniformly approximated on $[a, b]$ by polygonal functions.

6.12.45 Exercise. Let $f : (a, b] \to \mathbb{R}$ be continuous. Is it true or false that f must attain at least one of its supremum and infimum on $(a, b]$? Prove your assertion.

6.12.46 Exercise. Let $I_n = \left[\frac{2}{2n+1}, \frac{1}{n}\right], \forall\, n \in \mathbb{N}$ and let $E = \{0\} \cup \bigcup_{n-1}^{\infty} I_n$. Then define a function $f : E \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0, & \text{if } x = 0 \\ \sqrt[n]{5}, & \text{if } x \in I_n. \end{cases}$$

Prove that f is continuous on E.

6.12.47 Exercise. Let $I_n = [\frac{1}{2^n}, \frac{1}{2^n} + \frac{1}{4^n}], \forall\, n \in \mathbb{N}$ and let $E = \{0\} \cup \bigcup_{n-1}^{\infty} I_n$. Then define a function $f : E \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0, & \text{if } x = 0 \\ \frac{1}{n} \sin\,(\frac{\pi}{x}), & \text{if } x \in I_n. \end{cases}$$

Prove that f is continuous on E.

6.12.48 Exercise. Let $((a_n, b_n))$ be sequence of disjoint open intervals and let $G = \bigcup_{n=1}^{\infty}(a_n, b_n)$. Given a sequence $v_n \in \mathbb{R}$, and define a function $f : G \to \mathbb{R}$ by $f(x) = v_n$, if $x \in (a_n, b_n)$. Prove that f is continuous on G.

6.12.49 Exercise. Many statements in mathematics involve the so-called logical quantiers 'for all' ($\forall$) and 'there exists' ($\exists$). The order of these quantifiers is essential as interchanging them can alter the logical meaning of the statement. As an example we give a definition of the continuity of a function $f : D \to \mathbb{R}$ at a point $a \in D$. The function f is continuous at the point $a \in D$ if

$$(\forall\, \epsilon > 0)(\exists\, \delta > 0)(\forall\, x \in D)(|x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon)$$
$$\text{i.e. } \forall\, \epsilon > 0\ \exists\, \delta > 0 \text{ such that } \forall\, x \in D \text{ and } |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon.$$

We often need to find the negation of this statement. It can be shown that this is equivalent to interchanging $(\forall)$ and $(\exists)$:

$$(\exists\, \epsilon > 0)(\forall\, \delta > 0)(\exists\, x \in D)(|x-a| < \delta \text{ and } |f(x) - f(a)| \geq \epsilon)$$

Let $x_0 \in \mathbb{R}$. Following are eight ϵ-δ conditions on a function $f : \mathbb{R} \to \mathbb{R}$. Which, if any, of these conditions imply continuity of f at x_0? Which, if any, are implied by continuity at x_0?

1. $\forall\, \epsilon > 0\ \exists\, \delta > 0$ such that $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$.
2. $\forall\, \epsilon > 0\ \exists\, \delta > 0$ such that $|f(x) - f(x_0)| < \delta \Rightarrow |x - x_0| < \epsilon$.
3. $\forall\, \epsilon > 0\ \exists\, \delta > 0$ such that $|x - x_0| < \epsilon \Rightarrow |f(x) - f(x_0)| < \delta$.
4. $\forall\, \epsilon > 0\ \exists\, \delta > 0$ such that $|f(x) - f(x_0)| < \epsilon \Rightarrow |x - x_0| < \delta$.
5. $\exists\, \epsilon > 0$ such that $\forall\, \delta > 0\ \ |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$.
6. $\exists\, \epsilon > 0$ such that $\forall\, \delta > 0\ \ |f(x) - f(x_0)| < \delta \Rightarrow |x - x_0| < \epsilon$.
7. $\exists\, \epsilon > 0$ such that $\forall\, \delta > 0\ \ |x - x_0| < \epsilon \Rightarrow |f(x) - f(x_0)| < \delta$.
8. $\exists\, \epsilon > 0$ such that $\forall\, \delta > 0\ \ |f(x) - f(x_0)| < \epsilon \Rightarrow |x - x_0| < \delta$.

For each of the eight conditions of the above exercises, describe in words which functions satisfy the condition. (Some of these conditions characterize familiar classes of functions, including the empty class.)

6.12.50 Exercise (**Universal Chord Theorem**)**.** Let $f : [0, 1] \to \mathbb{R}$ and $f(0) = f(1)$, i.e. there exists a horizontal chord of length 1, then there are horizontal chords of lengths 1/2,1/3,1/4,..., but not necessarily a horizontal chord of any given length that is not the reciprocal of an integer.

6.12.51 Exercise. Give an example of a continuous function $f : [0, 1] \to \mathbb{R}$ for which the set $\{x \in [0, 1]; f(x) = v\}$ is infinite for at least one $v \in \mathbb{R}$.

6.12.52 Exercise.

1. Let $f : [0, 1] \to [0, 1]$ be monotone and surjective. Show that f is continuous.
2. Let $g : [0, 1] \to [0, 1]$ be continuous and injective. Show that g is strictly monotone.

6.12.53 Exercise. Let f be a continuous function on R and define

$$\omega(\delta) = \sup\{|f(x) - f(y)|; |x - y| < \delta\},$$

$\delta > 0$, to be the modulus of continuity of f. Show that $\omega(\delta)$ decreases 0 as δ decreases to 0 and that f is uniformly continuous if and only if $\omega(\delta) \to 0$ as $\delta \to 0$.

6.12.54 Exercise.

1. Prove that $\mathbb{R} \setminus \mathbb{Q}$ is a G_δ set.
2. Find a function $f : \mathbb{R} \to \mathbb{R}$ whose set of points of discontinuity is $\mathbb{R} \setminus \mathbb{Q}$.

6.12.55 Exercise. Give an example of a non-constant continuous function $f : \mathbb{R} \to \mathbb{R}$ and an open set G such that $f(G)$ is not open.

6.12.56 Exercise. Let $((a_n, b_n))$ be sequence of disjoint open intervals and let $G = \bigcup_{n=1}^{\infty}(a_n, b_n)$. Given a sequence $v_n \in \mathbb{R}$, and define a function $f : G \to \mathbb{R}$ by $f(x) = v_n$, if $x \in (a_n, b_n)$. Prove that f is continuous on G.

6.12.57 Exercise. Definition(Periodic Function): A function $f : \mathbb{R} \to \mathbb{R}$ is said to be **periodic** if $\exists$ a positive integer p such that $f(x + p) = f(x), \forall x \in \mathbb{R}$.
Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a periodic function. Prove that f is uniformly continuous on $\mathbb{R}$.

6.12.58 Exercise. Let $f : I \to \mathbb{R}$ be a function defined on an interval I is continuous on some open interval I, and c is a maximum point for f (i.e. points where it takes on its maximum value on I), inside this interval. Common sense suggests f should be increasing immediately to the left of c and decreasing immediately to the right of c. Is this true? Either prove it, or give a counterexample. (Note that a constant function is considered to be both increasing and decreasing.)

6.12.59 Exercise. Let $f : I \to \mathbb{R}$ be continuous on the compact interval I, and suppose that f has an infinity of maximum points (i.e. points where it takes on its maximum value on I), an infinity of minimum points on I, and that between any two maximum points lies at least one minimum point. Prove f is constant on I. Why isn't f defined by $f(x) = \sin(1/x)$ a counterexample?

6.12.60 Exercise. Let f be a non-constant, periodic and continuous at least one point. Prove that this function cannot have arbitrarily small positive periods.

6.12.61 Exercise. Let $f : \mathbb{R} \to \mathbb{R}$ and $t, a \neq 0 \in \mathbb{R}$. Define $g, h : \mathbb{R} \to \mathbb{R}$ bt $g(x) = f(x + t)$ and $h(x) = f(ax)$. Prove that, if f is continuous(uniformly continuous) then show that g, h are also continuous(uniformly continuous).

6.12.62 Exercise.

Let $f : \mathbb{R} \to \mathbb{R}$ be such that f satisfies Cauchy functional equation

$$f(x + y) = f(x) + f(y) \ \forall \ x, y \in \mathbb{R}.$$

1. If f is continuous at a point $x_0 \in \mathbb{R}$, show that f is continuous on $\mathbb{R}$ and $\exists$ a constant $c \in \mathbb{R}$ such that
$$f(x) = cx \ \forall \ x \in \mathbb{R}.$$
2. If is bounded above on some interval or f is monotonic on $\mathbb{R}$ then also $\exists$ a constant $c \in \mathbb{R}$ such that
$$f(x) = cx \ \forall \ x \in \mathbb{R}.$$

6.12.63 Exercise. Let $f : (0, \infty) \to \mathbb{R}$ be such that f satisfies

$$f(xy) = f(x) + f(y) \ \forall \ x, y \in \mathbb{R}.$$

If f is continuous at a point $x_0 \in (0, \infty)$, show that f is continuous on $\mathbb{R}$ and $\exists$ a constant $c \in \mathbb{R}$ such that

$$f(x) = c \log x \ \forall \ x \in (0, \infty).$$

6.12.64 Exercise. Let $f : (0, \infty) \to \mathbb{R}$ be such that f satisfies

$$f(xy) = f(x)f(y) \ \forall \ x, y \in \mathbb{R}.$$

If f is continuous at a point $x_0 \in (0, \infty)$, show that f is continuous on $\mathbb{R}$. Find all such continuous functions.

6.12.65 Exercise. Find all such continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) - f(y)$ is rational for rational $x - y$.

6.12.66 Exercise. For $|q| < 1$, find all such continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that f is is continuous at 0 and satisfies

$$f(x) + f(qx) = 0.$$

6.12.67 Exercise. Let $f : \mathbb{R} \to \mathbb{R}$ be such that f satisfies functional equation

$$f(x + y) = f(x)f(y) \ \forall \ x, y \in \mathbb{R}.$$

If f is continuous at 0, show that f is continuous on $\mathbb{R}$ and $\exists$ a constant $c \in \mathbb{R}$ such that

$$f(x) = e^{cx} \ \forall \ x \in \mathbb{R}.$$

6.12.68 Exercise. Find all such continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x)$ satisfies Jensen equation

$$f\left(\frac{x + y}{2}\right) = \frac{f(x) + f(y)}{2} \ \forall \ x, y \in \mathbb{R}.$$

6.12.69 Exercise. Find all such continuous functions $f : (a, b) \to \mathbb{R}$ such that $f(x)$ satisfies Jensen equation

$$f\left(\frac{x + y}{2}\right) = \frac{f(x) + f(y)}{2} \ \forall \ x, y \in \mathbb{R}.$$

6.12.70 Exercise. Let $f : \mathbb{R} \to \mathbb{R}$ be such that either $f(x+)$ or $f(x-)$ exists finitely then show that the set of discontinuities of f is countable. The result holds even if the limit exists infinitely.

6.12.71 Exercise. Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$. Suppose f has a local minima (maxima) at two different points $x_1, x_2 \in (a, b)$. Show that there exists a point ξ with $x_1 < \xi < x_2$ where f has a local maximum(minimum).

6.12.72 Exercise. Let $f : [a, b] \to \mathbb{R}$ be continuous such that for each $c \in (a, b)$ there is no nbhd. U of c such that either $f(x) \geq f(c)$ or $f(x) \leq f(c), \forall x \in U$. Prove that f is either strictly increasing or strictly decreasing on $[a, b]$.

6.12.73 Exercise. Let $f : [0, 1] \to \mathbb{R}$ be continuous such that $f(0) = f(1)$. Prove that for each positive integer $n, \exists \ x \in [0, -1/n]$ such that $f(x) = f(x + 1/n)$.

6.12.74 Exercise. Let $f : (0, 1] \to \mathbb{R}$ be continuous and bounded. Is it true or false that f must attain at least one of its sup of inf on $(a, b]$? Justify.

6.12.75 Exercise. Let $f : \mathbb{R} \to \mathbb{R}$ be bounded. And in every closed interval it itattains supremum and infimum there. Is it continuous on $\mathbb{R}$? Justify.

6.12.76 Exercise. Let $f : [a, b] \to \mathbb{R}$ be monotone, then show that f can be uniformly approximated on $[a, b]$ by step-functions.

6.12.77 Exercise. Let $f : [a, b] \to \mathbb{R}$ be continuous, show that f can be uniformly approximated on $[a, b]$ by polygonal functions.

6.12.78 Exercise. Prove that the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \inf \left\{ \left| x - \frac{1}{n} \right|, \forall\, n \in \mathbb{N} \right\}$$

is continuous on $\mathbb{R}$.

6.12.79 Exercise. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous.

1. If $f(0) > 0$, show that $\exists a \in \mathbb{R}$ such that $f(x) > 0, \forall x \in (-a, a)$.
2. If $f(x) \geq 0$ for all rational x, show that $f(x) \geq 0, \forall x \in \mathbb{R}$. Is it true when "$\geq 0$" replaced by "$> 0$".

6.12.80 Exercise. Let $f : [a, b] \to \mathbb{R}$ be continuous at $c \in [a, b]$ and suppose that $f(c) > 0$. Prove that $\exists\, m > 0$ and an interval $[u, v] \subseteq [a, b]$ such that $c \in [u, v]$ and $f(x) \geq m, \forall x \in [u, v]$.

6.12.81 Exercise. Let (r_n) be the enumeration of $\mathbb{Q}$ and (v_n) be a sequence of non-zero real numbers that converges to 0. Define a function

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is irratioal,} \\ v_n, & \forall n \in \mathbb{N} \end{cases}$$

Show that f is continuous everywhere except for the set $\mathbb{Q}$.

6.12.82 Exercise. (Characterization of Monotone Functions).

1. Show that $f : \mathbb{R} \to \mathbb{R}$ is monotone if and only if $f^{-1}(I)$ is an interval for every interval $I \subseteq \mathbb{R}$.
2. Give an example of a continuous function $f : \mathbb{R} \to \mathbb{R}$ and an interval $I \subseteq \mathbb{R}$ such that $f^{-1}(I)$ is not an interval.

6.12.83 Exercise. Show that there can be no continuous function $f : \mathbb{R} \to \mathbb{R}$ that satisfies

$$f(x) \in \mathbb{Q} \Leftrightarrow f(x+1) \in \mathbb{Q}^C.$$

6.12.84 Exercise. Suppose that $f : [a, b] \to \mathbb{R}$ is monotone. Prove that f is continuous a.e. everywhere on $[a, b]$.

6.12.85 Exercise. For the function $f(x) = \begin{cases} \sin \frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$,
find $\omega_f(0)$.

6.12.86 Exercise. Suppose that $f : [a, b] \to \mathbb{R}$ is a bounded function and let $c \in [a, b]$. Prove that f is continuous at c iff $\omega_f(c) = 0$.

6.12.87 Exercise. Let $S \subseteq \mathbb{R}, f : S \to \mathbb{R}$. Consider the statements

1. $\forall\, y \in S$ and $\forall\, \epsilon > 0$, $\exists\, \delta > 0$ such that $x \in S$ and $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$.
2. $\forall\, \epsilon > 0\ \exists\, \delta > 0$ such that $\forall\, y \in S\ x \in S\ |x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$.

One of these says that f is continuous, the other that f is uniformly continuous. Which is which?

6.12.88 Exercise. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$ and let $S = \{x \in \mathbb{R}; f(x) = 0\}$ be the "zero set" of f. If (x_n) is in S and $x = \lim x_n$, show that $x \in S$.

6.12.89 Exercise. Let $A \subseteq B \subseteq \mathbb{R}$, let $f : \mathbb{R} \to \mathbb{R}$ and let g be the restriction of f to A (that is, $g(x) = f(x)$ for $x \in A$).

1. If f is continuous at $c \in A$, show that g is continuous at c.
2. Show by example that if g is continuous at c, it need not follow that f is continuous at c.

6.12.90 Exercise. Let $f : (0, 1) \to \mathbb{R}$ be bounded but such that $\lim_{x\to 0} f(x)$ does not exist. Show that there are two sequences (x_n) and (y_n) in (0,1) with $\lim(x_n) = 0 = \lim(y_n)$, but such that $(f(x_n))$ and $(f(x_n))$ exist but are not equal.

6.12.91 Exercise. Let $x_1, x_2, ..., x_n$ be real numbers, each in the domain of some function f. Show that f is uniformly continuous on the set $X = \{x_1, x_2, .., x_n\}$.

6.12.92 Exercise. Let $X = \{x_1, x_2, .., x_n\}$. What property must X have so that every function continuous on X is uniformly continuous on X?

6.12.93 Exercise. Suppose f is uniformly continuous on each of the sets $X_1, X_2, ..., X_n$ and let $X = \bigcup_{i=1}^{n} X_i$. Show that f need not be continuous on X. Show that, even if f is continuous on X, f need not be uniformly continuous on X.

6.12.94 Exercise. Suppose f is uniformly continuous on each of the compact sets $X_1, X_2, ..., X_n$. Prove that f is uniformly continuous on the set $X = \bigcup_{i=1}^{n} X_i$. Show that this need not be the case if the sets X_k are not closed and need not be the case if the sets X_k are not bounded.

6.12.95 Exercise. Give an example of a function f that is continuous on $\mathbb{R}$ and a sequence of compact intervals $X_1, X_2, ..., X_n, ...$ on each of which f is uniformly continuous, but for which f is not uniformly continuous on $X = \bigcup_{i=1}^{n} X_i$.

6.12.96 Exercise. Suppose that $f : E \to \mathbb{R}$ is continuous. If E is compact, then f must be uniformly continuous on E. Conversely, if every continuous function $f : E \to \mathbb{R}$ is uniformly continuous, then E must be compact.

6.12.97 Exercise. Show that, if $f \in C(\mathbb{R})$ is strictly decreasing, then $f(x) = x$ for a unique $x \in \mathbb{R}$ i.e., f has a unique fixed point.

6.12.98 Exercise. If $f \in C[0, 1]$ satisfies $f([0, 1]) \subset \mathbb{Q}$, what can you say about f?

6.12.99 Exercise. Show that no function $f \in C[0, 1]$ assumes each of its values exactly twice.

6.12.100 Exercise. Let I and J be (nonempty) intervals.

1. Show that, if $f, g : I \to \mathbb{R}$ are uniformly continuous (on I), then so is f. Show that the same is true for fg provided f and g are both bounded (on I). Give an example to show that, if one of f, g is unbounded, then fg need not be uniformly continuous.
2. Show that, if $f : I \to \mathbb{R}$ and $g : J \to \mathbb{R}$ are uniformly continuous and $f(I) \subseteq J$, then $g \circ f$ is also uniformly continuous.

6.12.101 Exercise. Show that, if $S \subseteq \mathbb{R}$ is bounded, then any uniformly continuous $f : S \to \mathbb{R}$ is bounded. Show (by example) that f need not be bounded if S is unbounded.

6.12.102 Exercise. Show that, if $f : [1, \infty) \to \mathbb{R}$ is uniformly continuous, then $f(x) = O(x)$ $(x \to \infty)$.

6.12.103 Exercise. Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$. Show that, if for each $\epsilon > 0$ there is a uniformly continuous function $g : I \to \mathbb{R}$ such that

$$|f(x) - g(x)| < \epsilon \; \forall \, x \in I, \forall \, x \in I,$$

then f is also uniformly continuous.

6.12.104 Exercise. Let $\emptyset \neq S \subseteq \mathbb{R}$. For all $x \in \mathbb{R}$, show that the function

$$d_S(x) = \inf\{|x - s|; s \in S\}$$

is Lipschitz: $|d_S(x) - d_S(y)| \leq |x - y| \; \forall \, x, y \in \mathbb{R}$.

6.12.105 Exercise.

1. Let $F \subseteq \mathbb{R}$ be a closed set. Show that there is a function $f \in C(\mathbb{R})$ such that $F = \{x; f(x) = 0\}$.
2. Let $E, F \subseteq \mathbb{R}$ be closed sets such that $E \cap F = \emptyset$. Show that there is a function $f \in C(\mathbb{R})$ such that $E = \{x; f(x) = 1\}$ and $F = \{x; f(x) = 0\}$. Hint: Use the functions d_E and d_F introduced in previous problem.

6.12.106 Exercise. (Kepler's Equation). Show that, for any constants $a \in (0, 1)$ and $b \in \mathbb{R}$, the equation $x = a \sin x + b$ has a unique solution.

6.12.107 Exercise. Let $a < b$ and suppose that $f : [a, b] \to [a, b]$ satisfies $|f(x) - f(y)| \leq |x - y|$ for all $x, y \in [a, b]$. Show that the sequence (x_n) defined recursively by $x_{n+1} = (x_n + f(x_n))/2$ and an arbitrary $x_1 \in [a, b]$, converges to a fixed point of f. Hint: Show that (x_n) is monotone by using the identity

$$x_{n+2} - x_{n+1} = \frac{1}{2} \left[f(x_{n+1}) - f(x_n) + x_{n+1} - x_n \right] \quad \forall \, n \in \mathbb{N}.$$

6.12.108 Exercise. Let $a < b$ and let $f : [a, b] \to [a, b]$ be Lipschitz with Lipschitz constant 1 i.e., $|f(x) - f(y)| \leq |x - y|$ for all $x, y \in [a, b]$. Show that the set of all fixed points of f is a subinterval $[\alpha, \beta] \subseteq [a, b]$ possibly reduced to a single point.

6.12.109 Exercise. (**Contractive Map**). Let $\emptyset \neq S \subseteq \mathbb{R}$. A map $f : X \to X$ is said to be **contractive** if

$$|f(x) - f(y)| < |x - y| \; \forall \, x, y \in X$$

1. Show that a contractive map has at most one fixed point.
2. Show that the function $f(x) = x + \frac{1}{x}$ is contractive on $[1, \infty)$ and does not have fixed points.
3. Show that $g(x) = \frac{1}{2}(x + \sin x)$ is contractive on $\mathbb{R}$. Is there a fixed point?
4. Show that, if f is contractive on a (nonempty) compact set $K \subseteq \mathbb{R}$, then it has a unique fixed point. Hint: Look at $\inf\{|f(x) - x|; x \in K\}$.

6.12.110 Exercise. (**Expansive Map**). Let $\emptyset \neq S \subseteq \mathbb{R}$. A map $f : X \to X$ is said to be **expansive** if

$$|f(x) - f(y)| \geq |x - y| \; \forall \; x, y \in X.$$

Show that if $f : \mathbb{R} \to \mathbb{R}$ is both continuous and expansive, then it is a homeomorphism with Lipschitz inverse $f : \mathbb{R} \to \mathbb{R}$.

6.12.111 Exercise. Let $f : [a, b] \to \mathbb{R}$ be continuous and satisfies $f(a) < f(b)$. Show that there are $c, d \in [a, b]$ such that $a \leq c < d \leq b$ and $f(a) = f(c) < f(x) < f(d) = f(b)$ for all $x \in (c, d)$.

6.12.112 Exercise.

1. Show that, if f has a local maximum at every point $x \in I$, then the range of f is countable. Hint: For each $y \in f(I)$, pick an interval J_y with rational endpoints such that $y = \max\{f(x); x \in I \cap J_y\}$. Assuming in addition that f is continuous on I, show that f must be constant.

2. Show that $M = \{x \in I;\; f \text{ has a strict local maximum at } x\}$ is countable.

3. Show that, if $f : [a, b] \to \mathbb{R}$ be continuous and does not have a local maximum or minimum at any point in (a, b), then it must be monotone.

6.12.113 Exercise.

1. (**Characterization of Monotone Functions**) Show that $f : \mathbb{R} \to \mathbb{R}$ is monotone if and only if $f^{-1}(J)$ is an interval for every interval $J \subseteq \mathbb{R}$.

2. Give an example of a continuous function $f : \mathbb{R} \to \mathbb{R}$ and an interval $J \subseteq \mathbb{R}$ such that $f^{-1}(J)$ is not an interval.

6.12.114 Exercise. Suppose that A is a nonempty subset of $\mathbb{R}$ consisting only of accumulation points. Can A be countable?

6.12.115 Exercise. Give three examples of a function f that fails to be continuous at a point x_0. The first should be discontinuous merely because f is not defined at x_0. The second should be discontinuous because $\lim_{x\to x_0} f(x)$ fails to exist. The third should have neither of these defects but should nonetheless be discontinuous.

6.12.116 Exercise. Prove or disprove: If f is a continuous function on [0,1) such that $\lim_{x\to\infty} \frac{f(x)}{x} = 1$, then f is uniformly continuous on [0,1).

6.12.117 Exercise. Define a function f on [0,1] by $f(x) = \begin{cases} 0, & \text{if } x = 0 \\ x \ln x, & \text{if } 0 < x \leq 1. \end{cases}$

Is f uniformly continuous on [0,1]?

6.12.118 Exercise. Prove that for every set S of real numbers, except for a subset of S of the first category, every point of S is a point of second category of S.

6.12.119 Exercise. Prove that the set S_c of points of the second category ot any set S is perfect.

6.12.120 Exercise. Prove that if S is a set of real numbers of the second category there is an interval every point of which is a point of the second category of S.

6.12.121 Exercise.

1. Prove that $\mathbb{R} \setminus \mathbb{Q}$ is a G_δ set.

2. Find a function $f : \mathbb{R} \to \mathbb{R}$ whose set of points of discontinuity is $\mathbb{R} \setminus \mathbb{Q}$.

6.12.122 Exercise. Let $E \subset \mathbb{R}$. Recall that

A set E is a G_δ **set** if E is a countable intersection of open sets.

A set E is an F_σ **set** if E is a countable union of closed sets. Show that

1. The complement of F_σ set is a G_δ set.

2. The complement of G_δ set is an F_σ set.

3. Countable union of F_σ sets is an F_σ set.

4. Countable intersection of G_δ sets is a G_δ set.

5. Intersection of two F_σ sets is an F_σ set.

6. Union of two G_δ sets is a G_δ.

7. Every open set is an F_σ set.

8. Every closed set is a G_δ set.

9. The set difference of two closed sets is an F_σ set.

10. The set $\mathbb{Q}$ is not a G_δ set, but it is an F_σ set.

6.12.123 Exercise. Let F be a closed nonempty subset of $\mathbb{R}$. Assume that $F = \bigcup_{n=1}^{\infty} F_n$, where each F_n is a closed set. Then there exists $n \in \mathbb{N}$ such that F_n has a nonempty interior relatively to F.

6.12.124 Exercise. $\mathbb{Q}$ cannot be written as the countable intersection of open subsets of $\mathbb{R}$.

6.12.125 Exercise. $\mathbb{R} \setminus \mathbb{Q} \neq D(f)$ for any $f : \mathbb{R} \to \mathbb{R}$, where $D(f)$ is the set of points of discontinuities of f.

6.12.126 Exercise. If $\mathbb{R} = \bigcup_{n=1}^{\infty} E_n$ where each E_n is closed, then some E_n contains an open interval.

6.12.127 Exercise. If $\mathbb{R} = \bigcup_{n=1}^{\infty} E_n$ then the closure of some E_n contains an interval; that is, $(\overline{E_n})^\circ \neq \emptyset$ for some n.

6.12.128 Exercise. If $\mathbb{R} \setminus \mathbb{Q} = \bigcup_{n=1}^{\infty} E_n$ then the closure of some E_n contains an interval; that is, $(\overline{E_n})^\circ \neq \emptyset$ for some n.

6.12.129 Exercise. Prove that any dense G_δ-set in $\mathbb{R}$ is necessarily uncountable.

6.12.130 Exercise. Prove that A has an empty interior in $\mathbb{R}$ if and only if A^C is dense in $\mathbb{R}$.

6.12.131 Exercise. If G is open and dense in $\mathbb{R}$, show that the same is true of $G \setminus \{x\}$ for any $x \in \mathbb{R}$.

6.12.132 Exercise. If $x_n \to x$ in $\mathbb{R}$, show that the set $\{x\} \cup \{x_n; n \geq 1\}$ is nowhere dense in $\mathbb{R}$.

6.12.133 Exercise. Let (r_n) be an enumeration of $\mathbb{Q}$. For each n, let I_n be the open interval centered at r_n of radius 2^{-n}, and let $\bigcup_{n=1}^{\infty} I_n$. Prove that U is a proper, open, dense subset of $\mathbb{R}$ and that U^C is nowhere dense in $\mathbb{R}$.

6.12.134 Exercise. Is there a dense, open set in $\mathbb{R}$ with uncountable complement? Explain.

6.12.135 Exercise. Show that any subset of a first category set is still first category, and that a countable union of first category sets is again first category.

6.12.136 Exercise. Prove that any superset of a second category set is itself a second category set.

6.12.137 Exercise. Show that $\mathbb{N}$ is first category in $\mathbb{R}$ but second category in itself.

6.12.138 Exercise. In $\mathbb{R}$, show that any open interval (and hence any nonempty, open set) is a second category set.

6.12.139 Exercise. Let E be an F_σ set in $\mathbb{R}$. Prove that E is a first category set in $\mathbb{R}$ if and only if E^C is dense in $\mathbb{R}$.

6.12.140 Exercise. Let $f : \mathbb{R} \to \mathbb{R}$. Show that f is discontinuous on a set of the first category in $\mathbb{R}$ if and only if f is continuous at a dense set of points.

6.12.141 Exercise. Show that the complement of a first category set in $\mathbb{R}$ is uncountable.

6.12.142 Exercise. When is a first category set an F_σ set? Equivalently, when is a set containing a dense G_δ set itself a G_δ set?

6.12.143 Exercise. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function that is non-constant on any interval. If A is a second category set in $\mathbb{R}$, show that $f(A)$ is also second category. [Hint: If B is closed and nowhere dense, show that $f^{-1}(B)$ is closed and nowhere dense.]

6.12.144 Exercise. Let S be a subset of $\mathbb{R}$. A point $x \in S$ is said to be a point of first category relative to S if, for some neighborhood U of x, the set $U \cap S$ is of first category in $\mathbb{R}$. If S_0 is the set of points of first category relative to S, show that S_0 is of first category in $\mathbb{R}$. [Hint: $\mathbb{R}$ has a countable open base.]

6.12.145 Exercise. Prove that if (E_n) is a sequence of closed sets in $\mathbb{R}$, each having empty interior, then $\bigcup_{n=1}^{\infty} E_n$ has empty interior.

6.12.146 Exercise. Is the complement of a first category set necessarily a second category set? Likewise, is the complement of a second category set necessarily a first category set? Explain.

6.12.147 Exercise. If A is either open or closed in $\mathbb{R}$, show that ∂A is nowhere dense in $\mathbb{R}$. Is the same true of any set A?

6.12.148 Exercise. Show that $\{x\}$ is nowhere dense in $\mathbb{R}$ iff x is not an isolated point of $\mathbb{R}$.

6.12.149 Exercise. Prove that a set $A \subseteq \mathbb{R}$ without any isolated points is uncountable. In particular, this gives another proof that Cantor ser C is uncountable.

6.12.150 Exercise. If $\mathbb{R} = \bigcup_{n=1}^{\infty} E_n$, where each E_n is closed, show that $D = \bigcup_{n=1}^{\infty} (E_n)^\circ$ is dense in $\mathbb{R}$. [Hint: Consider $\mathbb{R} \setminus D$.]

6.12.151 Exercise. In $\mathbb{R}$, show that any subset of a first category set is still first category, and that a countable union of first category sets is again first category.

6.12.152 Exercise. In $\mathbb{R}$, prove that any superset of a second category set is itself a second category set.

6.12.153 Exercise. In $\mathbb{R}$, show that any open interval (and hence any nonempty, open set) is a second category set.

6.12.154 Exercise. Let E be an F_σ set in $\mathbb{R}$. Prove that E is a first category set in $\mathbb{R}$ iff E is dense in $\mathbb{R}$.

6.12.155 Exercise. Show that the complement of a first category set in $\mathbb{R}$ is a dense set of the second category in $\mathbb{R}$. In particular, a first category set in $\mathbb{R}$ must have empty interior.

6.12.156 Exercise. Show that the subset $\left\{\left(x, x^{-1}\right); x > 0\right\} \cup \{(0, y); y \in \mathbb{R}\}$ of $\mathbb{R}^2$ is not connected.

6.12.157 Exercise. Which of the following sets are of type F_σ?

1. $\mathbb{N}$.
2. $\left\{\frac{1}{n}; n \in \mathbb{N}\right\}$.
3. The set $\{C_n; n \in \mathbb{N}\}$ of midpoints of intervals complementary to the Cantor set.
4. A finite union of intervals (that need not be open or closed).

6.12.158 Exercise. Prove that a set of type F_σ in $\mathbb{R}$ is either first category or contains an open interval.

6.12.159 Exercise. If f and g are functions such that $f + g$ is continuous, does it follow that at least one of f or g must be continuous?

6.12.160 Exercise. If $|f|$ is continuous, does it follow that f is continuous?

6.12.161 Exercise. If $e^{f(x)}$ is continuous, does it follow that f is continuous?

6.12.162 Exercise. If $f(f(x))$ is continuous, does it follow that f is continuous?

6.12.163 Exercise. Let $f : [0, 1] \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} x, & \text{if } x \text{ is rational,} \\ x^2, & \text{if } x \text{ is irrational.} \end{cases}$$

Show that f is continuous at 0 and at 1, but is not continuous at any point in (0,1).

6.12.164 Exercise. Let f be continuous on $[a, b]$ and suppose that it takes every real value at most once. Show that f is monotonic.

6.12.165 Exercise. Let $f : (-1, 1) \to \mathbb{R}$ be continuous at 0, and suppose that $f(x) = f(x^2)\ \forall\, x \in (-1, 1)$. Show that $f(x) = f(0)\ \forall\, x \in (-1, 1)$.

6.12.166 Exercise. Let $a, b, c \in \mathbb{R}$, with $a < b < c$. Show that if f is uniformly continuous on $[a, b]$ and also on $[b, c]$, then it is uniformly continuous on $[a, c]$.

6.12.167 Exercise. Suppose f has the IVP on (a, b) and is discontinuous at $x_0 \in (a, b)$. Prove that there exists $y \in \mathbb{R}$ such that $\{x; f(x) = y\}$ is infinite.

6.12.168 Exercise. If $f : A \to B$ and $C \subseteq B$, what is $\chi_C \circ f$ (χ_C as a characteristic function)?

6.12.169 Exercise. Is there a continuous characteristic function on $\mathbb{R}$? If $A \subseteq \mathbb{R}$, show that χ_A is continuous at each point of A°. Are there any other points of continuity?

6.12.170 Exercise. Let A and B be subsets of $\mathbb{R}$, and let $f : \mathbb{R} \to \mathbb{R}$. Prove or disprove the following statements:

1. If f is continuous at each point of A and f is continuous at each point of B, then f is continuous at each point of $A \cup B$.
2. If $f|_A$ is continuous, relative to A and $f|_B$ is continuous, relative to B, then $f|_{A\cup B}$ is continuous, relative to $A \cup B$.

If either statement is not true in general, what modifications are necessary to make it so?

6.12.171 Exercise. Let (r_n) be an enumeration of the rationals in [0,1] and define f on [0,1] by $f(x) = \sum_{r_n<x} 2^{-n}$. Show that f is everywhere discontinuous on [0,1] but that f is everywhere continuous when considered as a function on only $[0, 1] \setminus \mathbb{Q}$.

6.12.172 Exercise. A continuous function on $\mathbb{R}$ is completely determined by its values on $\mathbb{Q}$. Use this to "count" the continuous functions $f : \mathbb{R} \to \mathbb{R}$.

6.12.173 Exercise. Let $0 < \alpha \le 1$. A function $f \in C^\alpha[0, 1], f$ is called **Hölder continuous** of order α if $\exists\, K \ge 0$ such that $|f(x) - f(y)| \le K|x - y|^\alpha\ \forall\, x, y \in [0, 1]$. Prove that

1. $C^\alpha[0, 1] \subseteq C^\beta[0, 1]$ for $0 < \beta \le \alpha \le 1$.
2. If f is Hölder continuous of order 1, i.e. f is Lipschitz, then f is absolutely continuous on $[0, 1]$.
3. Let $a > 0$. Define $g(x) = x^a \sin(x^{-a})$ for $x \neq 0$ and $f(0) = 0$. Prove that $g \notin BV[0, 1]$ and also prove that $g \in C^\alpha[0, 1]$ with $\alpha = a/(1 + a)$, and $g \notin C^\beta[0, 1]$ for $\beta > a/(1 + a)$.

6.12.174 Exercise. Let $f : [a, b] \to \mathbb{R}$ be continuous such that for each $c \in (a, b)$ there is no nbhd. U of c such that either $f(x) \ge f(c)$ or $f(x) \le f(c), \forall x \in U$. Prove that f is either strictly increasing or strictly decreasing on $[a, b]$.

6.12.175 Exercise. Let $f : [0, 1] \to \mathbb{R}$ be continuous such that $f(0) = f(1)$. Prove that for each positive integer n, $\exists\, x \in [0, -1/n]$ such that $f(x) = f(x + 1/n)$.

6.12.176 Exercise. Let $f : (0, 1] \to \mathbb{R}$ be continuous and bounded. Is it true or false that f must attain at least one of its sup of inf on $(a, b]$? Justify.

6.12.177 Exercise. Let $f : \mathbb{R} \to \mathbb{R}$ be bounded. And in every closed interval it itattains supremum and infimum there. Is it continuous on $\mathbb{R}$? Justify.

6.12.178 Exercise. Exhibit an example of function $f : [0, 1] \to \mathbb{R}$ that maps the convergent sequences into convergent sequences and which admits discontinuities.

6.12.179 Exercise. Let (r_n) be an enumeration of the rational numbers in [0,1]. Prove that the function

$$f(x) = \sum_{\{n; r_n < x\}} \frac{1}{2^n}$$

is strictly increasing and its discontinuities form a countable dense set.

It is worth noticing that every set $A \subseteq \mathbb{R}$ of type F_σ is the set of discontinuity of some function $f : \mathbb{R} \to \mathbb{R}$. See Gelbaum and Olmsted [3]. The particular case of closed sets makes the object of exercise below.

6.12.180 Exercise. Let A be a closed subset of $\mathbb{R}$. Verify that the characteristic function of the set $B = \partial A \cup (A^\circ \cap \mathbb{Q})$ is an example of function $f : \mathbb{R} \to \mathbb{R}$ whose set of discontinuities is exactly A.

6.12.181 Exercise. (Froda's Theorem). The purpose of this exercise is to sketch a proof of the fact that every function $f : \mathbb{R} \to \mathbb{R}$ may have only countably many points of discontinuity of first kind.

1. Prove that if $\epsilon > 0$ and ξ is limit of a sequence (a_n) of discontinuities of the first kind, distinct two by two, for which $\omega_f(a_n) > \epsilon$, then ξ is a point of discontinuity of the second kind. (Hint: Suppose that the sequence (a_n) is increasing. Then, in every interval $\left(\frac{a_{n-1}+a_n}{2}, \frac{a_n+a_{n+1}}{2}\right)$, one can choose points u_n and v_n such that $f(u_n) - f(v_n) > \epsilon/2$. If ξ would be a discontinuity of first kind, then $\lim_{n\to\infty} f(u_n) = \lim_{n\to\infty} f(v_n) = \lim_{x\to\xi-} f(x)$, a contradiction.)
2. Prove that the set D_n, of all points a of discontinuity of the first kind such that $\omega_f(a) > 1/n$, is countable.
3. Conclude that every function $f : \mathbb{R} \to \mathbb{R}$ may have only countably many points of discontinuity of first kind.

6.12.182 Exercise. Show that a continuous mapping $f : [0,1] \to [0,1]$ which satisfies $f(f(x)) = x$ for each $x \in [0,1]$, and for which $f(x) \neq x$ for at least one $x \in [0,1]$, must have exactly one fixed point.

6.12.183 Exercise. Does a continuous mapping $f : \mathbb{R} \to \mathbb{R}$ which satisfies $f(f(x)) = x$ for each $x \in \mathbb{R}$ necessarily have a fixed point?

6.12.184 Exercise. Describe a continuous mapping $f : [0,1] \to [0,1]$ for which $f(f(x)) = x$ and $f(x) \neq x$ for more than one $x \in [0,1]$.

6.12.185 Exercise. Let $f : \mathbb{R} \to [0,\infty)$ an arbitrary non-negative function. Assume that

$$\inf\{f(x) + f(y); |x - y| \geq \epsilon\} > 0$$

for any $\epsilon > 0$. Prove that each sequence (x_n) in $\mathbb{R}$ such that $f(x_n) \to 0$, converges to one and the same point $x \in M$.

6.12.186 Exercise. Let $A \subseteq \mathbb{R}$ and $f : \mathbb{R} \to [0,\infty)$ an arbitrary non-negative function. Assume that

$$\inf\{f(x); \inf\{|x - y|; y \in A\} \geq \epsilon\} > 0$$

for any $\epsilon > 0$. Prove that each sequence (x_n) in $\mathbb{R}$ such that $f(x_n) \to 0$, contains a subsequence which converges to some point $x \in A$.

6.12.187 Exercise. Let $f : [0, \infty) \to \mathbb{R}$ be a continuous surjective function. Prove that for every $a \in \mathbb{R}$, the equation $f(x) = a$ has infinitely many solutions.

6.12.188 Exercise. Consider a real parameter a and the even function $c_a : \mathbb{R} \to \mathbb{R}$ such that $c_a(0) = a$, and for every $n \in \mathbb{Z}$, $c_a(2^n) = (-1)^n$ and c_a is affine on $[2^n, 2^{n+1}]$.

1. Sketch the graph of this function.
2. Prove that the function c_a has a discontinuity of second kind at the origin, whenever $a \in \mathbb{R}$.
3. Prove that the function c_a has the intermediate value property if and only if $a \in [-1, 1]$. □

6.12.189 Exercise. Prove that a function with the intermediate value property may have only discontinuities of the second kind.

6.12.190 Exercise.

1. Let $M \subseteq \mathbb{R}$. If $f : M \to \mathbb{R}$ is continuous and $a \in \mathbb{R}$, show that the sets $\{x : f(x) > a\}$ and $\{x : f(x) < a\}$ are open subsets in M.
2. Conversely, if the sets $\{x : f(x) > a\}$ and $\{x : f(x) < a\}$ are open for every $a \in \mathbb{R}$, show that f is continuous.
3. Show that f is continuous even if we assume only that the sets $\{x : f(x) > a\}$ and $\{x : f(x) < a\}$ are open for every rational a.

6.12.191 Exercise. Let $f : \mathbb{R} \to \mathbb{R}$ is continuous.

1. If $f(0) > 0$, show that $f(x) > 0$ for all x in some interval $(-a, a)$.
2. If $f(x) \geq 0$ for every rational x, show that $f(x) \geq 0$ for all real x. Will this result hold with "≥ 0" replaced by "> 0"? Explain.

6.12.192 Exercise. Let $A = (0, 1] \cup \{2\}$, considered as a subset of $\mathbb{R}$. Show that every function $f : A \to \mathbb{R}$ is continuous, relative to A, at 2.

6.12.193 Exercise. Let A and B be subsets of $\mathbb{R}$, and let $f : \mathbb{R} \to \mathbb{R}$. Prove or disprove the following statements:

1. If f is continuous at each point of A and f is continuous at each point of B, then f is continuous at each point of $A \cup B$.
2. If $f|_A$ is continuous at each point of A and $f|_B$ is continuous at each point of B, then $f|_{A\cup B}$ is continuous at each point of $A \cup B$.

If either statement is not true in general, what modifications are necessary to make it so?

6.12.194 Exercise. Is there a nonempty subset of $\mathbb{R}$ that is open when considered as a subset of $\mathbb{R}^2$? closed?

6.12.195 Exercise. Show that the function $(x, y) \mapsto d(x, y)$, from $S \times S \to \mathbb{R}$, is uniformly continuous.

6.12.196 Exercise. If $f : [a, b] \to [a, b]$ is a homeomorphism, show that f carries endpoints to endpoints.

6.12.197 Exercise. Let $f : \mathbb{R} \to \mathbb{R}$ be bounded and continuous. Define $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$d(x, y) = \sup\{f(t - x) - f(t - y); t \in \mathbb{R}\}.$$

Show that d is a semimetric on $\mathbb{R}$. Show that d is a metric iff f is not periodic.

6.12.198 Exercise. Show that the function $f : [0, 1] \to [0, 1]$ defined by

$$f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 1 - x, & \text{if } x \text{ is irrational} \end{cases}$$

is onto and thus satisfies the intermediate value property. Show, however, that f is continuous only at $x = 1/2$.

6.12.199 Exercise. Let $D \subseteq \mathbb{R}$. If $f : D \to \mathbb{R}$ is uniformly continuous and (x_n) is a Cauchy sequence in D, show that $(f(x_n))$ is Cauchy. Can uniform continuity be replaced by continuity? Is the converse true?

6.12.200 Exercise. If $f : (a, b) \to \mathbb{R}$ is uniformly continuous, then show that $\lim_{x\to a+} f(x)$ and $\lim_{x\to b-} f(x)$ exist.

6.12.201 Exercise. Show that an increasing function $f : [a, b] \to \mathbb{R}$ can have at most countably many points of discontinuity. Hint: If x is a point of discontinuity, pick a rational r_x between $\lim_{t\to x-} f(t)$ and $\lim_{t\to x+} f(t)$ and consider the map $x \mapsto r_x$.

6.12.202 Exercise. Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous and $f(t) \neq 0 \ \forall\ t \in \mathbb{R}$. If $f^2(t) = g^2(t) \ \forall\ t$, show that either $f - g = 0$ or $f + g = 0$. Can continuity be dropped?

6.12.203 Exercise. Let $f : [0, 1] \times [0, 1] \to \mathbb{R}$ be continuous. Let

$$g(x) = \sup\{f(x, y); 0 \le y \le x\}.$$

Show that g is continuous.

6.12.204 Exercise. Prove or disprove: A function $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous on $\mathbb{R}$ iff f is uniformly continuous on every bounded interval.

6.12.205 Exercise. Let $f : [0, \infty)$ be continuous, $f(0) = 0$, and $\lim_{x\to\infty} f(x) = 1$. If $0 < \xi < 1$, show that $\exists\ x$ such that $f(x) = \xi$.

6.12.206 Exercise. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and $\lim_{|x|\to\infty} f(x) = 0$. Show that f has either a maximum or a minimum value on $\mathbb{R}$.

6.12.207 Exercise. Show that if $f : [a, b] \to \mathbb{R}$ is monotonic and has the intermediate value property, then f is continuous.

6.12.208 Exercise. Let $f : I \to \mathbb{R}$ have the property that for every $\epsilon > 0$, there is a uniformly continuous function $g : I \to \mathbb{R}$ such that $f(t) - g(t)| < \epsilon$. Show that f is uniformly continuous.

6.12.209 Exercise. Let $f : (0, 1] \to \mathbb{R}$ be uniformly continuous. If $\lim_{k\to\infty} f(1/k) = L$, show that $\lim_{x\to 0} f(x) = L$. Can "uniformly continuous" be replaced by "continuous"?

6.12.210 Exercise. If $f : (0, 1] \to \mathbb{R}$ is continuous and $\epsilon > 0$, show that there is a piecewise linear function g such that $f(t) - g(t)| < \epsilon$ for all $t \in [a, b]$.

6.12.211 Exercise. Let $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$. Show that f is continuous iff f carries convergent sequences into convergent sequences. Hint: If $x_n \to x$, consider the sequence (y_n) where $y_n = \begin{cases} x_n, & \text{if } n \text{ is odd} \\ x, & \text{if } n \text{ is even.} \end{cases}$

6.12.212 Exercise. Prove or disprove:

1. For a continuous function f on the real line the set of points
$$\{x : f(x) = 1\}$$
is closed.
2. For a continuous function f on the real line the set of points
$$\{x : f(x) = 1\}$$
can have no points of accumulation.
3. For a continuous function f on the real line the set of points
$$\{x : f(x) = 1\}$$
can have no interior points.
4. There is a closed set E that has infinitely many points but no points of accumulation.
5. There is a closed set E that has countably many points of accumulation.
6. There is an open set E that has no points of accumulation.
7. There is an uncountable set E that has no points of accumulation.

6.12.213 Exercise. Show that if $f : [0, 1] \to [0, 1]$ is a continuous function then f has a fixed point. Show that this is not necesarily true for discontinuous functions.

6.12.214 Exercise. Let $f : [0, 1] \to [0, 1]$ be a continuous function and define $g = f \circ f$. Show that g must have a fixed point too and that every fixed point of f is also a fixed point of g but not conversely.

6.12.215 Exercise. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous. Prove or disprove:

1. f must be unbounded.
2. f cannot be uniformly continuous unless f is constant.
3. $\lim_{n\to\infty} f(\sqrt[n]{n}) = f(1)$.
4. $f^{-1}(E)$ is compact if E is compact.
5. $f^{-1}(E)$ is open if E is open.

6. $f^{-1}(E)$ is finite if E is finite.
7. $f^{-1}(E)$ is countable if E is countable.
8. $f^{-1}(E)$ is bounded if E is bounded.

6.12.216 Exercise. Let f be a continuous function on an interval $[a, b]$ and let

$$E = \{x \in [a, b]; f(x) < 0\}$$

1. If c is a point of accumulation of the set E show that $f(c) \leq 0$.
2. If d is a boundary point of the set E and $d \neq a, b$ show that $f(d) = 0$.
3. Show that E can have no isolated points.
4. Is it possible that $[a, b] \setminus E$ has isolated points?

6.12.217 Exercise. Define what is meant by saying that f is a continuous function on an interval I. Define what is meant by saying that f is a uniformly continuous function on an interval I.

6.12.218 Exercise. Define uniform continuity.

1. Give an example of a function which is continuous but not uniformly continuous on the interval $(-1, 1)$. Prove that your example is not uniformly continuous.
2. Prove directly from your definition of uniform continuity that the function $f(x) = x^3$ is uniformly continuous on $(-1, 1)$.
3. Prove directly from your definition of uniform continuity that if a function f is uniformly continuous on $(-1, 1)$ then it must be bounded there.

6.12.219 Exercise. Define what is meant by a closed set of real numbers and an open set of real numbers. Prove that for a continuous function f on the real line the set of points

$$\{x : 0 < f(x) < 1\}$$

is open and that the set of points

$$\{x : 0 \leq f(x) \leq 1\}$$

is closed.

6.12.220 Exercise. Let $f : \mathbb{R} \to \mathbb{R}$ be an increasing function. Suppose that there are sequences (x_n), (y_n) such that $x_n < 0 < y_n \ \forall \ n \in \mathbb{N}$ and $f(y_n) - f(x_n) \to 0$ as $n \to \infty$. Prove that f is continuous at 0.

6.12.221 Exercise. Suppose $f : [0, 1] \to [0, 1]$ is a continuous function. Then the sequence of iterates $x_{n+1} = f(x_n)$ converges if and only if $\lim_{n\to\infty}(x_{n+1} - x_n) = 0$.

6.12.222 Exercise. Is it true that if $f : [0, 1] \to [0, 1]$ is

1. Monotonically increasing
2. Monotonically decreasing

then there is an $x \in [0,1]$ for which $f(x) = x$?

6.12.223 Exercise. Let f be continuous and nowhere monotone on [0,1]. Show that the set of points on which f attains local minima is dense in [0,1].

6.12.224 Exercise. Suppose $f, g : \mathbb{R} \to \mathbb{R}$ are two functions such that f is continuous, g is monotonic, and $f(x) = g(x)$, $\forall\, x \in \mathbb{Q}$. Show that $f(x) = g(x)$, $\forall\, x \in \mathbb{R}$.

6.12.225 Exercise. Consider a function $f : [a,b] \to [a,b]$ such that $|f(x) - f(y)| < |x - y|\ \forall\, x, y \in [a,b]$. Let $a_1 \in [a,b]$ and define $a_{n+1} = (a_n + f(a_n))/2$, for $n = 1, 2, ...$ Show that $\lim_{n\to\infty} a_n = a_0$ and $f(a_0) = a_0$.

6.12.226 Exercise. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying

$$f(x) = f\left(x + \frac{1}{n}\right), \forall\, x \in \mathbb{Q} \text{ and } n \in \mathbb{N}.$$

Show that the function f is constant.

6.12.227 Exercise. (Cauchy) Let $\phi : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying

$$\phi(x+y) = \phi(x) + \phi(y),\ \forall\, x, y \in \mathbb{R}.$$

Find the set of continuous functions fulfilling the above condition.

6.12.228 Exercise. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying, for a given real a,

$$\phi(x+y) = \phi(x) + \phi(y) + a,\ \forall\, x, y \in \mathbb{R}.$$

Find the set of continuous functions fulfilling the above condition.

6.12.229 Exercise. Let $\phi : (0,\infty) \to \mathbb{R}$ be a continuous function satisfying

$$\phi(xy) = \phi(x) + \phi(y),\ \forall\, x, y \in (0,\infty).$$

Find the set of continuous functions fulfilling the above condition.

6.12.230 Exercise. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a non-zero continuous function satisfying

$$\phi(x+y) = \phi(x)\phi(y) + a,\ \forall\, x, y \in \mathbb{R}.$$

Find the set of continuous functions fulfilling the above condition.

6.12.231 Exercise. Let $\phi : (0,\infty) \to \mathbb{R}$ be a non-zero continuous function satisfying

$$\phi(xy) = \phi(x)\phi(y),\ \forall\, x, y \in (0,\infty).$$

Find the set of continuous functions fulfilling the above condition.

6.12.232 Exercise. Let $\phi : \mathbb{R} \setminus \{-1, 1\} \to \mathbb{R}$ be a continuous function satisfying

$$\phi(x) + \phi(y) = \phi\left(\frac{x+y}{1+xy}\right),\ \forall\, x, y \in \mathbb{R} \setminus \{-1, 1\},\ 1 + xy \neq 0.$$

Find the set of continuous functions fulfilling the above condition.

6.12.233 Exercise. Find all continuous functions $f : \mathbb{R} \to [0, \infty)$ such that

$$f^2(x+y) - f^2(x-y) = 4f(x)f(y),$$

for all real numbers x, y.

6.12.234 Exercise. Prove the following version of amalgamation (only minor adjustments should be necessary): Suppose $-\infty < b < \infty$, f is uniformly continuous on $(-\infty, b]$, g is uniformly continuous on $[b, \infty)$, and $f(b) = g(b)$. Then there is a unique function h such that

1. h is uniformly continuous on $(-\infty, \infty)$.
2. $h(x) = f(x)$ for $x \in (-\infty, b]$.
3. $h(x) = g(x)$ for $x \in [b, \infty)$.

6.12.235 Exercise. (Volterra's Theorem). Let $f, g : [a, b] \to \mathbb{R}$ and let C_f and C_g denote the continuity sets of f and g; respectively. Thus $C_f = \{x \in (a, b); f \text{ is continuous at } x\}$ and C_g defined similarly. Show that if C_f and C_g are both dense, then so is $C_f \cap C_g$. Deduce that there is no function that is continuous on $\mathbb{Q}$ and discontinuous on $\mathbb{Q}^C$.

6.12.236 Exercise. Prove that if f is uniformly continuous on I and J, where these are two intervals which overlap, but are not assumed to be compact, then f is uniformly continuous on the interval $I \cup J$.

6.12.237 Exercise. Is $f : [0, \infty) \to \mathbb{R}$ defined by $f(x) = \sqrt{x}$ uniformly continuous on $[0, \infty)$? Justify your answer. (Note that its slope at 0 is infinite. There are several approaches to this exercise; one of them uses the preceding exercise.)

6.12.238 Exercise. Assuming the laws of logarithms, and that $\ln x$ is continuous at 1, prove $\ln x$ is uniformly continuous on $[1, \infty)$, a non-compact interval.

6.12.239 Exercise. Let $f : I \to \mathbb{R}$ be a function defined on an interval I (not assumed compact), and assume that its secants have bounded slope, i.e., for any two distinct points on the graph of f defined by $gr(f) = \{(x, f(x)); x \in I\}$ over I, the slope λ of the line joining them is bounded: $|\lambda| < K$ where K is some fixed number not depending on the two points selected.

1. Prove f is uniformly continuous on I.
2. Does f defined by $f(x) = \sqrt{x}$ on [0,1] satisfy the above hypotheses on f? Is it uniformly continuous on the interval?

6.12.240 Exercise. Determine necessary and sufficient conditions on a pair of sets A and B so that they will have the property that there exists a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = 0 \; \forall \, x \in A$ and $f(x) = 1 \; \forall \, x \in B$.

6.12.241 Exercise. (Extensions of continuous functions) If $f : A \to \mathbb{R}, g : B \to \mathbb{R}, A \subseteq B$, and $f(x) = g(x) \; \forall \, x \in A$, then the function g is said to be an extension of the function f. Prove each of the following:

1. A function that is continuous on a closed set A can be extended to a function that is continuous on $\mathbb{R}$.

2. A function that is uniformly continuous on a set A can be extended to a function that is uniformly continuous on $\overline{A}$.

3. A function that is uniformly continuous on an arbitrary nonempty subset of $\mathbb{R}$ can be extended to a function that is uniformly continuous on all of $\mathbb{R}$.

4. Give an example of a function f that is continuous on (0,1) but that cannot be extended to a function continuous on [0,1].

6.12.242 Exercise. (**Absolute Continuity**). A function $F : [a, b] \to \mathbb{R}$ is said to be absolutely continuous (on $[a, b]$) if for each $\epsilon > 0$ there is a $\delta > 0$ such that, given any collection $\{[a_k, b_k]; 0 \leq k \leq n\}$ of pairwise disjoint open subintervals of $[a, b]$, we have

$$\sum_{k=1}^{n}(b_k - a_k) < \delta \Rightarrow \sum_{k=1}^{n}|F(b_k) - F(a_k)| < \epsilon.$$

1. Show that an absolutely continuous function on $[a, b]$ is uniformly continuous there. To show that the converse is false, consider the function

$$f(x) = \begin{cases} x \sin \frac{\pi}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

   Show that f is uniformly continuous but not absolutely continuous on $[0, 1]$ Hint: Let $\epsilon = 1$ and, for each $\delta > 0$, choose $M, N \in \mathbb{N}$ with $1/\delta < M < N$ such that $\sum_{k=M}^{N} a_k > 1$, where $a_k = 2/(4k + 1)$. Now let $b_k = 2/(4k)$ and consider the disjoint intervals (a_k, b_k) with $M \leq k \leq N$.

2. Show that, if $F \in Lip([a, b])$, then it is absolutely continuous. The converse is false again. Indeed, as we know, the function $f(x) = \sqrt{x}$ is not Lipschitz on $[0, 1]$. Show, however, that it is absolutely continuous on $[0, 1]$ as follows. Given $\epsilon > 0$. let $\delta = \epsilon^2/2$ and let $(a_j, b_j) \subset [0, 1]$ be pairwise disjoint with $\sum_{j=1}^{n}(b_j - a_j) < \delta$. If $\delta/2 \in (a_j, b_j)$ for some j, then insert it as an endpoint, getting two subintervals (a_j, δ) and (δ, b_j). Now write $\sum_{j=1}^{n}(\sqrt{b_j} - \sqrt{a_j}) = \sum_1 + \sum_2$ where $\sum_1$ is over all j with $b_j \leq \delta/2$ and $\sum_2$ is over the other j's. Finally, observe that $\sum_1 \leq \epsilon/2$ and

$$\sum_2 \leq \frac{1}{\epsilon}\sum(b_j - a_j) < \epsilon/2.$$

6.12.243 Exercise. Prove that the following are equivalent:

1. f is lower semicontinuous.

2. $f^{-1}(c, \infty)$ is open $\forall\ c \in \mathbb{R}$.

3. $f^{-1}(\infty, c]$ is closed $\forall\ c \in \mathbb{R}$.

4. If $x_n \to x$, then $f(x) \leq \sup_n f(x_n)$

5. Let A be open in $\mathbb{R}$ and for each $y \in A$ and for each $\epsilon > 0$ $\exists$ a nbhd. V of y such that $f(x) \geq f(y) - \epsilon$ for all $x \in V$.

6.12.244 Exercise. Let $A \subseteq \mathbb{R}$ and $g : A \to \mathbb{R}$ and define

$$f(x) = \limsup_{y \to x} g(y).$$

Prove that f is lower semicontinuous.

6.12.245 Exercise. If $E \subset \mathbb{R}$, then χ_E is lower semicontinuous iff E is open.
The function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0, & \text{if } \in \mathbb{Q}^C \\ \frac{1}{n}, & \text{if } x = \frac{m}{n}; n \in \mathbb{N} \text{ and } m \in \mathbb{Z}; (m,n) = 1 \end{cases}$$

is upper semicontinuous.

6.12.246 Exercise. Let $\{f_\alpha\}_{\alpha \in \Lambda}$ be a family of lower semicontinuous functions in $E \subset \mathbb{R}$. Prove that $f(x) = \sup_{\alpha \in \Lambda} f_\alpha(x)$ is lower semicontinuous.

6.12.247 Exercise. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and for every rational number q, there exists an integer n such that the n-th iterate of f, $f^{(n)} = f \circ f \circ ... \circ f$, evaluated at q is 0. Prove or disprove: for every real number r, there exists an integer n, such that $f^{(n)}(r) = 0$.

6.12.248 Exercise. Let $r_1, r_2, .,$ be an enumeration of the rationals in the interval [0,1]. For each $x \in [0,1]$, define the function ϕ_x on [0,1] by

$$\phi_x(y) = \begin{cases} 1, & \text{if } y \leq x \\ 0, & \text{if } y > x. \end{cases}$$

Let f be defined on [0,1] by

$$f(x) = \sum_{n=1}^{\infty} \phi_x(r_n)/2^n$$

Prove that

1. f is strictly increasing,
2. f is continuous at $x \in [0,1]$ if and only if $x \notin \mathbb{Q}$.

6.12.249 Exercise. Define a relation $\sim$ on the circle $S^1 = \{z \in \mathbb{C}; |z| = 1\}$, by $z \sim \alpha$ if $z = \alpha e^{2\pi\iota\theta}; \theta \in \mathbb{Q}$. Prove that the set of all limit points of any eqivalence class coincides with S^1.

6.12.250 Exercise. Show that there can be no continuous function $f : \mathbb{R} \to \mathbb{R}$ that satisfies

$$f(x) \in \mathbb{Q} \Leftrightarrow f(x+1) \in \mathbb{Q}^C.$$

6.12.251 Exercise. Show that there can be no continuous function $g : \mathbb{R} \to \mathbb{R}$ that maps rational numbers to irrational ones and vice versa.

6.12.252 Exercise. (Volterra's Theorem). Let $f, g : (0,1) \to \mathbb{R}$ and let C_f and C_g denote the continuity sets of f and g; respectively. Thus

$$C_f = \{x \in (0,1); f \text{ is continuous at } x\}$$

and C_g defined similarly. Show that if C_g and C_g are both dense, then so is $C_f \cap C_g$. Deduce that there is no function that is continuous on $\mathbb{Q}$ and discontinuous on $\mathbb{Q}^C$.

6.12.253 Exercise. The following exercise shows that any countable set in (a, b) is the set of discontinuities of a monotonically increasing function on (a, b).
Let $S = \{s_1, s_2, s_3, ...\}$ be a countable set on (a, b). Take any sequence (c_n) of positive constants such that $\sum c_n$ converges. For each $x \in (a, b)$ let

$$A_x = \{n \in \mathbb{N}; a < s_n < x\}.$$

Define the function f by

$$f(x) = \begin{cases} \sum_{n \in A_x} c_n, & \text{if } A_x \neq \emptyset \\ 0, & \text{if } A_x = \emptyset. \end{cases} \tag{6.2}$$

Then

1. f is monotonically increasing on (a, b).
2. $f(s_n+) - f(s_n-) = c_n$; that is, f is discontinuous at each point of S.
3. $f(-x) = f(x)$ for each $x \in (a > b)$.
4. is continuous at each $x \in (a, b) \setminus S$.

6.12.254 Exercise. Show that the following statement is false: if $f : \mathbb{R} \to \mathbb{R}$ is continuous at 0 and $f(0) = 0$, then the inequality $|f(x)| \leq |x|^\alpha$ holds for some $\alpha > 0$ at least in some neighborhood of 0.

6.12.255 Exercise. Provide a counterexample to the statement: if $f + g$ is continuous at a, then both f and g are continuous at a. What about other arithmetic operations?

6.12.256 Exercise. Give a counterexample to the statement: if $\cos \circ f$ is continuous at a, then f is also continuous. What about $\cosh \circ f$? $((\cos \circ f)(x) = \cos(f(x)))$

6.12.257 Exercise. Verify if the following statement is true: if $g \circ f$ is continuous at a, then f is also continuous. What if the condition of continuity of g at $f(a)$ is added?

6.12.258 Exercise. Verify if the following statement is true: if $g \circ f$ and f are continuous at a, then g is continuous at $f(a)$.

6.12.259 Exercise. Suppose f and g are such that both $f \circ g$ and $g \circ f$ are defined in a neighborhood of a and $f \circ g$ is continuous at a. Is $g \circ f$ also continuous?

6.12.260 Exercise. Give a counterexample to the statement that: a continuous on $[a, b]$ function can not possess infinitely many local extrema on this interval.

6.12.261 Exercise. Provide a counterexample to the following statement: if f is increasing and continuous on the interval $[a, c]$ and on the interval $(c, b]$, then it is increasing on $[a, b]$.

6.12.262 Exercise. Give a counterexample to the statement: a function defined on $\mathbb{R}$ cannot be continuous at each point $x \in \mathbb{Z}$ and discontinuous at all other points.

6.12.263 Exercise. It is known that the convexity condition on an open interval $(a, b) - f(\alpha x_1 + (1-\alpha)x_2) \leq \alpha f(x_1) + (1-\alpha)f(x_2), \forall\, x_1, x_2 \in (a, b)$ and $\forall\, \alpha \in (0, 1)$ implies continuity of f on (a, b). Show that this implication fails in the case of a closed interval. Hint: Consider $f(x) = \begin{cases} 0 & \text{if } x \in (-1, 1) \\ 1 & \text{if } x = 0, 1. \end{cases}$

6.12.264 Exercise. It is known that any function f that maps continuously a closed interval on itself has a fixed point, that is, such point c in the closed interval that $f(c) = c$. Show that this is not true for an open interval. (Hint: consider $f(x) = x^2$ on (0,1).)

6.12.265 Exercise. Provide a counterexample to the following statement: if f is continuous and invertible on domain X, then the inverse function f^{-1} is continuous on $Y = f(X)$. Hint: Consider
$f(x) = \begin{cases} x & \text{if } x \leq 0 \\ x-1 & \text{if } x > 1. \end{cases}$

6.12.266 Exercise. Provide a counterexample to the following statement: if f is continuous on $X \subset \mathbb{R}, X \neq \mathbb{R}$, then there is a continuous extension of f from X onto $\mathbb{R}$, that is, such function g continuous on $\mathbb{R}$, that $g(x) = f(x)$, $\forall\, x \in X$. Hint: Consider $f(x) = \sin \frac{1}{x}, X = (0, \infty)$, notice that the statement is true for a closed set X.

6.12.267 Exercise. Find an example of a function f discontinuous on $\mathbb{Q}$ and another function g discontinuous at only one point, but $g \circ f$ is nowhere continuous.

6.12.268 Exercise. Show that the statement: if f is continuous and has a compact image, then its domain is a bounded set, is false.

6.12.269 Exercise. Show that the following statement is false: if f is continuous and bounded on a closed set, then it attains its global minimum and maximum on this set.

6.12.270 Exercise. Provide a counterexample to the following statement: if f is continuous on $[a, b]$ except for only one point, then f is bounded on $[a, b]$.

6.12.271 Exercise. Give a counterexample to the statement: if f is defined, but not continuous on $[a, b]$, it cannot attain its global extrema on this interval.

6.12.272 Exercise. Verify if the following "definition" of the uniform continuity is correct: for every $\epsilon > 0$ and every $\delta > 0$ whenever $x_1, x_2 \in S$ and $|x_1 - x_2| < \delta$ it follows that $|f(x_1) - f(x_2)| < \epsilon$. If not, provide a counterexample.

6.12.273 Exercise. Since the violation of uniform continuity for one of the functions in the composition can lead to loss of uniform continuity by the composite function, it may be tempting to state that: if f is non-uniformly continuous on X, and g is non-uniformly continuous on $f(X)$, then the composite function $g \circ f$ is non-uniformly continuous on X. Show that it is not true by providing a counterexample.

6.12.274 Exercise. Is it possible to have a point-wise discontinuous function which has the points of discontinuity at a set which is perfect and not everywhere dense? Give reasons for your conclusion.

6.12.275 Exercise. Could one have a point-wise discontinuous function whose points of continuity form a discrete set? a set of first species? an enumerable set? a set of first category? a closed set? a perfect set? a closed set of measure zero?

6.12.276 Exercise. If f is a rational integral function on $[a, b]$, does it take all values between $f(a)$ and $f(b)$? Is it uniformly continuous on $[a, b]$? Does it have a maximum or minimum in $[a, b]$? Can it be represented as the limit of a sequence of continuous functions?

6.12.277 Exercise. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(t) = \frac{p+\sqrt{2}}{q+\sqrt{2}}$ if $t = \frac{p}{q}; p, q \in \mathbb{Z}$ and $(p, q) = 1$ and $f(t) = 0$ if t is irrational. Answer the following:

1. Find the set of irrational numbers t where f is continuous.

2. Find the set of rational numbers t where f is continuous.

6.12.278 Exercise. Let f, g be two continuous functions on $\mathbb{R}$. For any $a \in \mathbb{R}$ we define $J_a(f, g)$ by

$$J_a(f,g)(t) = \begin{cases} f(t) & \text{if } t \leq a \\ g(t) & \text{if } t > a. \end{cases}$$

For what values of a, $J_a(f, g)$ is a contunuous function.

6.12.279 Exercise. Let A, B be two finite subsets of $\mathbb{R}$. Describe the necessary and sufficient condition for the spaces $\mathbb{R} \setminus A$ and $\mathbb{R} \setminus B$ to be homeomorphic.

6.12.280 Exercise. Let $f : \mathbb{R} \to \mathbb{R}$ be a function and let J be a bounded open interval in $\mathbb{R}$. Define $W(f, J) = \sup\{f(x); x \in J\} - \inf\{f(x); x \in J\}$. Which one of the following is false?

1. $W(f, J_1) \subseteq W(f, J_2)$ if $J_1 \subseteq J_2$.

2. If f is a bounded function in J and $J \supseteq J_1 \supseteq J_2 .... \supseteq J_n \supseteq ...$ such that the length of the interval J_n tends to 0 as $n \to \infty$, then $\lim_{n\to\infty} W(f, J_n) = 0$

3. If f is discontinuous at a point $a \in J$, then $W(f, J) \neq 0$.

4. If f is continuous at a point $a \in J$, then for any given $\epsilon > 0$ there exists an interval $I \subseteq J$ such that $W(f, I) < \epsilon$.

6.12.281 Exercise. For $x \in \mathbb{R}$, let $f(x) = \begin{cases} x^3 \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$

Then which one of the following is false?

1. $\lim_{x\to 0} \frac{f(x)}{x} = 0$

2. $\lim_{x\to 0} \frac{f(x)}{x^2} = 0$

3. $\frac{f(x)}{x^2}$ has infinitely many maxima and minima on the interval (0,1).

4. $\frac{f(x)}{x^4}$ is continuous at $x = 0$ and not differentiable at $x = 0$.

6.12.282 Exercise. For $x \in \mathbb{R}$, let

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \cup \{0\} \\ 1 - \frac{1}{p} & \text{if } x = \frac{n}{p}, n \in \mathbb{Z} \setminus \{0\}, p \in \mathbb{N}, \gcd(n,p) = 1 \end{cases}$$

Then which one of the following is true?

1. all $x \in \mathbb{Q} \setminus \{0\}$ are strict local minima for f.

2. f is continuous at all $x \in \mathbb{Q}$.

3. f is not continuous at all $x \in \mathbb{R} \setminus \mathbb{Q}$.

4. f is continuous at $x = 0$.

6.12.283 Exercise. Let $f : [0,1] \to [0,1]$, be a non-constant continuous function such that $f \circ f = f$. Define

$$E_f = \{x \in [0,1]; f(x) = x\}.$$

Then which one of the following is true?

1. E_f is neither open nor closed.
2. F_f is empty.
3. E_f need not be an interval.
4. E_f is an interval.

6.12.284 Exercise. Let $f : \mathbb{R} \to \mathbb{R}$ with the property that for every $y \in \mathbb{R}$, the value of the expression

$$\sup_{x \in \mathbb{R}} [xy - f(x)]$$

is finite. Define $g(y) = \sup_{x \in \mathbb{R}} [xy - f(x)],\ y \in \mathbb{R}$. Then which one of the following is true?

1. g is even if f is even.
2. g is odd if f is even.
3. f must satisfy $\lim_{|x| \to \infty} \frac{f(x)}{|x|} = +\infty$
4. f must satisfy $\lim_{|x| \to \infty} \frac{f(x)}{|x|} = -\infty$.

6.12.285 Exercise. In the following let $f : A \cup B \to \mathbb{R}$ and $f|_A : A \to \mathbb{R}$ and $f|_B : B \to \mathbb{R}$ denote the restrictions of f to the sets A and B respectively.
(a) Show that f need not be continuous if $f|_A$ and $f|_B$ are continuous.
(b) If A and B are open and $f|_A, f|_B$ are continuous, show that f is continuous.
(c) If A and B are closed and $f|_A, f|_B$ are continuous, show that f is continuous.

Chapter 7

Differentiablity

To understand mathematics means to be able to do mathematics.
And what does it mean doing mathematics? In the first place it
means to be able to solve mathematical problems.
–George Pólya (1887–1985)

7.1 Introduction

We owe the word derivative and the prime symbol to denote the derivative to a French mathematician Joseph Louis Lagrange (1736–1813). They appear in his article in 1770. He also wrote $du = u'dx$ in his article in 1772 , but the symbols dx, dy, and $\frac{dy}{dx}$ were introduced by Leibniz in 1675. Lagrange was probably the best French mathematician of the 18th century. Among his many accomplishments is a book M'ecanique analytique (The Analytical Mechanics) published in 1788, which offered a revolutionary view of mechanics as the four-dimensional geometry. Lagrange himself was reported to have said that "mechanics was really a branch of pure mathematics" and that his book "does not contain a single diagram". Sir William Rowan Hamilton said the work could be described only as a scientific poem. Hamilton (1805–1865) was the leading Irish physicist, astronomer, and mathematician, who made important contributions to classical mechanics.

7.2 Differentiablity

7.2.1 Definition. Let f be a real-valued function defined on an interval I. Let $a \in I^\circ$. We say f is **differentiable** at a if the limit

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a} = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} \tag{7.1}$$

exists. In this case we write $f'(a)$ or $D_x f(a) = f'(a)$ for this limit. If the function f is differentiable at each point of the set $S \subseteq I$, then f is said to be **differentiable** on S and the function $f' : S \to \mathbb{R}$ is called the **derivative** or **differential coefficient**[1] of f on S. When f is differentiable at a, the tangent line to f at a is the linear function $L(x) = f(a) + f'(a)(x - a)$.

[1]see Special Topics

Notice that a is not an endpoint of I, In particular, if f is differentiable on $[a,b]$, then

$$f'(a)=\lim_{h\to 0+}\frac{f(a+h)-f(a)}{h} \text{ and } f'(b)=\lim_{h\to 0-}\frac{f(b+h)-f(b)}{h}$$

It is important to notice that a function which is differentiable on two sets is not necessarily differentiable on their union.

7.2.2 Example. $f(x)=|x|$ is differentiable on $A=[-1,0]$ and on $B=[0,1]$ but not on $A\cup B=[-1,1]$. Because 0 is the end-point of A and B but in $A\cup B$, 0 becomes as an interior point of $A\cup B$.

The following criterion for the existence of a derivative is an immediate consequence of (7.2.1).

7.2.3 Theorem (Caratheodory). Let Let f be a real-valued function defined on an interval I. Let $a\in I^\circ$. Then f is **differentiable** at a if and only if the function

$$f_p(x)=\begin{cases}\frac{f(x)-f(a)}{x-a} & \text{if } x\neq a\\ f'(a) & \text{if } x\neq a\end{cases}$$

is continuous at $x=a$.

7.2.4 Theorem. Let $I\subseteq\mathbb{R}$ be an interval, let $c\in I$, and let $u:I\to\mathbb{R}$ and $v:I\to\mathbb{R}$ be functions such that u' and v' exist at c. Then:

1. If $a\in\mathbb{R}$, then the function au is differentiable at c, and $(au)'(c)=au'(c)$.
2. The function $u+v$ is differentiable at c, and $(u+v)'(c)=u'(c)+v'(c)$.
3. (Product Rule) The function uv is differentiable at c, and

$$(uv)'(c)=u'(c)v(c)+u(c)v'(c).$$

4. (Quotient Rule) If $v(c)\neq 0$, the function u/v is differentiable at c, and

$$(u/v)'(c)=\frac{u'(c)v(c)-u(c)v'(c)}{(v(c))^2}.$$

7.2.5 Theorem. (Chain Rule) Let I,J be intervals in $\mathbb{R}$, let $g:I\to\mathbb{R}$ and $f:J\to\mathbb{R}$ be functions such that $f(J)\subseteq I$ and let $c\in J$. If f is differentiable at c and if g is differentiable at $f(c)$, then the composite function $g\circ f$ is differentiable at c and

$$(g\circ f)'(c)=g'(f(c))f'(c).$$

7.2.6 Theorem. Let $I\subseteq\mathbb{R}$ be an interval, let $c\in I$, and let $f:I\to\mathbb{R}$ differentiable at c. Then:

1. If $f(c)>0$, then there is a number $\delta>0$ such that $f(x)>f(c)$ for $x\in I$ such that $c<x<c+\delta$.
2. If $f(c)<0$, then there is a number $\delta>0$ such that $f(x)>f(c)$ for $x\in I$ such that $c-\delta<x<c$.

7.2.7 Theorem (Darboux Intermediate Value Theorem). If f is differentiable on $I=[a,b]$ and if k is a number between $f'(a)$ and $f'(b)$, then there is at least one point $c\in(a,b)$ such that $f'(c)=k$.

7.2.8 Theorem. Let $I \subseteq \mathbb{R}$ be an interval, and let $f : I \to \mathbb{R}$ differentiable on I. Then $f'(I)$ is an interval.

7.2.9 Theorem. Let $I \subseteq \mathbb{R}$ be an interval, and let $f : I \to \mathbb{R}$ differentiable on I. Then the derived functicn f' cannot have a jump discontinuity on I.

7.2.10 Note.

1. A derived function on an interval can have a discontinuity of second kind.
2. If f' exists in some deleted neighbourhood of c and $\lim_{x\to c-} f'(x) \neq \lim_{x\to c+} f'(x)$, then f cannot be differentiable at c.
3. If a function f be differentiable on an interval I and f' is monotonic on I, then f' is continuous on I.

It follows from the property that a monotone fUnction can have jump discontinuities in its domain.

7.3 Dini Derivatives

7.3.1 Definition. Let f be a real-valued function defined on a nonempty open subset U of $\mathbb{R}$. Let $x \in U$. Define

$$D^+f(x) = \limsup_{y\to x+} \frac{f(y)-f(x)}{y-x};\ D_+f(x) = \liminf_{y\to x+} \frac{f(y)-f(x)}{y-x};$$
$$D^-f(x) = \limsup_{y\to x-} \frac{f(y)-f(x)}{y-x};\ D_-f(x) = \liminf_{y\to x-} \frac{f(y)-f(x)}{y-x}.$$

The above quantities are referred to as **Dini derivatives** of f at x. They exist as elements in $\mathbb{R} \cup \{\pm\infty\}$.

7.3.2 Definition. Let f be a real-valued function defined on a nonempty open subset U of $\mathbb{R}$. Let $x \in U$. If $D^+f(x) = D^+f(x)$, then we write

$$f'_+(x) = D^+f(x) = D_+f(x) = \lim_{y\to x+} \frac{f(y)-f(x)}{y-x}$$

and refer to this quantity as the **right-hand-side derivative** of f at x. Similarly, if $D^-f(x) = D^-f(x)$, then we write

$$f'_-(x) = D^-f(x) = D_-f(x) = \lim_{y\to x-} \frac{f(y)-f(x)}{y-x}$$

and refer to this quantity as the **left-hand-side derivative** of f at x.

The following result is due to H. Lebesgue.

7.3.3 Theorem. Let f be a real-valued monotone function defined on a general open interval I. Then $f(x)$ exists (a.e.) in I as a real number.

7.4 Mean Value Theorems

7.4.1 Definition. (Local Extrema). Let $f : I \to \mathbb{R}$ and let $x_0 \in I^\circ$. We say that f has a local maximum (resp., local minimum) at x_0 if there exists $\delta > 0$ such that $f(x) \leq f(x_0)$ (resp., $f(x) \geq f(x_0)$) for all $x \in (x_0 - \delta, x_0 + \delta) \cap I$. We say that f has a local extremum at x_0 if it has a local maximum or a local minimum at x_0.

7.4.2 Theorem (Fermat's Theorem). Let $f : I \to \mathbb{R}$. and let $x_0 \in I^\circ$ be an interior point. If f has a local extremum at x_0 and is differentiable at x_0, then $f'(x_0) = 0$. In other words, the tangent line to the graph of f at the point $(x_0, f(x_0))$, is horizontal.

Pierre de Fermat (1601–1665) was a French lawyer and an amateur mathematician. He never published any results, but communicated most of his work in letters to friends, often with little or no proof of his theorems. He is best known for **"Fermat's Last Theorem"** which was discovered by his son in the margin of Diophantus's Arithmetica and became widely known in 1670 when the son published this book with his father's notes.
Fermat's Theorem shows that, if at some point c, $f'(c)$ exists and is different from 0, then f cannot attain its extreme value at c. However, this result does not shed any light on the issue whether f is increasing or decreasing. For that we will need another theorem. Here is a stepping stone in that direction.

7.4.3 Theorem. (Rolle's Theorem): Suppose that f is a function defined and continuous on an interval $[a, b]$, that it is differentiable in (a, b), and that $f(a) = f(b)$. Then there exists $c \in (a, b)$ such that $f'(c) = 0$.

Michel Rolle (1652–1719) was a French mathematician. In 1691, he gave the first known formal proof of Theorem 7.4.3. The name Rolle's theorem was first used by Moritz Wilhelm Drobisch (1802–1896), a German mathematician, in 1834. Rolle is also remembered for popularizing the symbol for equality =, which had been invented by a Welsh doctor and mathematician Robert Recorde, and the symbol for the n-th root $\sqrt[n]{\ }$, although it had been suggested (for the cube root) by Albert Girard. Rolle was an outspoken critic of calculus, and his opposition had a positive effect on the new discipline. Eventually, he formally recognized its value by 1706. Geometrically, Rolle's Theorem says that, under the listed assumptions, if f takes the same value at the endpoints, then somewhere in between the tangent line to the graph of f is horizontal. In other words, somewhere in between there is a point at which the tangent line is parallel to the (horizontal) line connecting $f(a)$ and $f(b)$. What if the latter line is not horizontal?

7.4.4 Theorem. (Mean Value Theorem due to Lagrange). Suppose that f is a function defined and continuous on an interval $[a, b]$, and that it is differentiable in (a, b). Then there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Another form of Mean Value Theorem due to Lagrange is known as

7.4.5 Theorem. Taylor's theorem of order 1. If $f : [a, b] \to \mathbb{R}$ is continuous and f' exists $c \in (a, b)$, then

$$f(c + h) = f(c) + hf'(c + \theta h),$$

where $0 < \theta < 1$.

7.4.6 Theorem. Taylor's theorem of order n If $f : [a, b] \to \mathbb{R}$ is continuous and $f', f'', .., f^n$ exists in (a, b), then

$$f(c+h) = f(c) + hf'(c) + \frac{h^2}{2!}f''(c) + ... + \frac{h^n}{n!}f^n(c+\theta h),$$

where $0 < \theta < 1$.

7.4.7 Theorem. L'Hôspital's Rule: Suppose the functions f and g are differentiable at every point, except possibly at c in an interval (a, b). If $g'(x) \neq 0$ for $x \neq c$, and if $\frac{f(x)}{g(x)}$ has either the undetermined form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ at $x = c$, then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$$

provided $\lim_{x\to c} \frac{f'(x)}{g'(x)}$ exists.

7.4.8 Theorem. (Cauchy's mean value theorem) Let $f, g : [a, b] \to \mathbb{R}$ be continuous and f', g' exist in (a, b), and $g'(x) \neq 0$ in (a, b) then prove that $\exists\, \xi \in (a, b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(\xi)}{g'(\xi)}.$$

7.5 Absolute maximum, minimum:

7.5.1 Definition. Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$. We say that f has an **absolute maximum** on A if there is a point $x^* \in A$ such that $f(x^*) \geq f(x)$ for all $x \in A$. We say that f has an **absolute minimum** on A if there is a point $x_* \in A$ such that $f(x_*) \leq f(x)$ for all $x \in A$. We say that x^* is an **absolute maximum point maximizer** for f on A, and that x_* is an **absolute minimum point** or **minimizer** for f on A, if they exist.

7.6 Relative maximum, minimum:

7.6.1 Definition. A function $f : I \to \mathbb{R}$, is said to have a **relative (local) maximum** [respectively, **relative (local) minimum**] at $c \in I$ if there exists a nbhd. $B(c; \delta)$ of c such that $f(x) \leq f(c)$ [respectively, $f(c) \leq f(x)$] for all $x \in B(c; \delta) \cap I$. We say that f has a relative extremum at $c \in I$, if it has either a relative maximum or a relative minimum at c.

7.6.2 Theorem. Let f be continuous at a point c and on $(c - \delta, c + \delta)$, for some $\delta > 0$, then f has a minimum at c if

$$\lim_{x\to c+} f'(x) = \infty, \ \lim_{x\to c-} f'(x) = -\infty$$

and f has a maximum at c if

$$\lim_{x\to c+} f'(x) = -\infty, \ \lim_{x\to c-} f'(x) = \infty.$$

7.6.3 Example. Consider $f(x) = x^{\frac{2}{3}}$ at 0.

7.6.4 Note. All the extreme values may not be given by solutions of $f'(x) = 0$, there may be extremes at points where $f'(x)$ does not exist.
A function f need not have an extreme at every point at which $f'(x) = 0$.

The term "primitive function" was introduced by Lagrange in 1797. We owe the "antiderivative" to a French mathematician Sylvestre-Francois Lacroix (1765–1843). He was known for a number of textbooks held in high esteem.

7.7 Problems and Solutions on Chapter 7

Warning! *Never Never Ever read this solution unless you tried problems quite long time and gave up. Doing so may impair your ability to think and solve problems.*

7.7.1 Problem. Universal Chord Theorem: An Application of Intermediate Value Theorem [R.P.Boas; Jr., Volume 13, Carus Mathematical Monographs, MAA.]
Let $f : [a, b] \to \mathbb{R}$ be continuous, and suppose that $f(a) = f(b)$. Then there exist real numbers c, d such that

1. $a < c < d < b$,
2. $d - c < \frac{b-a}{2}$ and
3. $f(c) = f(d)$.

7.7.1.1 Solution. Let $h = \frac{b-a}{3}$, and define $g : [a, a + 2h] \to \mathbb{R}$ by

$$g(x) = f(x + h) - f(x).$$

Since g is continuous on $[a, a + 2h]$ and that

$$\begin{aligned}
&g(a) + g(a + h) + g(a + 2h) \\
&= f(a + h) - f(a) + f(a + 2h) - f(a + h) + f(a + 3h) - f(a + 2h) \\
&= f(a + 3h) - f(a) \\
&= f(b) - f(a) = 0.
\end{aligned}$$

Since the sum of three real numbers is zero, then two cases arise (i) all three are zero,(ii) two of them must have opposite in signs.
In case (i), in particular, $0 = g(a + h) = f(a + 2h) - (a + h)$ implies $c = a + h, d = a + 2h$.
On the other hand, two of them have opposite signs, then by IVT $\exists\ p \in (a, a + 2h)$ such that $g(p) = 0$. Hence $f(p) = f(p + h)$ shows that $c = p, d = p + h$. Hence the theorem. □

7.7.2 Problem. Let $f : [a, b] \to \mathbb{R}$ be continuous, and suppose that $f(a) = f(b)$. Then there exist two sequences $(a_n), (b_n)$ in $[a, b]$ such that $\forall\ n \in \mathbb{N}$ and

1. $a < a_n < a_{n+1} < b_{n+1} < b_n < b$,
2. $b_n - a_n < \frac{b-a}{2^n}$ and
3. $f(a_n) = f(b_n)$.

7.7.2.1 Solution. Apply the "Universal Chord Theorem". □

7.7.3 Problem. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function with continuous derivative such that

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} f'(x) = \infty.$$

Prove that the function $g : \mathbb{R} \to \mathbb{R}$, defined by $g(x) = \sin f(x)$ is not periodic.

7.7.3.1 Solution. Assume by way of contradiction that $g : \mathbb{R} \to \mathbb{R}$, defined by $g(x) = \sin f(x)$ is periodic. In that case, $g'(x) = f'(x) \cos f(x)$ is also periodic, and since it is continuous (as both f and f' are continuous), $g'(x)$ is bounded. Consider the sequence $y_n = 2n\pi$. Because f is continuous and $\lim_{x\to\infty} f(x) = \infty$, there is some positive integer n_0 such that if $n \geq n_0$, there is x_n such that $f(x_n) = y_n$. Note that $\lim_{n\to\infty} x_n = \infty$. We obtain

$$\lim_{n\to\infty} g'(x_n) = \cos 2n\pi.f'(x_n) = 1.\infty$$

This contradicts the fact that g is bounded. Hence our assumption was false, and g is not a periodic function. In particular, $\sin x^2$ or $\cos x^2$ is not periodic. $\square$

7.7.4 Problem. (Mean Value Theorem due to Lagrange). Suppose that f is a function defined and continuous on an interval $[a, b]$, and that it is differentiable in (a, b). Then there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

7.7.4.1 Solution. Define the function g as follows

$$g(x) = f(a) - f(x) + \frac{f(b) - f(a)}{b - a}(x - a) \text{ for } x \in [a, b].$$

We see that g is continuous on $[a, b]$ and differentiable on (a, b). Then

$$g'(x) = -f'(x) + \frac{f(b) - f(a)}{b - a} \text{ for } x \in (a, b).$$

Furthermore, $g(a) = g(b) = 0$. Then by the above problem there exists a sequence $([a_n, b_n])$ of nested closed intervals such that $g(a_n) = g(b_n)$ for each $n \in \mathbb{N}$, and $\bigcap_{n\in\mathbb{N}}[a_n, b_n] = \{p\}$, where $p \in (a, b)$. Since g is differentiable at p, there a unique function $g_p : [a, b] \to \mathbb{R}$ defined by $g_p(x) = \frac{g(x)-g(p)}{x-p}$ $\forall\, x \in [a, b]$. In particular, for each $n \in \mathbb{N}$, we have

$$\begin{aligned} g(a_n) &= g(p) + (a_n - p)g_p(a_n) \text{ and} \\ g(b_n) &= g(p) + (b_n - p)g_p(b_n) \end{aligned}$$

But $g(a_n) = g(b_n)\ \forall\, n \in \mathbb{N}$ and hence $(a_n - p)g_p(a_n) = (b_n - p)g_p(b_n)\ \forall\, n \in \mathbb{N}$. Since $a_n - p < 0$ and $b_n - p > 0$ we see that $g_p(a_n)$ and $g_p(b_n)$ have opposite in signs. So the continuity of g at p implies that $g'(p) = g_p(p) = 0$. Hence

$$f'(p) = \frac{f(b) - f(a)}{b - a} \text{ for } x \in (a, b). \quad \square$$

7.7.1 Note. Note that the Mean Value Theorem is proved above without using Rolle's theorem, and Rolle's theorem now becomes as a corollary of the Mean Value Theorem.

7.7.5 Problem. Let a function $f : \mathbb{R} \to \mathbb{R}$ be continuous at 0 and let $\lambda, \mu \in \mathbb{R}^+$. Prove that

$$\lim_{x\to 0} \frac{f(\lambda x) - f(\mu x)}{x}$$

exists and finite if and only if f is differentiable at 0.

7.7.5.1 Solution. Let $\lambda > \mu$ and let $y = \mu x$. Then

$$\lim_{x\to 0} \frac{f(\lambda x) - f(\mu x)}{x} = \lim_{y\to 0} \frac{f\left(\frac{\lambda}{\mu} y\right) - f(y)}{\frac{y}{\mu}} = A,$$

so

$$\lim_{y\to 0} \frac{f(\alpha y) - f(y)}{y} = \frac{A}{\mu} = a,$$

where $\alpha = \frac{\lambda}{\mu} > 1$. Let $\epsilon > 0$. Then $\exists\, \delta > 0$ such that for any $|y| < \delta$, we have

$$a - \epsilon < \frac{f(\alpha y) - f(y)}{y} < a + \epsilon.$$

Substituting $\frac{y}{\alpha^k}$ for $y; k = 1, 2, .., n$ in the above relation, we obtain

$$\begin{aligned}
&\frac{1}{\alpha}(a-\epsilon) < \frac{f(y) - f\left(\frac{y}{\alpha}\right)}{y} < \frac{1}{\alpha}(a+\epsilon)\\
&\frac{1}{\alpha^2}(a-\epsilon) < \frac{f\left(\frac{y}{\alpha}\right) - f\left(\frac{y}{\alpha^2}\right)}{y} < \frac{1}{\alpha^2}(a+\epsilon)\\
&\ldots\ldots\\
&\frac{1}{\alpha^n}(a-\epsilon) < \frac{f\left(\frac{y}{\alpha^{n-1}}\right) - f\left(\frac{y}{\alpha^n}\right)}{y} < \frac{1}{\alpha^n}(a+\epsilon).
\end{aligned}$$

Summing up these inequalities, we get

$$\frac{1}{\alpha^n} \frac{1 - \frac{1}{\alpha^n}}{1 - \frac{1}{\alpha}}(a-\epsilon) < \frac{f\left(y\right) - f\left(\frac{y}{\alpha^n}\right)}{y} < \frac{1}{\alpha^n} \frac{1 - \frac{1}{\alpha^n}}{1 - \frac{1}{\alpha}}(a+\epsilon).$$

Since f is continuous at 0, we get

$$\lim_{n\to\infty} f\left(\frac{y}{\alpha^n}\right) = f(0),$$

and so as $n \to \infty$ we have

$$\frac{1}{\alpha - 1}(a-\epsilon) < \frac{f\left(y\right) - f\left(0\right)}{y} < \frac{1}{\alpha - 1}(a+\epsilon).$$

It follows that

$$f'(0) = \lim_{y\to 0} \frac{f\left(y\right) - f\left(0\right)}{y} = \frac{a}{\alpha - 1} = \frac{A}{\lambda - \mu},$$

so f is differentiable at 0.
Conversely, if f is differentiable at 0, then

$$\lim_{x\to 0}\frac{f(\lambda x)-f(0)}{\lambda x}=f'(0) \text{ and } \lim_{x\to 0}\frac{f(\mu x)-f(0)}{\mu x}=f'(0).$$

Hence

$$\lim_{x\to 0}\frac{f(\lambda x)-f(\mu x)}{x}=(\lambda-\mu)f'(0). \quad \square$$

7.7.6 Problem. Let $f:\mathbb{R}\to\mathbb{R}$ be differentiable and assume there is no $x\in\mathbb{R}$ such that $f(x)=f'(x)=0$. Show that $S=\{x;x\in[a,b],f(x)=0\}$ is finite.

7.7.6.1 Solution. Consider $f^{-1}(\{0\})$. Since $\{0\}$ is closed and f continuous, $f^{-1}(\{0\})$ is closed. Therefore $S=[0,1]\cap f^{-1}(\{0\})$ is a closed and bounded subset of $\mathbb{R}$. Hence, S is compact. Assume that S is infinite. Then there is a limit point $x\in S$; i.e., there is a sequence (x_n) of distinct points in S which converges to x. Also, as all points are in S, $f(x_n)=f(x)=0\ \forall\ n\in\mathbb{N}$. We now show that $f'(x)=0$, which will give us our desired contradiction. Since $|x_n-x|\to 0$, we can write the derivative of f as follows:

$$f'(x)=\lim_{n\to\infty}\frac{f(x+(x_n-x))-f(x)}{x_n-x}=\lim_{n\to\infty}\frac{f(x_n)-f(x)}{x_n-x}=0.$$

The last equality holds since $f(x)=f(x_n)=0$ holds for all $n\in\mathbb{N}$. $\square$

7.7.7 Problem. Use the concept of Lipschitz property of functions to give an alternative proof to the fact that $f(x)=1/x$ is continuous at every $x_0>0$.

7.7.7.1 Solution. Hint. We have $f'(x)=-1/x^2$. So, $|f'(x)|\le 4/x_0^2$, if $x\ge 1/2x_0$. Thus, the function is Lipschitz on $[\frac{1}{2}x_0,\infty)$ with the constant $4/x_0^2$. So put $\delta=\min\left\{\frac{1}{2}x_0,\frac{\epsilon x_0^2}{4}\right\}$ and use the Mean Value Theorem. $\square$

7.7.8 Problem. Let $f:\mathbb{R}\to\mathbb{R}$ be a twice-differentiable function with $f''(x)>0\ \forall\ x\in[0,1]$. Assume that $f(0)>0$ and $f(1)=1$. Prove that there exists $x_0\in(0,1)$ such that $f(x_0)=x_0$ if and only if $f'(1)>1$.

7.7.8.1 Solution. Suppose that there exists $x_0\in(0,1)$ such that $f(x_0)=x_0$. Hence, using Taylor's expansion with Lagrange remainder, we get

$$x_0=f(x_0)=f(1)+(x_0-1)f'(1)+\frac{(x_0-1)^2f''(\theta)}{2!}$$

for some $\theta\in(x_0,1)$. Rearranging

$$f'(1)=\frac{x_0-f(1)}{x_0-1}-\frac{(x_0-1)^2f''(\theta)}{2!}.$$

Now, since $f(1)=1,x_0<1$ and $f''(\theta)>0$, then we get that $f'(1)>1$ as desired.
Conversely, Now, assume that $f'(1)>1$. Suppose by way of contradiction that $\nexists x\in(0,1)$ such that $f(x)=x$. Hence, since, f is continuous, then, on (0,1), f lies on one side of the line $y=x$

(by applying the intermediate value theorem to the function $f(x) - x$). But, since $f(0) > 0$ by assumption, then, $f(x) > x$, $\forall\, x \in [0,1)$. But, then, we have the following: $\forall\, x \in [0,1)$

$$1 = \frac{f(1) - x}{1 - x} > \frac{f(1) - f(x)}{1 - x}.$$

Hence, taking the limit as $x \to 1-$ and using the continuity of f', we get that $1 \geq f'(1)$ contrary to our assumption. Hence, there exists $x_0 \in (0,1)$ such that $f(x_0) = x_0$. $\square$

7.7.2 Note. The derived function f' may in turn have a derivative, which is called **second order derivative** of f and is denoted symbolically by $\frac{d^2y}{dx^2}$ or $f''(x)$ at the point x. If f' exists at each point in an interval $(x_0 - \delta, x_0 + \delta)$ and is continuous at x_0, then the second order derivative at $x = x_0$ is given by the limit

$$\lim_{h \to 0} \frac{f'(x_0 + h) - f'(x_0)}{h},$$

or by the iterated limit

$$\lim_{k \to 0} \lim_{h \to 0} \frac{f(x_0 + h + k) - f(x_0 + h) - f(x_0 + k) + f(x_0)}{hk}.$$

But one should not infer from the above that it is permissible to put $h = k$ to obtain $f''(x_0)$ as

$$\lim_{h \to 0} \frac{f(x_0 + 2h) - 2f(x_0 + h) + f(x_0)}{h^2}.$$

This is not always possible, as for example,

7.7.3 Example. Given that $f(x) = \begin{cases} x^3 \cos \frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$

Consider the existence of $f''(0)$. The first derivative of $f(x)$ is

$$f'(x) = \begin{cases} 3x^2 \cos \frac{1}{x} + x \sin \frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

The derived function f' is continuous at $x = 0$ and exists in the nbhd. of this point; hence if $f''(0)$ exists, it is given by the limit

$$\lim_{h \to 0} \frac{f'(0 + h) - f'(0)}{h} = \lim_{h \to 0} 3h \cos \frac{1}{h} + \sin \frac{1}{h}.$$

But this limit does not exist, since $\lim_{h \to 0} \sin \frac{1}{h}$ does not exist. Again, we have

$$\lim_{h \to 0} \frac{f(0 + 2h) - 2f(0 + h) + f(0)}{h^2} = \lim_{h \to 0} 8h \cos \frac{1}{2h} + 2h \cos \frac{1}{h} = 0,$$

and consequently $f''(0)$ is not given by this limit. We may, however, state that: Let $f : [a,b] \to \mathbb{R}$ and $f'(x), f''(x)$ exists and finite in (a,b), then

$$f''(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - 2f(x_0) + f(x_0 - h)}{h^2}$$
$$\text{or } \; f''(x_0) = \lim_{h \to 0} \frac{f(x_0 + 2h) - 2f(x_0 + h) + f(x_0)}{h^2}.$$

The proof is left to the reader.

7.7.9 Problem. Show that the function $f_a(x) = \begin{cases} \cos\dfrac{1}{x} & \text{for } x \neq 0 \\ a & \text{for } x = 0 \end{cases}$ has the intermediate value property if $a \in [-1, 1]$ but is the derivative of a function only if $a = 0$.

7.7.9.1 Solution. The function is not continuous at 0, so it maps any interval that does not contain 0 onto an interval. Any interval containing 0 is mapped onto $[-1, 1]$, which proves that f has the intermediate value property for any $a \in [-1, 1]$. For the second part of the problem, we introduce the function F defined by

$$F(x) = \begin{cases} x^2 \sin\dfrac{1}{x} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0. \end{cases}$$

Now, we can show that

$$F'(x) = \begin{cases} 2x \sin\dfrac{1}{x} - \cos\dfrac{1}{x} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0. \end{cases}$$

7.7.10 Problem. Suppose that $f : \mathbb{R} \to [0, \infty)$ is twice continuously differentiable. Let K be the support of f. In other words, let K be the closure of $\{x \in \mathbb{R}; f(x) \neq 0\}$. Suppose that K is compact. Prove that there is a constant C (depending on f), such that for each $x \in \mathbb{R}$, we have

$$f'^2(x) \leq Cf(x).$$

7.7.10.1 Solution. On $\mathbb{R} \setminus K, f$ is identically zero so all derivatives of f are zero. Furthermore f'' attains a maximum on K since K is compact. Thus for all $x \in \mathbb{R}$ one has $f''(x) \leq C$ for some constant C. Using Taylor's Theorem with Lagrange remainders we have for all $x \in \mathbb{R}$ and for all $h > 0$,

$$f(x + h) = f(x) + f'(x)h + \frac{f''(c)}{h^2}$$

for some $c \in (x, x + h)$. Now it follows that for all $h > 0$ and for any fixed x, $0 \leq f(x + h) \leq f(x) + f'(x)h + \dfrac{C}{2}h^2$. Now $\frac{C}{2}h^2 + f'(x)h + f(x)$ is a quadradic polynomial in h that is always nonnegative so it's discriminant $(f'(x))^2 - 2Cf(x) \leq 0$, so $(f'(x))^2 \leq Cf(x)$. Since x is arbitrary, this proves the claim. $\square$

7.7.11 Problem. Suppose that f is a C^1 function on $\mathbb{R}$ which has the properties that

$$\lim_{x\to\infty} f(x) = A \text{ and } \lim_{x\to\infty} f'(x) = B$$

for some real numbers A and B. Show that $B = 0$.

7.7.11.1 Solution. Let $\epsilon > 0$ be fixed. Since $f(x) \to A$ as $x \to \infty$, there exists $M_1 > 0$ such that for all $x > M_1$, we have that $|f(x) - A| < \epsilon/4$. Thus, for all $x, y > M_1$, we have

$$|f(x) - f(y)| \leq |f(x) - A| + |A - f(y)| < \epsilon/2.$$

Similarly, since $f'(x) \to B$, as $x \to \infty$, there exists $M_2 > 0$ such that for all $x, y > M_2$, we have $|f'(x) - f'(y)| < \epsilon/2$. Let $M = \max\{M_1, M_2\}$. Let $x > M$ be arbitrary. We wish to show that $|f'(x)| < \epsilon$. By the mean value theorem, we have

$$\frac{f(x+1) - f(x)}{(x+1) - x} = f'(\theta)$$

for some $\theta \in (x, x+1)$. Moreover, we have that since $x, x+1 > M$, then $|f(x+1) - f(x)| = |f'(\theta)| < \epsilon/2$. Also, since $x, \theta > M$, then we have $|f'(x) - f'(\theta)| < \epsilon/2$. But, then we get

$$|f'(x)| < |f'(\theta)| + \epsilon/2 < \epsilon/2 + \epsilon/2 = \epsilon$$

as desired.

7.7.12 Problem. Let $f : (1, \infty) \to \mathbb{R}$ be differentiable and define $g, h : (1, \infty) \to \mathbb{R}$ by

$$g(x) = \frac{f'(x)}{x}, \text{ and } h(x) = \frac{f(x)}{x}.$$

Suppose g is bounded. Prove that h is uniformly continuous.

7.7.12.1 Solution. It suffices to show that h' is bounded on $(1, \infty)$, for if a differentiable function is has bounded derivative then it is uniformly continuous by the Mean Value Theorem. Toward that end observe that by the quotient rule,

$$h'(x) = \frac{f'(x)}{x} - \frac{f(x)}{x^2}$$

We will show that $\frac{f(x)}{x^2}$ is bounded. We first check that it is bounded on the interval (1,2] by showing f is bounded on (1,2]. By our assumption on (1,2], we have that $f'(x)$ is bounded by a constant. If we suppose for contradiction that f is unbounded on (1,2] then we may construct a decreasing sequence x_n, with $2 > x_1$ and $x_n \to 1$ and $f(x_n) \to \infty$ (increasing sequence). Now by the mean value theorem for any x_n we have that there is an $l_n \in (x_n, 2)$ so that $\dfrac{f(x_n) - f(2)}{x_n - 2} = f'(l_n)$. Now as n goes to infinity, we have $|f'(l_n)|$ goes to infinity as well since $x_n - 2$ goes to -1, but this contradicts $f'(x)$ being bounded on (1,2]. We next verify that $\dfrac{f(x)}{x^2}$ is bounded on $[2, \infty)$. By the Mean Value Theorem for any $x \in (2, \infty)$ one has

$$\left|\frac{f(x) - f(2)}{x - 2}\right| = |f'(y_x)|$$

for some $y_x \in (2, x)$. By our assumption $|f'(y_x)| \leq cy_x$ for some constant c and since $y_x < x$ we have

$$\left|\frac{f(x) - f(2)}{x - 2}\right| \leq |cx|.$$

Thus,

$$\left|\frac{f(x) - f(2)}{x^2}\right| \leq c.$$

And it follows that $\left|\dfrac{f(x)}{x^2}\right| \leq c + \dfrac{|f(2)|}{x^2} \leq c + \dfrac{|f(2)|}{4}$. It now follows that $h'(x)$ is bounded and therefore h is uniformly continuous. □

7.7.13 Problem. Show that the set of points where a continuous function is differentiable is a countable intersection of F_σ-sets.

7.7.13.1 Solution. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. We now consider the set

$$S = \{a \in \mathbb{R}; f \text{ is not differentiable at } a\}.$$

Take $c \in \mathbb{R}$. For each $n, k \in \mathbb{N}$, put

$$U_{n,k} = \left\{a \in \mathbb{R};\ \exists\, x \in \mathbb{R} \text{ such that } 0 < |x-a| < \frac{1}{n} \text{ and } \frac{f(x)-f(a)}{x-a} > c - \frac{1}{k}\right\}.$$

We see that $U_{n,k}$ is open. It is easy to see that $\overline{\lim}_{x\to a}(f(x)-f(a))/(x-a) \geq c$ if and only if a is contained in all of the sets $U_{n,k}$. Thus, for every $c \in \mathbb{R}$ the set

$$\left\{a;\ \overline{\lim}_{x\to a}\frac{f(x)-f(a)}{x-a} \geq c\right\}$$

is a G_δ. Similarly we can prove that $\underline{\lim}_{x\to a}(f(x)-f(a))/(x-a) \leq d$ is a G_δ. for every $d \in \mathbb{R}$. Now the set S of points where f is not differentiable can be written as

$$\bigcup_{c,d\in\mathbb{Q}} \left\{a :\ \underline{\lim}_{x\to a}\frac{f(x)-f(a)}{x-a} \leq d < c \leq \overline{\lim}_{x\to a}\frac{f(x)-f(a)}{x-a}\right\}$$

and therefore is a countable union of G_δ-sets. Thus the set of points where a continuous function is differentiable is a countable intersection of F_σ-sets. □

7.7.14 Problem. Let f be a function on $[a, b]$ that is differentiable at $c \in (a, b)$. Let $L(x)$ be the tangent line to f at c. Prove that L is the unique linear function with the property that

$$\lim_{x\to c}\frac{f(x)-L(x)}{x-c} = 0.$$

7.7.14.1 Solution. Let $x - c = h$, then

$$\begin{aligned}\lim_{x\to c}\frac{f(x)-L(x)}{x-c} &= \lim_{h\to 0}\frac{f(c+h)-(f(c)+hf'(c))}{h}\\ &= \lim_{h\to 0}\frac{f(c+h)-f(c)}{h} - f'(c)\\ &= 0\end{aligned}$$

Now if we have another function K that satisfies this limit, then by the continuity of K and f we have

$$\begin{aligned}f(c)-K(c) &= \lim_{x\to c} f(x) - K(c)\\ &= \lim_{x\to c}(x-c)\frac{f(x)-K(c)}{x-c}\\ &= \lim_{x\to c}(x-c).\lim_{x\to c}\frac{f(x)-K(c)}{x-c}\\ &= 0.\end{aligned}$$

Thus $K(x) = f(c) + m(x - c)$ where m is the slope and

$$\begin{aligned} m = K'(c) &= \lim_{h\to 0} \frac{K(c+h) - K(c)}{h} \\ &= \lim_{h\to 0} \frac{K(c+h) - f(c+h)}{h} + \lim_{h\to 0} \frac{f(c+h) - f(c)}{h} \\ &= 0 + f'(c) = f'(c). \end{aligned}$$

Thus the line K that goes through the point $(c, f(c))$ has the same slope as L. Therefore $K = L$. $\square$

7.7.15 Problem. Suppose a differentiable function $f : \mathbb{R} \to \mathbb{R}$ has a uniformly continuous derivative on $\mathbb{R}$. Show that

$$\lim_{n\to\infty} n \left[f\left(x + \frac{1}{n}\right) - f(x)\right] = f'(x).$$

7.7.15.1 Solution. Since f' is uniformly continuous on $\mathbb{R}$, given $\epsilon > 0$, there exists $\delta > 0$ such that $|f'(x) - f'(y)| < \epsilon$ for any $x, y \in \mathbb{R}$ for which $|x - y| < \delta$. Let $N \in \mathbb{N}$ such that for any $n \geq N$ we have $1/n < \delta$. Then for any $x \in \mathbb{R}$ we have

$$|f'(t) - f'(x)| < \epsilon \text{ for any } t \in \left(x, x + \frac{1}{n}\right).$$

Since f is differentiable, we can use the Mean Value Theorem to obtain

$$\begin{aligned} \left| n \left[f\left(x + \frac{1}{n}\right) - f(x)\right] - f'(x) \right| &= \left| \frac{f\left(x + \frac{1}{n}\right) - f(x)}{\frac{1}{n}} - f'(x) \right| \\ &= |f'(t_n) - f'(x)| < \epsilon \end{aligned}$$

for some $t_n \in \left(x, x + \frac{1}{n}\right)$, which yields

$$\lim_{n\to\infty} n \left[f\left(x + \frac{1}{n}\right) - f(x)\right] = f'(x). \quad \square$$

7.7.16 Problem. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable at $x = 1, f(1) = 1$ and $k \in \mathbb{N}$. Show that

$$\lim_{n\to\infty} n \left[f\left(1 + \frac{1}{n}\right) + f\left(1 + \frac{2}{n}\right) + .. + f\left(1 + \frac{k}{n}\right)\right] = \frac{k(k+1)}{2} f'(1).$$

7.7.16.1 Solution. Let $\frac{1}{n} = h$, then the expression becomes

$$\begin{aligned} E =& n \left[f\left(1 + \frac{1}{n}\right) + f\left(1 + \frac{2}{n}\right) + .. + f\left(1 + \frac{k}{n}\right) - k\right] \\ =& \frac{1}{h} \left[f\left(1 + h\right) + f\left(1 + 2h\right) + .. + f\left(1 + kh\right) - kf(1)\right] \\ =& \frac{f(1+h) - f(1)}{h} + 2\frac{f(1+h) - f(1)}{2h} + k\frac{f(1+kh) - f(1)}{kh}. \end{aligned}$$

$$\text{Hence } \lim_{h\to 0} E =(1 + 2 + .. + k)f'(1) = \frac{k(k+1)}{2} f'(1). \quad \square$$

7.7.17 Problem. If $f : \mathbb{R} \to \mathbb{R}$ be differentiable at $x = c \in \mathbb{R}$. Show that

$$f'(c) = \lim_{n\to\infty} n \left\{ f\left(c + \frac{1}{n}\right) - f(c) \right\}.$$

However, show by example that the existence of the limit of this sequence does not imply the existence of $f'(c)$.

7.7.17.1 Solution. First part is left to the reader. Example: $f(x) = |x|$ at $x = 0$. For

$$\lim_{n\to\infty} n \left\{ \frac{1}{n} - 0 \right\} = 1$$

exists but $f'(0)$ does not exist. □

7.7.18 Problem. Let $f(x)$ be differentiable at a. Find

$$\lim_{n\to\infty} \frac{a^n f(x) - x^n f(a)}{x - a}.$$

7.7.18.1 Solution. We have

$$\begin{aligned} \frac{a^n f(x) - x^n f(a)}{x - a} &= \frac{a^n f(x) - a^n f(a) + a^n f(a) - x^n f(a)}{x - a} \\ &= a^n \frac{f(x) - f(a)}{x - a} + f(a) \frac{x^n - a^n}{x - a} \\ &= a^n f'(a) - f(a) n a^{n-1}. \quad \square \end{aligned}$$

7.7.19 Problem. We say a function $f : (a, b) \to \mathbb{R}$ is **uniformly differentiable** if f is differentiable on (a, b) and for each $\epsilon > 0$ there exists a $\delta > 0$ such that $0 < |x - y| < \delta$ and $x, y \in (a, b)$ implies

$$\left| \frac{f(x) - f(y)}{x - y} \right| < \epsilon.$$

Prove that if f is uniformly differentiable, then f' is continuous. Then give an example of a function that is differentiable but not uniformly differentiable.

7.7.19.1 Solution. Suppose that f is uniformly differentiable and $\epsilon > 0$ be given. Then there exists some $\delta > 0$ such that

$$0 < |x - y| < \delta \Rightarrow \left| \frac{f(x) - f(y)}{x - y} - f'(x) \right| < \frac{\epsilon}{2}.$$

Hence,

$$|f'(x) - f'(y)| = \left| f'(x) - \frac{f(x) - f(y)}{x - y} \right| + \left| \frac{f(x) - f(y)}{x - y} - f'(y) \right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

shows that f' is continuous.

For the next part, consider the function f defined by

$$f(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Then the function is differentiable everywhere but $f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}$ for $x \neq 0$ and $\lim_{x\to 0} f'(x)$ does not exist. Thus, f' exists but is not continuous. □

7.7.20 Problem. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Consider the sequence $x_0 \in \mathbb{R}$ and $x_{n+1} = f(x_n)$. Assume that $\lim_{n\to\infty} x_n = l$ and $f'(l)$ exists. Show that $|f'(l)| \leq 1$.

7.7.20.1 Solution. Assume not, i.e., $|f'(l)| > 1$. First note that since $f(x)$ is continuous and $x_{n+1} = f(x_n)$, for any $n \geq 1$, we get by letting $n \to \infty$, $f(l) = l$. Since

$$\lim_{x\to l} \frac{f(x) - f(l)}{x - l} = f'(l)$$
$$\text{then } \lim_{x\to l} \left|\frac{f(x) - l}{x - l}\right| = |f'(l)| > 1$$

Take $\epsilon = \frac{|f'(l)|-1}{2} > 0$. Then there exists $\delta > 0$ such that for any $x \in (l - \delta, l + \delta)$, we have

$$|f'(l)| - \epsilon < \left|\frac{f(x) - l}{x - l}\right| < |f'(l)| + \epsilon,$$

or

$$\frac{|f'(l)| + 1}{2} < \left|\frac{f(x) - l}{x - l}\right|.$$

In particular, since $1 < |f'(l)|$, we have

$$|x - l| < |x - l|\frac{|f'(l)| + 1}{2} < |f(x) - l|$$

for any $x \in (l - \delta, l + \delta)$,. Since (x_n) converges to l, there exists $N \geq 1$ such that for any $n \geq N$, we have $x_n \in (l - \delta, l + \delta)$,. So

$$|x_n - l| < |f(x_n) - l| = |x_{n+1} - l|$$

for any $n \geq N$. In particular, we have $|x_N - l| < |x_{N+1} - l| < |x_n - l|$ for any $n > N$. If we let $n \to \infty$, we will get $|x_N - l| < |x_{N+1} - l| \leq 0$ which generates the desired contradiction. So we must have $|f'(l)| \leq 1$. Note that one may think that maybe $|f'(l)| < 1$. That is not the case in general. Indeed, take $f(x) = \sin x$. Then (x_n) converges to 0 but $f'(0) = \cos 0 = 1$. □

7.7.21 Problem. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and differentiable everywhere in (a, b) except possibly at $c \in (a, b)$. Assume that $\lim_{x\to c} f'(x) = l$. Show that $f(x)$ is differentiable at c and $f'(c) = l$.

7.7.21.1 Solution. Let $x \neq c$. Since Then by MVT, we get

$$\frac{f(x) - f(c)}{x - c} = f'(\xi),\ \xi \in (a, c) \text{ or } (c, b).$$

Since $\lim_{x\to c} f'(x)$ exists, given $\epsilon > 0$, there is a $\delta > 0$ such that as $0 < |x - c| < \delta$, we have $l - \epsilon < f'(x) < l + \epsilon$. So, if we choose $x \in (c - \delta, c + \delta), x \neq c$, we then have

$$l - \epsilon < \frac{f(x) - f(c)}{x - c} = f'(\xi) < l + \epsilon$$

That is, $f'(c)$ exists and equals l. □

7.7.22 Problem. Let f be a continuous function on $[a, b]$, differentiable on (a, b), and $f'(x) \neq 0$ for any $x \in (a, b)$. Show that f is one-to-one. Then show that $f'(x) > 0$ for every $x \in (a, b)$, or $f'(x) < 0$ for every $x \in (a, b)$. Deduce from this, that f' satisfies the Intermediate Value Theorem without using of continuity of f'.

7.7.22.1 Solution. Assume that $f(x)$ is not one-to-one. Then there exists $x_1 < x_2$ such that $f(x_1) = f(x_2)$. Rolle's theorem will imply the existence of $c \in (x_1, x_2)$ such that $f(c) = 0$ which contradicts our assumption. Next we will prove that $f(x)$ is monotone. Without loss of generality, assume $f(a) < f(b)$. Let $x \in (a, b)$. Assume $f(a) < f(b) < f(x)$. Since $f(x)$ is continuous on $[a, b]$, the Intermediate Value Theorem implies the existence of $c \in (a, x)$ such that $f(c) = f(b)$. Clearly $c \neq b$ which generates a contradiction with $f(x)$ being one-to-one. The same ideas will imply that $f(x) < f(a) < f(b)$ does not hold. Therefore we must have $f(a) < f(x) < f(b)$ for any $x \in (a, b)$. Next let $x, y \in (a, b)$ such that $x < y$. Assume $f(y) < f(x)$. Then we have $f(y) < f(x) < f(b)$. Again the Intermediate Value Theorem implies the existence of $c \in (y, b)$ such that $f(c) = f(x)$. Clearly $c \neq x$ because $x < y$. This is a contradiction with $f(x)$ being one-to-one. Therefore, for any $x, y \in (a, b)$ with $x < y$, we have $f(x) < f(y)$, i.e., $f(x)$ is increasing. This will imply that $f'(x) \geq 0$ but since $f'(x) \neq 0$, we get $f'(x) > 0$ for any $x \in (a, b)$. Finally let us prove that $f'(x)$ satisfies the conclusion of the Intermediate Value Theorem. Indeed, let $x_1, x_2 \in (a, b)$ and $\alpha \in \mathbb{R}$ such that $f'(x_1) < \alpha < f'(x_2)$. Without loss of generality we assume that $x_1 < x_2$. Next define $g(x) = f(x) - \alpha x$. The function $g(x)$ inherits all the properties of $f(x)$. In particular, we have $g'(x) = f'(x) - \alpha$. Assume that $g'(x) \neq 0$ for any $x \in (x_1, x_2)$. From the first part, we deduce that $g'(x) > 0$ or $g'(x) < 0$ for any $x \in (x_1, x_2)$. It is easy to check that this conclusion still holds at x_1 and x_2. Hence $g'(x) > 0$ or $g'(x) < 0$ for any $x \in [x_1, x_2]$. In other words, we have $f'(x) < \alpha$ or $f'(x) > \alpha$ for any $x \in [x_1, x_2]$. This is a contradiction with $f'(x_1) < \alpha < f'(x_2)$. Therefore there exists $c \in (x_1, x_2)$ such that $g'(c) = 0$ or $f'(c) = \alpha$. This completes the proof of our statement. □

7.7.23 Problem. Let $f : \mathbb{R} \to \mathbb{R}$. Assume that for any $x, t \in \mathbb{R}$ we have

$$|f(x) - f(t)| \leq |x - t|^{1+\alpha}$$

where $\alpha > 0$. Show that $f(x)$ is constant.

7.7.23.1 Solution. For any $x \neq t$, we have

$$\frac{|f(x) - f(t)|}{|x - t|} < |x - t|^{\alpha}.$$

Since $\lim_{x \to t} |x - t|^{\alpha} = 0$, we deduce that

$$\lim_{x \to t} \left| \frac{f(x) - f(t)}{x - t} \right| = \lim_{x \to t} \frac{|f(x) - f(t)|}{|x - t|} = 0.$$

Hence

$$\lim_{x \to t} \frac{f(x) - f(t)}{x - t} = 0.$$

therefore $f'(t)$ exists for any $t \in \mathbb{R}$. Since

$$f'(t) = \lim_{x \to t} \frac{f(x) - f(t)}{x - t} = 0.$$

we deduce that $f(x)$ is constant. □

7.7.24 Problem. Let $f : [0,\infty) \to \mathbb{R}$ differentiable everywhere. Assume that $\lim_{x\to\infty}(f(x) + f'(x)) = 0$. Show that $\lim_{x\to\infty} f(x) = 0$.

7.7.24.1 Solution. Let $g(x) = e^x f(x)$. Then, we have $g'(x) = e^x\left(f(x) + f'(x)\right)$. Let $\epsilon > 0$. There exists $A > 0$ such that for any $x > A$ we have $|f(x)+f'(x)| < \epsilon/2$. Let $x > A$, the generalized Mean Value Theorem implies the existence of $c \in (A, x)$ such that

$$\frac{g'(c)}{e^c} = \frac{g(x) - g(A)}{e^x - e^A},$$

or

$$f(c) + f'(c) = \frac{g(x) - g(A)}{e^x - e^A}.$$

In particular, we have

$$\begin{aligned} &|g(x) - g(A)| < \frac{\epsilon}{2}|e^x - e^A| \\ \Rightarrow &|g(x)| < \frac{\epsilon}{2}|e^x - e^A| + |g(A)| \end{aligned}$$

or

$$|f(x)| < \frac{\epsilon}{2}|1 - e^{A-x}| + |f(A)e^{A-x}| < \epsilon/2 + |f(A)e^{A-x}|$$

because $0 < e^{A-x} < 1$. Since limx $\lim_{x\to\infty} |f(A)e^{A-x}| = 0$, there exists $B > 0$ such that for any $x > B$ we have $|f(A)e^{A-x}| < \dfrac{\epsilon}{2}$. Let $A^* = \max\{A, B\}$, then for any $x > A^*$ we have

$$|f(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This completes the proof. □

7.7.25 Problem. Let $f : [0,1] \to \mathbb{R}$ be continuous and differentiable on (0,1) such that

1. $f(0) = 0$, and
2. there exists $M > 0$ such that $|f'(x)| \leq M|f(x)|$, for $x \in (0,1)$.

Show that $f(x) = 0$ for $x \in [0,1]$.

7.7.25.1 Solution. Let $D = \{x \in [0,1]; f(t) = 0 \; for \; t \in [0,x]\}$. Since $0 \in D, D$ is a nonempty subset of [0,1]. So the supremum of D exists. Let $s = \sup D$. Note that continuity of $f(x)$ implies $s \in D$. In order to complete the proof of our statement, we want to show that $s = 1$. Assume otherwise that $s < 1$. Then there exists $a_0 > 0$ such that $s + a_0 < 1$ and $a_0 M < 1$. For any $x \in (s, s + a_0)$, the Mean Value Theorem ensures the existence of $c \in (s, x)$ such that

$$f(x) - f(s) = f'(c)(x - s).$$

Since $f(s) = 0$, we get $f(x) = f(c)(x - s)$. Hence

$$|f(x)| \leq |f'(c)||x - s| \leq Ma_0 \max_{s\leq t\leq s+a_0} |f(t)|$$

for any $x \in [s, s + a_0]$. Hence

$$\max_{s\leq x\leq s+a_0} |f(x)| \leq Ma_0 \max_{s\leq t\leq s+a_0} |f(t)|$$

Since $a_0 M < 1$, we get

$$\max_{s \leq x \leq s+a_0} |f(x)| \leq \max_{s \leq t \leq s+a_0} |f(t)|$$

which is the desired contradiction. So $s = 1$ or $f(x) = 0$ for any $x \in [0, 1]$. $\square$

7.7.26 Problem. Consider a function $f : (a, b) \to \mathbb{R}$ be continuous whose second derivative $f''(x)$ exists and is continuous on (a, b). Let $c \in (a, b)$. Show that

$$\lim_{h \to 0} \frac{f(c+h) - 2f(c) + f(c-h)}{h^2} = f''(c).$$

Is the existence of the second derivative necessary to prove the existence of the above limit?

7.7.26.1 Solution. Fix $x, c \in (a, b)$ with $x \neq c$. Let

$$F(t) = f(t) + f'(t)(x - t) + M(x - t)^2$$

where M is chosen such that $F(c) = f(x)$. Then we have $F(x) = f(x) = F(c)$. The Mean Value Theorem implies the existence of θ between x and c such that $F'(\theta) = 0$. But

$$F'(\theta) = f'(\theta) + f''(\theta)(x - \theta) - f'(\theta) - 2M(x - \theta) = 0$$

which implies $M = f''(\theta)/2$. So for $x, c \in (a, b)$, there exists θ between x and c such that

$$f(x) = f(c) + f'(c)(x - c) + \frac{f''(\theta)}{2}(x - c)^2.$$

So for any $h > 0$, there exist $\theta_1 \in (c, c+h)$ and $\theta_2 \in (c-h, c)$ such that

$$f(c+h) = f(c) + f'(c)h + \frac{f''(\theta_1)}{2}h^2$$

and

$$f(c-h) = f(c) - f'(c)h + \frac{f''(\theta_2)}{2}h^2.$$

So

$$\lim_{h \to 0} \frac{f(c+h) - 2f(c) + f(c-h)}{h^2} = \frac{f''(\theta_1) + f''(\theta_1)}{2}.$$

It is clear that when $h \geq 0$, then $\theta_i \to c$ for $i = 1, 2$. And since $f''(x)$ is continuous at c, we get

$$\lim_{h \to 0} \frac{f(c+h) - 2f(c) + f(c-h)}{h^2} = f''(c).$$

For the converse, the answer is in the negative. Indeed, take $f(x) = x|x|$ and $c = 0$. Then we have

$$\lim_{h \to 0} \frac{f(0+h) - 2f(0) + f(0-h)}{h^2} = 0.$$

but $f''(0)$ does not exist. $\square$

7.7.27 Problem. Consider a function $f : [0,1] \to \mathbb{R}$ whose second derivative $f''(x)$ exists and is continuous on [0,1]. Assume that $f(0) = f(1) = 0$ and suppose that there exists $A > 0$ such that $|f(x)| \leq A$ for $x \in [0,1]$. Show that

$$\left|f'\left(\frac{1}{2}\right)\right| \leq \frac{A}{4} \text{ and } |f'(x)| \leq \frac{A}{2} \;\forall\, x \in (0,1).$$

7.7.27.1 Solution. Let $a \in (0,1)$. The proof of the previous problem or Taylor expansion with remainder of $f(x)$ at a will imply

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(c)}{2}(x-a)^2$$

where c is between a and x. Let $x = 0$ and $x = 1$ in the above equation results in

$$f(0) = f(a) + f'(a)(0-a) + \frac{f''(c)}{2}(0-a)^2$$

and

$$f(1) = f(a) + f'(a)(1-a) + \frac{f''(c)}{2}(1-a)^2$$

where $0 < c_1 < a$ and $a < c_2 < 1$. Subtract the first equation from the second to get

$$0 = f'(a) + \frac{f''(c_2)}{2}(1-a)^2 - \frac{f''(c_1)}{2}(a)^2$$

Hence

$$f'(a) = \frac{f''(c_1)}{2}(a)^2 - \frac{f''(c_2)}{2}(1-a)^2$$

If we use the fact $|f''(a)| \leq A$, we get

$$|f'(a)| \leq \frac{A}{2}\left(a^2 + (1-a)^2\right).$$

Consider $a = 1/2$ to get $\left|f'\left(\frac{1}{2}\right)\right| \leq \frac{A}{4}$. Also if we note that $a^2 + (1-a)^2 \leq 1$, we get $|f'(a)| \leq \frac{A}{2}$. $\square$

7.7.28 Problem. Prove that the equation

$$1 - x + \frac{x^2}{2} - \frac{x^3}{3} + ... + (-1)^n \frac{x^n}{n} = 0$$

has exactly one solution in $\mathbb{R}$ if n is odd and no solutions if n is even.

7.7.28.1 Solution. Let $P_n(x)$ be the left-hand side of the above equation. Suppose n is even. Note that as $x \to \pm\infty, P_n(x) \to \infty$. In particular, $P_n(x)$ attains its infimum. If the infimum is positive, then $P_n(x)$ has no roots. The derivative is

$$P_n'(x) = -1 + x - x^2 + ... + (-1)^n x^{n-1} = \frac{1-x^n}{1-x}$$

The second equality holds only if $x \neq -1$. Note that -1 is not a zero of $P_n'(x)$, since then every term of $P_n'(x)$ will be negative. The only real zero of $P_n'(x)$ is $x = 1$, such that $P_n(x)$ must obtain its infimum at $x = 1$. We see that

$$P_n(1) = (1-1) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{n-2} - \frac{1}{n-1}\right) + \frac{1}{n} > 0$$

Since the infimum of $P_n(x)$ is positive, it follows that $P_n(x)$ has no zero if n is even. Now let n be odd. Note that lim $\lim_{n\to-\infty} P_n(x) = \infty$ and $\lim_{n\to\infty} P_n(x) = -\infty$. Since $P_n(x)$ is a polynomial it is continuous. The intermediate value theorem implies that $P_n(x)$ has at least one zero. If $P_n(x)$ has another zero, it must have either a local maximum or local minimum by the continuity of the derivative (it is also a polynomial) and Rolle's theorem. The derivative is

$$P_n'(x) = -1 + x - x^2 + ... + (-1)^n x^{n-1} = -\frac{1+x^n}{1+x}$$

where the second equality holds only if $x \neq 1$. Note that -1 is not a zero of $P_n'(x)$ since, as above, every term in $P_n'(x)$ will be negative. Since this is the only possible real zero of $1 + x^n, P_n(x)$ has no real zeros and thus $P_n(x)$ has no local maxima or minima. This implies $P_n(x)$ has exactly one real zero when n is odd. This proves the claim. $\square$

7.7.29 Problem. Let $f : (0, \infty) \to \mathbb{R}$ be a function and let $c > 0$. Show that the following are equivalent:

1. f is differentiable at c.
2. $\lim_{k\to1} \dfrac{f(kc) - f(c)}{k-1}$ exists.

Moreover, show that if either (1) or (2) hold, then $f'(c) = \dfrac{1}{c} \lim_{k\to1} \dfrac{f(kc) - f(c)}{k-1}$.

7.7.29.1 Solution. (1) $\Rightarrow$ (2): Let $\epsilon > 0$. Let $\delta > 0$ be such that whenever $0 < |x - c| < \delta$, we have

$$\left|\frac{f(x) - f(c)}{x - c} - f'(c)\right| < \epsilon,$$

Now take $\delta_1 = \delta/c$. Then $\delta_1 > 0$ and whenever $0 < |k - 1| < \delta_1$, we have $0 < |kc - c| < \delta$, so that

$$\left|\frac{f(kc) - f(c)}{kc - c} - f'(c)\right| < \epsilon,$$

that is,

$$\left|\frac{f(kc) - f(c)}{k - 1} - cf'(c)\right| < c\epsilon, \text{ so } \lim_{k\to1} \frac{f(kc) - f(c)}{k-1} = cf'(c).$$

(2) $\Rightarrow$ (1): Suppose that

$$\lim_{k\to1} \frac{f(kc) - f(c)}{k-1} = L.$$

Let $\epsilon > 0$. Let $\delta > 0$ be such that whenever $0 < |k-1| < \delta$, we have

$$\left|\frac{f(kc)-f(c)}{k-1} - L\right| < \epsilon.$$

Now let $\delta_1 = \delta/c$. Then $\delta_1 > 0$ and whenever $0 < |x-c| < \delta_1$, we have $k = x/c$ and that $0 < c\left|\frac{x}{c} - 1\right| < c\delta$, that is, $0 < |k-1| < \delta$, so that

$$\left|\frac{f(\frac{x}{c}c)-f(c)}{\frac{x}{c}-1} - L\right| < \epsilon.$$

So if $0 < |x-c| < \delta_1$, then we have

$$\left|\frac{f(x)-f(c)}{x-c} - \frac{L}{c}\right| < \frac{\epsilon}{c}.$$

Hence

$$\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$$

exists and equal to $\dfrac{L}{c}$. So f is differentiable at c and

$$f'(c) = \frac{L}{c} = \frac{1}{c}\lim_{k\to 1}\frac{f(kc)-f(c)}{k-1}. \quad \square$$

7.7.30 Problem. Let $f : (0,\infty) \to \mathbb{R}$ be such that for all $x, y \in (0,\infty)$, $f(xy) = f(x) + f(y)$. If f is differentiable at 1, then show that f is differentiable at every $c \in (0,\infty)$ and that $f'(c) = f'(1)/c$. Conclude that f is infinitely differentiable. If $f'(1) = 2$, then find $f^{(n)}(3), n \in \mathbb{N}$.

7.7.30.1 Solution. First we note that $f(1) = 0$ because $f(1) = f(1.1) = f(1) + f(1)$. We have

$$\begin{aligned} f'(c) &= \lim_{k\to 1}\frac{f(kc)-f(c)}{kc-c} = \frac{1}{c}\lim_{k\to 1}\frac{f(k)+f(c)-f(c)}{k-1} \\ &= \frac{1}{c}\lim_{k\to 1}\frac{f(k)}{k-1} = \frac{1}{c}\lim_{k\to 1}\frac{f(k)-f(1)}{k-1} \\ &= \frac{f'(1)}{c}. \end{aligned}$$

So $f'(x) = f'(1)/x, x \in (0,\infty)$. So f' is infinitely differentiable, and

$$f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!f'(1)}{x^n}, x \in (0,\infty).$$

In particular, if $f'(1) = 2$, then $f^{(n)}(3) = \dfrac{(-1)^{n-1}(n-1)!2}{3^n}$. $\square$

7.7.31 Problem. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable, $|f'(x)| \leq 1 \;\forall\; x \in \mathbb{R}$, and that there exists an $a > 0$ such that $f(-a) = -a, f(a) = a$. Find $f(0)$. Give an example of such a function f.

7.7.31.1 Solution. Suppose that $f(0) > 0$. Then by the Mean Value Theorem applied to $[-a, 0]$, we have

$$\frac{f(0) - f(-a)}{0 - (-a)} = f'(c)$$

for some c between $-a$ and 0. Hence

$$1 = 0 + 1 < \frac{f(0)}{a} + 1 = \frac{f(0) - (a)}{a} = \frac{f(0) - f(-a)}{0 - (-a)} = f'(c) < 1,$$

a contradiction. Next suppose that $f(0) < 0$. Then by the Mean Value Theorem applied to $[0, a]$, we have we have

$$\frac{f(a) - f(0)}{a - 0} = f'(d)$$

for some d between 0 and a. Hence

$$1 = 1 - 0 < 1 - \frac{f(0)}{a} = \frac{a - f(0)}{a} = \frac{f(a) - f(0)}{a - 0} = f'(d) < 1,$$

a contradiction.
An example of such a function f is $f(x) = x, x \in \mathbb{R}$. Then $|f'(x)| = 1 \leq 1 \ \forall \ x \in \mathbb{R}$, and for any $a > 0, f(-a) = -a, f(a) = a$. And we must have $f(0) = 0$. $\square$

7.7.32 Problem. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable and that there are $L, L' \in \mathbb{R}$ such that

$$\lim_{x \to \infty} f(x) = L$$
$$\lim_{x \to \infty} f'(x) = L'.$$

Prove that $L' = 0$.

7.7.32.1 Solution. Given $\epsilon > 0$, let $R' > 0$ be such that for all $x > R', |f'(x) - L'| < \epsilon$. Take any $R > R'$ such that for all $x \in \mathbb{R}$ with $x > R, |f(x) - L| < \epsilon/2$. Then by the Mean Value Theorem applied to f in $[R + 1, R + 2]$,

$$\frac{f(R + 2) - f(R + 1)}{1} = f'(\xi)$$

for some $\xi \in (R + 1, R + 2)$. So

$$|f'(\xi)| = |f(R + 2) - f(R + 1)| = |f(R + 2) - L + L - f(R + 1)| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence

$$|L'| = |L' - f'(\xi) + f'(\xi)| \leq |L' - f'(\xi)| + |f'(\xi)| \leq \epsilon + \epsilon = 2\epsilon.$$

As $\epsilon > 0$ was arbitrary, it follows that $L' = 0$. $\square$

7.7.33 Problem. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that f is differentiable on $\mathbb{R}$ and $\forall \ x \in \mathbb{R}$ and $\forall \ n \in \mathbb{N}$, $f'(x) = \dfrac{f(x + n) - f(x)}{n}$.

7.7.33.1 Solution. Since the mapping $x \mapsto f(x+n)$ is differentiable, it follows that

$$x \mapsto \frac{f(x+n)-f(x)}{n} = f'(x)$$

is differentiable. Also, for all $m \in \mathbb{N}$, we have

$$\begin{aligned} f'(x+m) &= \frac{f(x+m+n)-f(x+m)}{n} \\ &= \frac{f(x+m+n)-f(x)+f(x)-f(x+m)}{n} \\ &= \frac{m+n}{n}.\frac{f(x+m+n)-f(x)}{m+n} - \frac{m}{n}.\frac{f(x+m)-f(x)}{m} \\ &= \frac{m+n}{n}.f'(x) - \frac{m}{n}.f'(x) = f'(x). \end{aligned}$$

Thus,

$$f''(x) = \frac{f'(x+n)-f'(x)}{n} = \frac{f'(x)-f'(x)}{n} = 0.$$

By the Mean Value Theorem applied to f', this gives that f' is a constant, that is, there is a $c \in \mathbb{R}$ such that for all $x \in \mathbb{R}, f'(x) = c$. Applying the Mean Value Theorem to f, we obtain for every nonzero real x that

$$\frac{f(x)-f(0)}{x-0} = f'(z)$$

for some z between 0 and x. But since f' is constant, it follows that $f(x) = f(0) + cx$ for all $x \in \mathbb{R}$. □

7.7.34 Problem. Suppose $f : [0,1] \to \mathbb{R}$ is differentiable and $f'(x) \leq 1$ at every point $x \in (0,1)$. If $f(0) = 0$ and $f(1) = 1$, show that $f(x) = x \;\forall\; x \in [0,1]$.

7.7.34.1 Solution. Let $p \in (0,1)$. Applying MVT in $[0,p]$, we get

$$f(p) - f(0) = f'(c)p \text{ for some } c \in (0,p), \text{ as } f'(c) \leq 1$$

i.e. $f(p) \leq p$. Again by MVT in $[p,1]$, we get

$$f(1) - f(p) = f'(d)(1-p) \leq 1-p \text{ for some } d \in (p,1), \text{ as } f'(d) \leq 1.$$

i.e. $-f(p) \leq -p$ implies $f(p) \geq p$ and hence $f(p) = p \;\forall\; p \in [0,1]$. □

7.7.35 Problem. Let $f(x) = \ln(1+x^{2n})$, find $f^{(2n)}(-1)$.

7.7.35.1 Solution. Hint:
$x^{2n}+1 = (x-a_1)(x-a_2)...(x-a_{n-1})(x-a_n)(x-b_1)(x-b_2)...(x-b_{n-1})(x-b_n)$, where $b_k = \overline{a_k}$ conjugate of a_k, then

$$f'(x) = \sum_{k=1}^{n} \left[\frac{1}{(x-a_k)} + \frac{1}{(x-b_k)} \right].$$

Find $f^{(r)}(x)$ and then proceed. □

7.7.36 Problem. Let $a, b \in \mathbb{R}, a < b$, and let f, g be continuous real-valued functions on $[a, b]$ that are differentiable on (a, b). Prove that there exists a number $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

7.7.36.1 Solution. Consider the function

$$F(x) = (f(x) - f(a))(g(b) - g(a)) - (g(x) - g(a))(f(b) - f(a)).$$

The function $F(x)$ is continuous on $[a, b]$ and differentiable on (a, b) since f and g are. Note that

$$F(a) = (f(a) - f(a))(g(b) - g(a)) - (g(a) - g(a))(f(b) - f(a)) = 0 \text{ and}$$
$$F(b) = (f(b) - f(a))(g(b) - g(a)) - (g(b) - g(a))(f(b) - f(a)) = 0.$$

By the Mean Value Theorem, there exists $c \in (a, b)$ such that $f'(c) = 0$. Computing the derivative of $F(x)$ yields the desired result. □

7.7.37 Problem. Show that if $f(a) = 0$ and $f(x) = o(x - a)$ as $x \to 0$, then $f'(a)$ exists. What is $f'(a)$?

7.7.37.1 Solution. By definition

$$\frac{|f(x) - f(a)|}{|x - a|}$$

has the limit 0, when $x \to a$, which shows that $f'(a) = 0$. □

7.7.38 Problem. Provide an example or state that no such example exists

1. A function $f : (a, b) \to \mathbb{R}$ that is differentiable on (a, b) but not uniformly continuous on (a, b).
2. A function $f : (a, b) \to \mathbb{R}$ that is uniformly continuous on (a, b) but not differentiable on (a, b).
3. A function $f : \mathbb{R} \to \mathbb{R}$ be continuous and a closed subset of $D \subseteq \mathbb{R}$ such that $f(D)$ is not closed.
4. A function $f : \mathbb{R} \to \mathbb{R}$ be continuous and an open subset of $E \subseteq \mathbb{R}$ such that $f(E)$ is not closed.
5. A function $f : \mathbb{R} \to \mathbb{R}$ be continuous and injective, an open subset of $D \subseteq \mathbb{R}$ such that $f(D)$ is not open.
6. A function $f : [a, b] \to \mathbb{R}$ be differentiable on $[a, b]$ but not Riemann integrable on $[a, b]$.
7. A function $f : [a, b] \to \mathbb{R}$ be differentiable on a, b but f' is not Riemann integrable on $[a, b]$. [2]
8. If $f : (a, b) \to \mathbb{R}$ is differentiable and f is strictly decreasing on (a, b), then $f'(x) < 0,\ \forall x \in (a, b)$.
9. A function $f : \mathbb{R} \to \mathbb{R}$ be continuous and bounded, then f assumes its maximum or minimum values.

[2] For Riemann integrable functions, consult [10]

10. A function $f : \mathbb{R} \to \mathbb{R}$ be differentiable, then $\forall c \in \mathbb{R}$, $\exists\ a, b \in \mathbb{R}$ such that $a < c < b$ and $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

11. Let $f, g : (a, b) \to \mathbb{R}$ and $c \in (a, b)$. If fg and g are both differentiable at c, then f is differentiable at c.

7.7.38.1 Solution.

1. Hint: $f(x) = \frac{1}{x}$ on $(0, 1)$.
2. Hint: $f(x) = |x|$.
3. Hint: $f(x) = e^x, D = \mathbb{R}$.
4. Hint: $f(x) = 1, E = \mathbb{R}$.
5. Hint: No such example exists.Why?
6. Hint: No such example exists.Why?
7. Hint: Let $f : [-1, 1] \to \mathbb{R}$ be defined as

$$f(x) = \begin{cases} 0, & \text{if } x = 0 \\ x^2 \sin \frac{1}{x^2}, & \text{if } x \neq 0. \end{cases}$$

   Now f' is unbounded on $[-1, 1]$. Hence, not R-integrable.

8. Counterexample: $f(x) = -x^3$ is strictly decreasing on $(-1, 1)$ and $f'(0) = 0$.
9. Counterexample:$f(x) = \tan^{-1} x$ is continuous and bounded on $\mathbb{R}$ with the property that the maximum or minimum values are both not attained/assumed; $tan^{-1}x$ is an increasing function bounded between its horizontal asymptotes $y = \pm\dfrac{\pi}{2}$.
10. Counterexample: $f(x) = x^3, f'(0) = 0$ and $\forall a < 0 < b$, $\dfrac{f(b) - f(a)}{b - a} > 0$, because f is strictly increasing.
11. Counterexample: Let $f(x) = |x|$ and $g(x) = x$ and both fg and g are differentiable, but f is not differentiable. □

7.7.39 Problem. If f assumes a local maximum or local minimum at the end point of the domain $[a, b]$, what can be said about the one sided derivative (assuming it exists)?

7.7.39.1 Solution. Suppose f assumes a local maximum at a, then we have $\delta > 0$

$$\frac{f(x) - f(a)}{x - a} \leq 0, x \in (a, a + \delta)$$

for $f(x) \leq f(a)$ for some $a < x < a + \delta$. Hence $\lim_{x \to a+} \dfrac{f(x) - f(a)}{x - a} = f'(a) \leq 0$. □

7.7.40 Problem. Prove that if f' is constant on $\mathbb{R}$, then f is an affine function.

7.7.40.1 Solution. Suppose $f' = c$(a constant) on $\mathbb{R}$, then by MVT we get

$$f(x) = f(x_0) + f'(y)(x - x_0) = f(x_0) + c(x - x_0).$$

i.e. f is an affine function. $\square$

7.7.41 Problem. Let $f : [0, 1] \to \mathbb{R}$ be differentiabile on $(0, 1)$. Show by Cauchy's mean value theorem that the equation

$$f(1) - f(0) = \frac{f'(x)}{2x}$$

has at least one solution in $(0, 1)$.

7.7.41.1 Solution. Consider $g(x) = x^2$ and applying Cauchy's mean value theorem on $f(x)$ and $g(x)$, we get $\exists\, x \in (0, 1)$ such that

$$\frac{f(1) - f(0)}{1 - 0} = \frac{f'(x)}{2x}.$$

Hence the result follows. $\square$

7.7.42 Problem. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined as follows,

$$f(x) = \begin{cases} x^2, & \text{if } x \text{ is rational} \\ -x^2, & \text{if } x \text{ is irrational} \end{cases}.$$

Show that f is continuous only at 0. Is f differentiable anywhere? Explain.

7.7.42.1 Solution. Suppose f is continuous at $x(\neq 0)$. Let (x_n) be a rational sequence and (y_n) be an irrational sequence both converging to x. So $\lim_{n\to\infty} f(x_n) = x^2$ and $\lim_{n\to\infty} f(y_n) = -x^2$ Since f is continuous at x, then $x^2 = -x^2$, which is only possible when $x = 0$. So f is not continuous when $x \neq 0$. Which mans that f is not differentiable at $x \neq 0$. Now we will show that $f'(0) = 0$. For all $x \neq 0$, we have

$$-\left|\frac{x^2}{x}\right| \leq \frac{f(x) - f(0)}{x - 0} \leq \left|\frac{x^2}{x}\right|.$$

Now,

$$-\lim_{x\to 0}\left|\frac{x^2}{x}\right| = \lim_{x\to 0}\left|\frac{x^2}{x}\right| = 0,$$

shows that

$$f'(0) = \lim_{x\to 0}\frac{f(x) - f(0)}{x - 0} = 0. \quad \square$$

Note: It is important to assume the existence of the one-sided derivative as the existence of local extrema is not sufficient to guarantee that the function is differentiable. For example, the function

$$f(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{if } x \in (0, 1]. \end{cases}$$

7.7.43 Problem. Show that given any continuous function $f : \mathbb{R} \to \mathbb{R}$, there exists $x_0 \in [0, 1]$ and $m \in \mathbb{Z} \setminus \{0\}$ such that $f(x_0) = mx_0$. In other words, the graph of f intersects some nonhorizontal line $y = mx$ at some point $x_0 \in [0, 1]$.

7.7.43.1 Solution. Hint: If $f(0) = 0$, take $x_0 = 0$ and any $m \in \mathbb{Z}\{0\}$. If $f(0) = 0$, then choose $N \in \mathbb{N}$ satisfying $N > f(1)$, and apply the intermediate value theorem to the continuous function $g(x) = f(x) - Nx$ on the interval [0,1]. If $f(0) < 0$, then first choose a $N \in \mathbb{N}$ such that $N > -f(1)$, and consider the function $g(x) = f(x) + Nx$, and proceed in a similar manner. □

7.7.44 Problem. Suppose f is defined in a nbhd. of x and f'' exists. Prove that

$$\lim_{h\to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = f''(x)$$

Give an example of an f in which the above limit exists but f'' does not exist.

7.7.44.1 Solution. Suppose for some $\delta > 0, f$ and f'' are defined on $N(x;\delta)$, so f' is defined on $N(x;\delta)$. For $h \in [0, \frac{\delta}{2}]$, define

$$p(h) = f(x+h) + f(x-h) - 2f(x),\ q(h) = h^2.$$

Now p, q are differentiable on $(0, \frac{\delta}{2}), q'(h) \neq 0\ \forall\ h \in (0, \frac{\delta}{2}), p(0) = 0 = q(0)$, and by L'hospital's rule

$$\begin{aligned}\lim_{h\to 0} \frac{p'(h)}{q'(h)} &= \lim_{h\to 0} \frac{f'(x+h) - f'(x-h)}{2h} \\ &= \lim_{h\to 0} \frac{f'(x+h) - f'(x)}{2h} + \lim_{h\to 0} \frac{f'(x) - f'(x-h)}{2h} = f''(x). \quad \square\end{aligned}$$

Example: Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} \frac{x^2}{2}, x > 0 \\ -\frac{x^2}{2}, x \leq 0. \end{cases}$$

7.7.45 Problem. Let $f, g : (a, b) \to \mathbb{R}$ and $c \in (a, b)$. If fg and g are both differentiable at c, then f is differentiable at c.

7.7.45.1 Solution (Counterexample: Let $f(x) = |x|$ and $g(x) = x$ and both fg and g are differentiable, but f is not differentiable.). □

7.7.46 Problem. If f assumes a local maximum or local minimum at the end point of the domain $[a, b]$, what can be said about the one sided derivative(assuming it exists)?

7.7.46.1 Solution. Suppose f assumes a local maximum at a, then we have $\delta > 0$

$$\frac{f(x) - f(a)}{x - a} \leq 0, x \in (a, a + \delta)$$

for $f(x) \leq f(a)$ for some $a < x < a + \delta$. Hence $\lim_{x\to a+} \dfrac{f(x) - f(a)}{x - a} = f'(a) \leq 0.$ □

7.7.47 Problem. Prove that if f' is constant on $\mathbb{R}$, then f is an affine function.

7.7.47.1 Solution. Suppose $f' = c$(a constant) on $\mathbb{R}$, then by MVT we get

$$f(x) = f(x_0) + f'(y)(x - x_0) = f(x_0) + c(x - x_0).$$

i.e. f is an affine function. □

7.7.48 Problem. Give an example of a set $A \subset \mathbb{R}$ such that $f'(x) = 0, \forall\ x \in A$, but $f(x)$ is not constant.

7.7.48.1 Solution. Consider $A = (-1, 0) \cup (0, 1)$ and $f : A \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} -1, & \text{if } -1 < x < 0 \\ 1, & \text{if } 0 < x < 1. \end{cases}$$

For any point $c \in A, f'(x) = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h} = 0$ but f is not constant. □

7.7.49 Problem. Give an example of a set $A \subset \mathbb{R}$ such that if $a, b \in A$ and $f(a) < 0; f(b) > 0$, but there is no $c \in A$ such that $f(c) = 0$.

7.7.49.1 Solution. Consider the set A as in (7.7.48) or consider $f : \mathbb{Q} \to \mathbb{R}$ defined by $f(x) = x^2 - 2; f(2) > 0, f(1) < 0.$ □

Note: All the pathological behavior of f in (7.7.48) and (7.7.49) can be remedied, if the dom$f$ were **connected**.

7.7.50 Problem.

1. A function can be continuous at a point c, but fails to be continuous at points near c. Justify.
2. If a function is differentiable at c, then it must exist in a nbhd. of c, but the function need not be differentiable or continuous in that nbhd. Justify.

7.7.50.1 Solution. Consider $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^2, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \in \mathbb{Q}^C. \end{cases}$$

Here f is differentiable at $x = 0$, but neither differentiable nor continuous at any other point. □

7.7.51 Problem. Let $f : [a, b] \to \mathbb{R}$. Prove the following statements:

1. If f is continuous and injective, then f is monotone.
2. If f is differentiable and $f'(x) \neq 0$ for all $x \in (a, b)$, then f is injective.
3. If f is differentiable and $f'(a) < 0 < f'(b)$, then there is $c \in (a, b)$ such that $f'(c) = 0$.
4. If f is differentiable and $f'(a) < d < f'(b)$, then there is $c \in (a, b)$ such that $f'(c) = d$.

7.7.51.1 Solution.

1. Suppose f is continuous and injective, yet f is not monotone. Then there exist $x, y, z \in [a, b]$, with $x < y < z$, such that either (a) or (b) below holds:

   (a) $f(x) < f(y)$ and $f(y) > f(z)$

   (b) $f(x) > f(y)$ and $f(y) < f(z)$.

   We will consider only case (a), as (b) is similar. If $f(z) < f(x)$, then by IVT there exists $w \in (y, z)$ such that $f(w) = f(x)$, contradiction to injectivity of f. If $f(z) > f(x)$, then there exists $w \in (x, y)$ such that $f(w) = f(z)$, again contrary to injectivity of f.

2. Assume f is differentiable on (a, b). Suppose f is not injective. Then there is $x < y \in [a, b]$ such that $f(x) = f(y)$. By MVT, there exists $c \in (x, y) \subseteq (a, b)$ such that $f'(c) = [f(y) - f(x)]/(y - x) = 0$.

3. Assume f is differentiable on (a, b), and $f'(x) \neq 0$ for all $x \in [a, b]$. Then by (2), f is injective, so by (1), f is monotone. If f is increasing, then for any $x \neq y \in (a, b)$, $[f(y)-f(x)]/(y-x) \geq 0$, which shows $f'(x) \geq 0$. If f is decreasing, then for any $x \neq y \in (a, b)$, $[f(y)-f(x)]/(y-x) \leq 0$, which shows $f'(x) \leq 0$.

4. Suppose f is differentiable and $f'(a) < d < f'(b)$. Define $g(x) = f(x) - xd$ for $x \in [a, b]$. Then g is differentiable and $g'(a) < 0 < g'(b)$, so by (3), there exists $c \in (a, b)$ such that $g'(c) = 0$. Thus $f'(c) = g'(c) + d = d$. □

7.7.52 Problem. Give an example of a function f such that f' exists on [0,1] but is not continuous on [0,1]. Note that by problem above, such f' will have the intermediate value property, despite being discontinuous.

7.7.52.1 Solution. Define $f : [0, 1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} x^2 \sin(1/x), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

Then $f'(x) = 2x \sin(1/x) - \cos(1/x)$ for $x > 0$ and

$$f'(0) = \lim_{h \to 0} \frac{x^2 \sin(1/x) - 0}{x - 0} = \lim_{h \to 0} x \sin(1/x) = .0$$

Thus f' exists on [0,1] but is not continuous at 0, since f' does not have a limit at 0. □

7.7.53 Problem. Suppose $1 < a < 1 + b$. Prove that the function

$$f(x) = \begin{cases} x^a \sin \dfrac{1}{x^b}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

is differentiable on [0,1], but its derivative is unbounded on [0,1].

7.7.53.1 Solution. Left to the reader.

7.7.54 Problem. If a function f has a finite derivative f' at every point of the finite or infinite interval (a, b) and such that

$$\lim_{x \to a+} f(x) = \lim_{x \to b-} f(x), \tag{7.2}$$

then there exists at least one point $c \in (a, b)$ satisfying $f'(c) = 0$.

7.7.54.1 Solution. First we shall assume that (a, b) is a finite interval and let us denote by C the limits in relation (7.2). Then the function

$$F(x) = \begin{cases} f(x) & \text{if } x \in (a, b) \\ C & \text{if } x \in \{a, b\}, \end{cases}$$

satisfies the conditions of Rolle's theorem, since it is continuous on the closed interval $[a,b]$, has finite first derivative on the open interval (a,b) such that $F(a) = F(b)$. Hence there exists a point $c \in (a,b)$ such that $F(c) = f(c) = 0$. If the interval (a,b) is infinite, but the above limits are finite, i.e.

$$\lim_{x \to a+} f(x) = \lim_{x \to b-} f(x) = C$$

then we consider the lines $y = C + \epsilon, y = C - \epsilon$. At least one of them intersects the curve f in (at least) two points with abscisses a_1, a_2, provided that ϵ is sufficiently small. The function f satisfies the conditions of Rolle's theorem on $[a_1, a_2] \subseteq (a,b)$ and this implies that there exists a point $c \in (a,b)$ such that $f'(c) = 0$. Similarly one can treat the case when the limits in relation (7.2) are ∞ or $-\infty$. $\square$

7.7.55 Problem. Let f be an n times continuously differentiable real-valued function on $[a,b]$, where $a, b \in \mathbb{R},\ a < b$. Suppose that the n-th derivative of f satisfies $f^{(n)}(x) > 0$ for each $x \in [a,b]$. Prove that f has at most n zeros in $[a,b]$.

7.7.55.1 Solution. The above follows from the general fact that if $g : [a,b] \to \mathbb{R}$ is differentiable and has at least k (distinct) zeros, then g' must have at least $k-1$ zeros. This is guaranteed by Rolle's theorem, which says that if $a \leq x < y \leq b$ are such that $g(a) = g(b)$, then there exists $c \in (a,b)$ such that $g'(c) = 0$. The fact that $x < c < y$ guarantees that distinct consecutive pairs of zeros will lead to distinct zeros of the derivative. If we iterate the above procedure, we see that if f has at least k zeros in $[a,b]$ and $k \geq n$, then $f^{(n)}$ has at least $k - n$ zeros. Since $f^{(n)}$ has no zeros on $[a,b]$, we must have that f has at most n zeros on $[a,b]$. $\square$

7.7.56 Problem. Prove or disprove: For each function $f : \mathbb{R} \to \mathbb{R}$ such that f is differentiable at 0, and for each strictly decreasing sequence (a_n) in (0,1) such that $\lim_{n\to\infty} a_n = 0$, and we have

$$\lim_{n \to \infty} \frac{f(a_n) - f(a_{n+1})}{a_n - a_{n+1}} = f'(0).$$

7.7.56.1 Solution. We claim this statement is not true. To see this, consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ for $x \in \mathbb{Q}$ and $f(x) = 0$ for $x \in \mathbb{R} \setminus \mathbb{Q}$. Then we claim f is differentiable at 0. Let $\epsilon > 0$ be given and suppose that $|x| < \epsilon$ and $x \neq 0$. If $x \in \mathbb{Q}$ then

$$\left| \frac{f(x) - f(0)}{x - 0} \right| = |x| < \epsilon,$$

If $x \in \mathbb{R} \setminus \mathbb{Q}$, then

$$\left| \frac{f(x) - f(0)}{x - 0} \right| = 0 < \epsilon,$$

Thus, f is differentiable at 0 with $f'(0) = 0$. Now construct a sequence (a_n) as follows. Let $a_1 = 1$ and let a_2 be some irrational number in the interval $\left(\frac{1}{2}, 1\right)$. Let $a_3 = \frac{1}{2}$ and let a_4 be some irrational number in the interval $\left(\frac{1}{3}, \frac{1}{2}\right)$ Continue in this way so that for each $k \in \mathbb{N}$ and $a_{2k-1} = \frac{1}{k}$ and $a_{2k} \in \left(\frac{1}{k+1}, \frac{1}{k}\right)$ is irrational. Note that (a_n) is a strictly decreasing sequence of positive numbers with limit 0. Now observe the following for some $k \in \mathbb{N}$ for each is irrational. Note that (an) is a

strictly decreasing sequence of positive numbers with limit 0. Now observe the following for some $k \in \mathbb{N}$:

$$\frac{f(a_{2k-1}) - f(a_{2k})}{a_{2k-1} - a_{2k}} = \frac{\frac{1}{k^2} - 0}{\frac{1}{k} - a_{2k}} > \frac{\frac{1}{k^2}}{\frac{1}{k} - \frac{1}{k+1}} = \frac{k+1}{k} > 1.$$

From this, it follows that we cannot have $\lim_{n\to\infty} \frac{f(a_n)-f(a_{n+1})}{a_n - a_{n+1}} = 0 = f'(0)$ as the sequence $\left(\frac{f(a_n)-f(a_{n+1})}{a_n-a_{n+1}}\right)$ contains a subsequence that is always greater than 1. □

7.7.57 Problem. The function $\phi : [a, b] \to \mathbb{R}$ defined by

$$\phi(x) = \begin{vmatrix} f_1(x) & g_1(x) & h_1(x) \\ f_2(x) & g_2(x) & h_2(x) \\ f_3(x) & g_3(x) & h_3(x) \end{vmatrix},$$

where each function f_i, g_i, h_i has a derivative at $x_0 \in (a, b)$. Prove that ϕ has a derivative at x_0 is equal to the sum of the three determinants obtained by taking derivatives of each row (column) at a time. (A similar results holds for an $n \times n$ determinant.)

7.7.57.1 Solution. Calculate $\phi(x+t) - \phi(x)$ using the rules of determinants, then taking the limit as $t \to 0$ get the required result.

7.7.58 Problem. Let $f, g : \mathbb{R} \to \mathbb{R}$ such that g' exists, and $f(x) = x + \lambda$ such that $f \circ g = g \circ f$. Prove that g' is periodic.

7.7.58.1 Solution. By the problem,

$$\begin{aligned} &(f \circ g)(x) = (g \circ f)(x) \\ \Rightarrow &g(x) + \lambda = g(x + \lambda) \\ \Rightarrow &g'(x) = g'(x + \lambda) \end{aligned}$$

shows that g' is periodic. □

7.7.59 Problem. Give an example of a pair of functions f, g with the properties that $g(t_0) = x_0$, that f has no derivative at x_0 and g has no derivative at t_0, and that $f \circ g$ has a derivative at t_0.

7.7.59.1 Solution. Take $f(x) = \begin{cases} t_0 + (x - x_0) & \text{if } x \geq x_0 \\ t_0 + 2(x - x_0) & \text{if } x < x_0 \end{cases}$ and take $g = f^{-1}$. □

7.7.60 Problem. Let $f : [a, b] \to \mathbb{R}$ and f' exists in (a, b) and $f(a) = f(b) = 0$ prove that for every real λ, $\exists\, \xi \in (a, b)$ such that $f'(\xi) = \lambda f(\xi)$.

7.7.60.1 Solution. Consider $x \mapsto e^{-\lambda x} f(x)$.

7.7.61 Problem. Let $f : [a, b] \to \mathbb{R}$ a continuous function, and twice differentiable on (a, b). If $f(a) = f(b) = 0$ and $a < c < b$, prove that there exists $\xi \in (a, b)$ such that

$$f(c) = \frac{1}{2}(c - a)(c - b)f''(\xi).$$

7.7.61.1 Solution. Consider the function $\phi : [a,b] \to \mathbb{R}$ defined by

$$\phi(x) = \frac{(x-a)(x-b)}{(c-a)(c-b)} f(c) - f(x).$$

We see that $\phi(a) = \phi(c) = \phi(b) = 0$. Now applying Rolle's theorem on $[a,c]$ and on $[c,b]$, we get $\exists\, \xi_1 \in []$, and $\xi_2 \in [c,b]$ such that $\phi'(\xi_1) = 0$ and $\phi'(\xi_2) = 0$. Again by Rolle's theorem on $[\xi_1, \xi_2]$ we get $\exists\, \xi \in (\xi_1, \xi_2)$ such that $\phi''(\xi) = 0$. Hence

$$\begin{aligned} \frac{2f(c)}{(c-a)(c-b)} - f''(\xi) &= 0 \\ \Rightarrow f(c) &= \frac{1}{2}(c-a)(c-b)f''(\xi). \quad \square \end{aligned}$$

7.7.62 Problem. A function f is said to satisfy a Lipschitz condition of order α at c if there exists a positive number M (which may depend on c) and a r-ball $B(c;r)$ such that

$$|(f(x) - f(c)| < M|x-c|^{\alpha}$$

whenever $x \in B(c;r), x \neq c$.

1. Show that a function which satisfies a Lipschitz condition of order α is continuous at c if $a > 0$, and has a derivative at c if $a > 1$.
2. Give an example of a function satisfying a Lipschitz condition of order 1 at c for which $f'(c)$ does not exist.

7.7.62.1 Solution.

1. As $\alpha > 0$, given $\epsilon > 0\ \exists\, \delta \leq (\epsilon/M)^{\frac{1}{\alpha}}$ such that as $x \in (c-\delta, c+\delta) \subseteq B(c;r)$ we have

   $$|f(x) - f(c)| < M|x-c|^{\alpha} \leq M\delta^{\alpha} \leq \epsilon.$$

   So, f is continuous at c.
   Again, as $\alpha > 0$, for $x \in B(c;r)$ and $x \neq c$ we have

   $$\left|\frac{f(x)-f(c)}{x-c}\right| < M|x-c|^{\alpha-1} \to 0 \text{ as } x \to 0$$

   shows that $f'(c) = 0$.
2. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|$ we see that

   $$||x| - |c|| \leq |x-c|,$$

   shows that $|x|$ is a function satisfying a Lipschitz condition of order 1 at 0 for which $f'(0)$ does not exist.

7.7.63 Problem. Given a function f defined and having a finite derivative in (a,b) and such that $\lim_{x\to b-} f(x) = +\infty$. Prove that $\lim_{x\to b-} f'(x)$ either fails to exist or is infinite.

7.7.63.1 Solution. Suppose that $\lim_{x\to b-} f'(x)$ exists and equal to l. So given $\epsilon > 0 \; \exists \; \delta > 0$ such that as $x \in (b-\delta, b)$, we have

$$|f'(x)| \le |l| + \epsilon$$

and for $c, x \in (b-\delta, b)$ and MVT we get

$$|f(x) - f(c)| = |f'(\xi)(x-c)| \le (|l| + \epsilon)\,|x-c|$$

where $\xi \in (x, c)$ which implies that

$$|f(x)| \le (|l| + \epsilon)\,\delta$$

which contradicts to $\lim_{x\to b-} f(x) = +\infty$ Hence, $\lim_{x\to b-} f'(x)$ either fails to exist or is infinite. □

7.7.64 Problem. Give an example of a pair of functions f and g having a finite derivatives in (0,1), such that

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = 0$$

0, but such that $\lim_{x\to 0} \frac{f'(x)}{g'(x)}$ does not exist, choosing g so that $g'(x)$ is never zero.

7.7.64.1 Solution. Let $f(x) = \sin(1/x)$ and $g(x) = 1/x$. Then it is trivial for that $g'(x)$ is never zero. In addition, we have $\lim_{x\to 0} \frac{f(x)}{g(x)} = 0$, and $\lim_{x\to 0} \frac{f'(x)}{g'(x)}$ does not exist. [In this exercise, it tells us that the converse of L'Hôspital Rule is NOT necessary true.] □

7.7.65 Problem. Suppose that g is real-valued function defined on $\mathbb{R}$, with bounded derivative, say $|g| \le M$. Fix $\epsilon > 0$, and define $f(x) = x + \epsilon g(x)$. Show that f is 1-1 if ϵ is small enough. (It implies that f is strictly monotonic.)

7.7.65.1 Solution. Suppose that $f(x) = f(y)$ i.e. $x + \epsilon g(x) = y + \epsilon g(y)$ which implies that $|x-y| = \epsilon|g(x) - g(y)| \le \epsilon M|x-y|$ by Mean Value Theorem, and hypothesis. So, as ϵ is small enough, we have $x = y$. That is, f is 1-1. □

7.7.66 Problem. Let f be continuous on (a, b) with a finite derivative f' everywhere in (a, b), except possibly at c. If $\lim_{x\to c} f'(x)$ exists and has the value A, show that $f'(c)$ must also exist and have the value A.

7.7.66.1 Solution. Consider, for $x \ne c$, by Mean Value Theorem, $\dfrac{f(x) - f(c)}{x - c} = f'(\xi)$ where $\xi \in (x, c)$ or (c, x), Since $\lim_{x\to c} f'(x)$ exists, given $\epsilon > 0$, there is a $\delta > 0$ such that for all x satisfying $0 < |x - c| < \delta$ we have $A - \epsilon < f'(x) < A + \epsilon$. And, we then have

$$A - \epsilon < \frac{f(x) - f(c)}{x - c} = f'(\xi) < A + \epsilon$$

That is, $f'(c)$ exists and equals A. □

7.7.67 Problem. Let f be continuous on [0,1], $f(0) = 0$ and $f'(x)$ exist in (0,1). Prove that if f' is an increasing function on (0,1), then so is too is the function g defined by the equation $g(x) = f(x)/x$.

7.7.67.1 Solution. Since f is an increasing function on (0,1), so for any $x \in (0,1)$ we have that

$$f'(x) - \frac{f(x)}{x} = f'(x) - \frac{f(x) - f(0)}{x - 0} = f'(x) - f'(\xi) \geq 0,\ \xi \in (0, x),$$

So, let $x > y$, we have

$$\begin{aligned} g(x) - g(y) &= g'(\eta)(x - y), \eta \in (y, x) \\ &= \frac{\eta f'(\eta) - f(\eta)}{\eta^2}(x - y) \geq 0 \end{aligned}$$

which implies that g is an increasing function on (0,1). □

7.7.68 Problem. Suppose that f is defined on (a, b) and has a derivative at $c \in (a, b)$. If $\{x_n\} \subseteq (a, c)$ and $\{y_n\} \subseteq (c, b)$ such that $(x_n - y_n) \to 0$ as $n \to \infty$. Then we have

$$f'(c) = \lim_{n\to\infty} \frac{f(x_n) - f(y_n)}{x_n - y_n}.$$

7.7.68.1 Solution. Left to the reader.

7.7.69 Problem. Prove the following:

1. $|\arctan x - \arctan y| \leq |x - y|; x, y \in \mathbb{R}$,
2. $\dfrac{x - y}{x} < \ln \dfrac{x}{y} < \dfrac{x - y}{y}; 0 < y < x, x, y \in \mathbb{R}$,
3. $n(b - a)a^{n-1} < b^n - a^n < n(b - a)b^{n-1}, 0 < a < b, n \in \mathbb{N}$;
4. $\dfrac{\alpha}{n^{\alpha+1}} < \left(\dfrac{1}{(n-1)^\alpha} - \dfrac{1}{n^\alpha}\right), n \in \mathbb{N}, \alpha > 0.$

7.7.69.1 Solution.

1. Use the function $f(t) = \arctan t$ in $[x, y]$.
2. Use $f(t) = \ln t$ in $[x, y]$.
3. Use $f(x) = x^n$ in the interval $[a, b]$.
4. Use $f(x) = 1/x^\alpha$ in the interval $[n - 1, n]$.

7.7.70 Problem. Let $g : [0, 1] \to \mathbb{R}$ be twice-differentiable (i.e., both g and g' are differentiable functions) with $g'' > 0$ for all $x \in [0, 1]$. If $g(0) > 0$ and $g(1) = 1$, show that $g(d) = d$ for some point $d \in (0, 1)$ if and only if $g'(1) > 1$.

7.7.70.1 Solution. ⇒ We first show that if $g(d) = d$ for some $d \in (0, 1)$, then $g'(1) > 1$. Assume there exists a d where $g(d) = d$. Applying the Mean Value Theorem to g on $[d, 1]$, we see that

$$g'(c) = \frac{g(1) - g(d)}{1 - d} = \frac{1 - d}{1 - d} = 1$$

for some $c \in (d, 1)$. Consider $g'(x)$ on $[c, 1]$. Since $g''(x) > 0$ on [0,1], g' is increasing. So we must have $g'(1) - g'(c) \geq 1$. If $g'(1) = 1$, then g' would be constant on $[c, 1]$, which would imply that $g'' = 0$, which is not true. Thus $g'(1) > 1$ as desired.
$\Leftarrow$ Now, we show that if $g'(1) > 1$, then $g(d) = d$ at some $d \in (0, 1)$. Let $f(x) = g(x) - x$. Since $g(0) > 0, f(0) = g(0) > 0$. Since $g'(1) > 1$, let $\epsilon = g0(1) - 1 > 0$. Then, as

$$g'(1) = \lim_{x \to 1} \frac{g(x) - g(1)}{x - 1} = \lim_{x \to 1-} \frac{g(1) - g(x)}{1 - x} = \lim_{x \to 1-} \frac{1 - g(x)}{1 - x}$$

there exists $\delta > 0$ such that $1 - \delta < x < 1$ implies $\left| \frac{g(1) - g(x)}{1 - x} \right| < \epsilon$. Thus

$$-\epsilon < \frac{g(1) - g(x)}{1 - x} < \epsilon$$

for $1 - \delta < x < 1$, so $g'(1) - \epsilon = g'(1) - (g'(1) - 1) = 1 < \frac{1 - g(x)}{1 - x}$ which implies that $1 - x < 1 - g(x)$ (because $x < 1$), so that $g(x) < x$ whenever $1 - \delta < x < 1$. Therefore, $f(x) = g(x) - x < 0$ for $1 - \delta < x < 1$, which implies that $f(d) = 0$ for some $d \in (0, 1)$ by the Intermediate Value Theorem. Hence, for this $d \in (0, 1)$, we have $g(d) = d$ as required. □

7.7.71 Problem. Let a, b, c be positive real numbers such that $abc = 1$. Prove that $a^2 + b^2 + c^2 \leq a^3 + b^3 + c^3$.

7.7.71.1 Solution. We prove that the function $f(t) = a^t + b^t + c^t$ is increasing for $t \geq 0$. Its first derivative is $f'(t) = a^t \ln a + b^t \ln b + c^t \ln c$, for which we can tell only that $f'(0) = \ln abc = \ln 1 = 0$. However, the second derivative is $f''(t) = a^t \ln^2 a + b^t \ln^2 b + c^t \ln^2 c$, which is clearly positive. We thus deduce that f' is increasing, and so $f'(t) \geq f'(0) = 0$ for $t \geq 0$. Therefore, f itself is increasing for $t \geq 0$, and the conclusion follows.

7.7.72 Problem. Prove that

$$A = \begin{vmatrix} 1 + a_1 & 1 & 1 & . & 1 \\ 1 & 1 + a_2 & 1 & . & 1 \\ 1 & 1 & 1 + a_3 & . & 1 \\ . & . & . & . & . \\ 1 & 1 & 1 & . & 1 + a_n \end{vmatrix} = a_1 a_2 .. a_n \left(1 + \frac{1}{a_1} + \frac{1}{a_2} + .. + \frac{1}{a_n} \right).$$

7.7.72.1 Solution. Consider the function

$$f(x) = \begin{vmatrix} x + a_1 & x & x & . & x \\ x & x + a_2 & x & . & x \\ x & x & x + a_3 & . & x \\ . & . & . & . & . \\ x & x & x & . & x + a_n \end{vmatrix},$$

then its derivative is equal to

$$f'(x) = \begin{vmatrix} 1 & 1 & 1 & . & 1 \\ x & x+a_2 & x & . & x \\ x & x & x+a_3 & . & x \\ . & . & . & . & . \\ x & x & x & . & x+a_n \end{vmatrix} + \begin{vmatrix} x+a_1 & x & x & . & x \\ 1 & 1 & 1 & . & 1 \\ x & x & x+a_3 & . & x \\ . & . & . & . & . \\ x & x & x & . & x+a_n \end{vmatrix}$$

$$+ \begin{vmatrix} x+a_1 & x & x & . & x \\ x & x+a_2 & x & . & x \\ 1 & 1 & 1 & . & 1 \\ . & . & . & . & . \\ x & x & x & . & x+a_n \end{vmatrix} + ... + \begin{vmatrix} x+a_1 & x & x & . & x \\ x & x+a_2 & x & . & x \\ x & x & x+a_3 & . & x \\ . & . & . & . & . \\ 1 & 1 & 1 & . & 1 \end{vmatrix}.$$

Proceeding one step further, we see that the second derivative of f consists of two types of determinants: some that have a row of 0's, and others that have two rows of 1's. In both cases the determinants are equal to zero, showing that $f''(x) = 0$. It follows that f must be a linear function, i.e. $f(x) = a + bx$. From where we get $a = f(0)$ and $b = f'(0)$ hence

$$f(x) = f(0) + xf'(0).$$

Again, we see that $f(0) = a_1 a_2 ... a_n$, and

$$f'(0) = \begin{vmatrix} 1 & 1 & 1 & . & 1 \\ 0 & a_2 & 0 & . & 0 \\ 0 & 0 & a_3 & . & 0 \\ . & . & . & . & . \\ 0 & 0 & 0 & . & a_n \end{vmatrix} + \begin{vmatrix} a_1 & 0 & 0 & . & 0 \\ 1 & 1 & 1 & . & 1 \\ 0 & 0 & a_3 & . & 0 \\ . & . & . & . & . \\ 0 & 0 & 0 & . & a_n \end{vmatrix}$$

$$+ \begin{vmatrix} a_1 & 0 & 0 & . & 0 \\ 0 & a_2 & 0 & . & 0 \\ 1 & 1 & 1 & . & 1 \\ . & . & . & . & . \\ 0 & 0 & 0 & . & a_n \end{vmatrix} + ... + \begin{vmatrix} a_1 & 0 & 0 & . & 0 \\ 0 & a_2 & 0 & . & 0 \\ 0 & 0 & a_3 & . & 0 \\ . & . & . & . & . \\ 1 & 1 & 1 & . & 1 \end{vmatrix}$$

expanding each determinant along the row of 1's we get

$$\begin{aligned} f'(0) &= a_2 a_3 ... a_n + a_1 a_3 ... a_n + ... + a_1 a_2 ... a_{n-1} \\ &= a_1 a_2 a_3 .. a_n \left[\frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n} \right] \\ \text{Hence, } f(x) &= a_1 a_2 ... a_n \left[1 + \left(\frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n} \right) x \right] \end{aligned}$$

Now, substituting $x = 1$, we get $f(1) = a_1 a_2 ... a_n \left[1 + \left(\frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n} \right) \right]$. Thus the result follows.

7.7.73 Problem. How many real solutions does the equation

$$f(x) = \sin(\sin(\sin(\sin(\sin x)))) = \frac{x}{3}$$

have?

7.7.73.1 Solution. Let $u = \sin x,\ v = \sin u,\ w = \sin v,\ t = \sin w$. Then $f(x) = \sin t$. The first obvious solution is $x = 0$. Now, $f'(x) = \cos t \cos w . \cos v . \cos u \cos x \Rightarrow f'(0) = 1 \geq 1/3$. Therefore, $f(x) > x/3$ in some neighborhood of 0. On the other hand, $f(x) < 1$, whereas $x/3$ is not bounded as $x \to \infty$. Therefore, $f(x_0) = x_0/3$ for some $x_0 > 0$. Because f is odd, $-x_0$ is also a solution. The second derivative of f is ......which is clearly nonpositive for $0 \leq x \leq 1$. This means that $f'(x)$ is monotonic. Therefore, $f'(x)$ has at most one root x in $[0, \infty)$. Then $f(x)$ is monotonic in $[0, x']$ and $[x', \infty)$ and has at most two nonnegative roots. Because $f(x)$ is an odd function, it also has at most two nonpositive roots. Therefore, $-x_0, 0, x_0$ are the only solutions. □

7.7.74 Problem. Two functions of x are differentiable and not identically equal to zero. Find an example of two such functions having the property that the derivative of their quotient is the quotient of their derivatives.(Indiana College Mathematics Competition' 66)

7.7.74.1 Solution. Hint: By the condition, we get,

$$\begin{aligned} \left(\frac{f(x)}{g(x)}\right)' &= \frac{f'(x)}{g'(x)} \\ \Rightarrow & g(x)g'(x)f'(x) - (g'(x))^2 f(x) = (g(x))^2 f'(x). \end{aligned}$$

If we know one of the functions, say $g(x)$, then we have a first order linear differential equation for the other function, that we can solve. Choosing $g(x) = x$ so $g'(x) = 1$, we find that $f(x)$ must satisfy

$$xf'(x) - f(x) = x^2 f'(x).$$

then solving the differential equation to get $f(x) = C\dfrac{x}{x-1}$ and $g(x) = x$. □

7.7.75 Problem. Let $f : [0, 1] \to \mathbb{R}$ be continuous. Suppose f is twice-differentiable on the open interval (0,1) and M is a real constant such that for each $x \in (0, 1)$, we have $f''(x) \leq M$. Let $a \in (0, 1)$. Prove that

$$|f'(a)| \leq |f(1) - f(0)| + \frac{M}{2}.$$

7.7.75.1 Solution. By Taylor's theorem, we can expand $f(x)$ at $x = a$. In particular, we have

$$\begin{aligned} f(0) &= f(a) - af'(a) + a^2 \frac{f''(\xi)}{2} \\ f(1) &= f(a) + (1-a)f'(a) + (1-a)^2 \frac{f''(\eta)}{2} \end{aligned}$$

for some $0 < \xi < a$ and $a < \eta < 1$. Subtracting and rearranging, we get

$$f'(a) = f(0) - f(1) + a^2 \frac{f''(\xi)}{2} - (1-a)^2 \frac{f''(\eta)}{2}.$$

Note that the quadratic function $g(x) = 2x^2 - 2x + 1$ has axis at $x = 1/2$ and has absolute maximum on [0,1] at x=0 and 1, which is 1. Since $f''(x) \leq M$ for all $x \in (0, 1)$, by triangle inequality

$$|f'(a)| \leq |f(0) - f(1)| + |a^2 + (1-a)^2| \frac{M}{2} \leq |f(0) - f(1)| + \frac{M}{2}.$$

we obtain as desired. □

7.7.76 Problem. Let f be a real-valued continuous function on [0,1] which is twice continuously differentiable on (0,1). Suppose that $f(0) = f(1) = 0$ and f satisfies the following equation

$$x^2 f''(x) + x^4 f'(x) - f(x) = 0.$$

1. If f attains its maximum at some point x_0 in the open interval (0,1), then prove that $M = 0$.

2. Prove that f is identically zero on [0,1].

7.7.76.1 Solution.

1. Since f attains its maximum M at x_0 so $M = f(x_0) \geq f(0) = f(1) = 0 \Rightarrow M \geq 0$. Let $M > 0$. Again, since f maximum at x_0 so $f'(x_0) = 0$ and $f''(x_0) < 0$, and hence $x_0^2 f''(x_0) + x_0^4 f'(x_0) - f(x_0) = 0$ implies $x_0^2 f''(x_0) - M = 0$ i.e. $M = x_0^2 f''(x_0) < 0$, a contradiction. Thus $M = 0$.

2. Left to the reader. □

7.7.77 Problem. Show that $|\log x - \log y| < |x - y| \ \forall\ x, y \in [1, \infty)$. Use this inequality to prove that log x is uniformly continuous on $[1, \infty)$. Also show that $\log x$ is not uniformly continuous on $(0, 1]$.

7.7.77.1 Solution. Let$f(x) = \log x$ and assume that $x \neq y$. The MVT implies $\exists\ \xi \in (x, y)$ such that $\frac{f(x)-f(y)}{x-y} = f'(\xi) = \frac{1}{\xi}$. When $\xi \geq 1, 0 \leq \frac{1}{\xi} \leq 1$. So $|f(x) - f(y)| = \frac{1}{\xi}|x - y| \leq |x - y| \ \forall\ x, y \in [1, \infty)$, with $x \neq y$. And this inequality holds trivially when $x = y$. Hence $|\log x - \log y| < |x - y| \ \forall\ x, y \in [1, \infty)$.
Let $\epsilon > 0$ then let $\delta = \epsilon$. So when $x, y \in [1, \infty)$ and $|x-y| < \delta = \epsilon$, we have $|\log x - \log y| < |x-y| < \epsilon$. Hence $\log x$ is not uniformly continuous on $[1, \infty)$.
Consider the Cauchy sequence $(\frac{1}{n})$ in $(0, 1]$ but $f(\frac{1}{n}) = -\log n$ is not Cauchy, hence $\log x$ is not uniformly continuous on $(0, 1]$. □

7.7.78 Problem. Let a, b be two positive numbers, and let $f : [a, b] \to \mathbb{R}$ be a continuous function, differentiable on (a, b). Prove that there exists $c \in (a, b)$ such that

$$\frac{1}{a-b}(af(b) - bf(a)) = f(c) - cf'(c).$$

7.7.78.1 Solution. If we apply Cauchy's MVT to the functions $f(x)/x$ and $1/x$, we conclude that $\exists\ c \in (a, b)$ with

$$\left(\frac{f(b)}{b} - \frac{f(a)}{a}\right)\left(-\frac{1}{c^2}\right) = \left(\frac{1}{b} - \frac{1}{a}\right)\left(\frac{cf'(c) - f(c)}{c^2}\right).$$

Hence

$$\frac{1}{a-b}(af(b) - bf(a)) = f(c) - cf'(c). \quad \square$$

7.7.79 Problem. Let $f, g : [a, b] \to \mathbb{R}$ be two continuous functions, differentiable on (a, b). Assume in addition that g and g' are nowhere zero on (a, b) and that $f(a)/g(a) = f(b)/g(b)$. Prove that there exists $c \in (a, b)$ such that

$$f(c)/g(c) = f'(c)/g'(c).$$

7.7.79.1 Solution. The function $h = f/g$ satisfies the conditions in the hypothesis of Rolle's theorem; hence there exists $c \in (a, b)$ with $h'(c) = 0$. Since

$$h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2},$$

we have $f'(c)g(c) - f(c)g'(c) = 0$; hence $f(c)/g(c) = f'(c)/g'(c)$. $\square$

7.7.80 Problem. (Gazeta Matematică (Mathematics Gazette, Bucharest), 1975; proposed by D. Andrica) Let $f : [a, b] \to \mathbb{R}$ be a continuous function, differentiable on (a, b). Assume in addition that g' are nowhere zero on (a, b). Prove that there exists $\theta \in (a, b)$ such that

$$\frac{f'(\theta)}{f(\theta)} = \frac{1}{a-\theta} + \frac{1}{b-\theta}.$$

7.7.80.1 Solution. Consider the function $g : [a, b] \to \mathbb{R}$ defined by $g(x) = (x - a)(x - b)f(x)$. We see that g is continuous on $[a, b]$, differentiable on (a, b), and that $g(a) = g(b) = 0$. Applying Rolle's theorem, we obtain a $\theta \in (a, b)$ with $g'(\theta) = 0$. But $g'(x) = (x - b)f(x) + (x - a)f(x) + (x - a)(x - b)f'(x)$. We have thus obtained a θ with $(\theta - b)f(\theta) + (\theta - a)f(\theta) + (\theta - a)(\theta - b)f'(\theta) = 0$. Dividing by $(\theta - a)(\theta - b)f(\theta)$ yields

$$\frac{f'(\theta)}{f(\theta)} = \frac{1}{a-\theta} + \frac{1}{b-\theta}. \quad \square$$

7.7.81 Problem. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Assume that $\lim_{x\to\infty} f(x) = a$ for some real number a and that $\lim_{x\to\infty} xf(x)$ exists. Evaluate this limit.

7.7.81.1 Solution. Let $n < N$ be two natural numbers. By applying the mean value theorem to the function $g(x) = xf(x)$ on the interval $[n, N]$, we deduce that there exists $c_n \in (n, N)$ such that $g'(c_n) = \dfrac{Nf(N) - nf(n)}{N - n}$. Since

$$\lim_{N\to\infty} \frac{Nf(N) - nf(n)}{N - n} = a$$

for sufficiently large n, $|g(c_n)-a| < 1/n$. Clearly, $\lim_{n\to\infty} c_n = \infty$. Also, since $g'(x) = f(x)+xf'(x)$, it follows that $g(x)$ has a limit at infinity. It follows that $\lim_{x\to\infty} g(x) = \lim_{n\to\infty} g(c_n) = a$; hence $\lim_{x\to\infty} xf(x) = \lim_{x\to\infty} g'(x) - \lim_{x\to\infty} f(x) = a - a = 0$. $\square$

7.7.82 Problem. (Romanian Mathematical Olympiad, 1999; proposed by M. Piticari and S. Rădulescu) The function $f : [0, 1] \to \mathbb{R}$ is differentiable and

$$f(x) = f\left(\frac{x}{2}\right) + \frac{x}{2}f'(x)$$

for every real number x. Prove that f is a linear function, that is, $f(x) = ax + b$ for some $a, b \in \mathbb{R}$.

7.7.82.1 Solution. To show that f is linear is to show that its derivative is constant. We will prove that the derivative at any point is equal to the derivative at zero. Fix x, which we assume to be positive, the case of x negative being completely analogous. Define the set

$$M = \{t; t \geq 0 \text{ and } f'(t) = f'(x)\}.$$

Clearly, M is bounded below, so let t_0 be the infimum of M. The relation in the statement implies that

$$\begin{aligned}\lim_{x\to 0} f'(x) &= \lim_{x\to 0} \frac{f(x)-f(x/2)}{x/2} = 2\lim_{x\to 0}\frac{f(x)-f(0)}{x} \\ &= \lim_{x\to 0}\frac{f(x/2)-f(0)}{x/2} = 2f'(0)-f'(0) = f'(0),\end{aligned}$$

which shows that f' is continuous at 0. It obviously is continuous everywhere else, being the ratio of two continuous functions. This implies that M is closed, and hence t_0 is in M. We want to prove that $t_0 = 0$. Suppose, by way of contradiction, that $t_0 > 0$. Since $f(t_0) - f(t_0/2)t_0/2 = f(t_0)$, the mean value theorem applied on the interval $[t_0/2, t_0]$ proves the existence of a $c \in (t_0/2, t_0)$ such that $f'(c) = f'(t_0) = f'(x)$, which contradicts the minimality of t_0. This shows that for all $x, f'(x) = f'(0) =$ constant, and we are done. □

7.7.83 Problem. Let $f : [a, b] \to \mathbb{R}$ a continuous positive function, differentiable on (a, b). Prove that there exists $c \in (a, b)$ such that

$$\frac{f(b)}{f(a)} = e^{(b-a)\frac{f'(c)}{f(c)}}.$$

7.7.83.1 Solution. Since f is positive, the function $\ln f(x)$ is well-defined and satisfies the hypothesis of the mean value theorem. Hence $\exists\, c \in (a, b)$ with

$$\frac{\ln f(b) - \ln f(a)}{b-a} = \frac{f'(c)}{f(c)}.$$

and the conclusion follows.

7.7.84 Problem. Suppose a and b are real numbers, $b > 0$, and f is defined on $[-1, 1]$ by

$$f(x) = \begin{cases} x^a \sin\left(\dfrac{1}{|x|^b}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Prove the following statements:

1. f is continuous if and only if $a > 0$.
2. f' exists if and only if $a > 1$.
3. f' is bounded if and only if $a \geq 1 + b$.
4. f' is continuous if and only if $a > 1 + b$.
5. f'' exists if and only if $a > 2 + b$.
6. f'' is bounded if and only if $a \geq 2 + 2b$.
7. f'' is continuous if and only if $a > 2 + 2b$.

7.7.84.1 Solution.

1. Suppose that $a > 0$. We have $f(0) = 0$ by definition. For f to be continuous at $x = 0$ it must be the case that $\lim_{x\to 0} f(x) = 0$. So if $a > 0$ we have $x^a \sin(x^{-b}) \to 0$ as $x \to 0$. which shows that $\lim_{x\to 0} f(x) = 0$, and therefore f is continuous at $x = 0$.

   Now let $a = 0$. Then, we see that a sequence (x_n) defined by $x_n = \left(\dfrac{1}{2n\pi + \pi/2}\right)^{\frac{1}{b}}$ converges to 0, but $f(x_n) = \sin x_n \to 1 \neq f(0)$. Thus f is not continuous at 0.

   Again, let $a < 0$, and the sequence (x_n) defined by $x_n = \left(\dfrac{1}{2n\pi + \pi/2}\right)^{\frac{1}{b}}$ converges to 0, but

$$f(x_n) = x_n^a \sin(x_n^{-b}) = \left(\frac{1}{2n\pi + \pi/2}\right)^{\frac{a}{b}} \sin(2n\pi + \pi/2) = \left(\frac{1}{2n\pi + \pi/2}\right)^{\frac{a}{b}}.$$

   Since, $b > 0$ so $a/b < 0$ and we have

$$f(x_n) = (2n\pi + \pi/2)^{-\frac{a}{b}}$$

   and $f(x_n) \to \infty$ therefore f is not continuous at $x = 0$. These cases show that f is continous iff $a > 0$.

2. From the definition of limit we have

$$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{h^a \sin(h^{-b})}{h} = \lim_{h\to 0} h^{a-1} \sin(h^{-b}) \tag{7.3}$$

   Now, we have for $a > 1$, $|h^{a-1} \sin(h^{-b})| \to 0$ as $h \to 0$.
   When $a = 1$, the limit $\lim_{h\to 0} h^{a-1} \sin(h^{-b}) = \lim_{h\to 0} \sin(h^{-b})$ doesnot exist.
   Again, if $a < 0$, then consider the sequences $(x_n), (y_n)$ defined by

$$x_n = \left(\frac{1}{2n\pi + \pi/2}\right)^{\frac{1}{b}},\ y_n = \left(\frac{1}{(2n+1)\pi + \pi/2}\right)^{\frac{1}{b}}$$

   When $a < 1$ we have $a - 1 < 0$ and therefore equation (6.2) gives us

$$\lim_{h\to 0} h^{a-1} \sin(h^{-b}) = \lim_{n\to\infty} x_n^{a-1} = \infty$$
$$\lim_{h\to 0} h^{a-1} \sin(h^{-b}) = \lim_{n\to\infty} -y_n^{a-1} = -\infty$$

   This means that the limit in (7.3) does not exist. These cases show that exists $f'(0)$ iff $a > 1$.

3. Thus if $a \geq 1 + b > 1$ implies

$$f'(x) = \begin{cases} ax^{a-1} \sin(x^{-b}) - bx^{a-b-1} \cos(x^{-b}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

   Thus f' is unbounded if $a < 1$ and $1 \leq a < 1 + b$. Consider a sequence (x_n) defined by $x_n = (2n\pi)^{-\frac{1}{b}}$ and

$$f'(x_n) = -b(2n\pi)^{\frac{1+b-a}{b}} \to -\infty \text{ as } n \to \infty$$

   f' is bounded when $a \geq 1 + b$ can be proved easily. These three cases show that f' is bounded iff $a \geq 1 + b$.

4. Left to the reader.
5. Left to the reader.
6. Left to the reader.
7. Left to the reader.

□

7.7.85 Problem. Suppose f is defined in $(-1,1)$ and $f'(0)$ exists. Suppose $-1 < \alpha_n < \beta_n < 1, \alpha_n \to 0$, and $\beta_n \to 0$ as $n \to \infty$. Define the difference quotients

$$D_n = \frac{f(\beta_n) - f(\alpha_n)}{\beta_n - \alpha_n}.$$

Prove the following statements:

1. If $\alpha_n < 0 < \beta_n$, then $\lim_{n\to\infty} D_n = f'(0)$.
2. If $0 < \alpha_n < \beta_n$, and $\left(\dfrac{\beta_n}{\beta_n - \alpha_n}\right)$ is bounded, then $\lim_{n\to\infty} D_n = f'(0)$.
3. If f' is continuous in $(-1,1)$, then $\lim_{n\to\infty} D_n = f'(0)$.
   Give an example in which f is differentiable in $(-1,1)$ (but f' is not continuous at 0) and in which α_n, β_n tend to 0 in such a way that $\lim_{n\to\infty} D_n$ exists but is different from $f'(0)$.

7.7.85.1 Solution.

1. We can write D_n as

$$D_n = \left[\frac{f(\beta_n) - f(0)}{\beta_n - 0}\frac{\beta_n}{\beta_n - \alpha_n}\right] + \left[\frac{f(0) - f(\alpha_n)}{0 - \alpha_n}\frac{-\alpha_n}{\beta_n - \alpha_n}\right]$$

Note that

$$f'(0) = \lim_{n\to\infty}\frac{f(\alpha_n) - f(0)}{\alpha_n} = \lim_{n\to\infty}\frac{f(\beta_n) - f(0)}{\beta_n}.$$

Thus for any $\epsilon > 0$, there exists N such that

$$L - \epsilon < \frac{f(\alpha_n) - f(0)}{\alpha_n} < L + \epsilon$$
$$L - \epsilon < \frac{f(\beta_n) - f(0)}{\beta_n} < L + \epsilon$$

whenever $n > N$ and $L = f'(0)$. Note that $\frac{\beta_n}{\beta_n - \alpha_n}$ and $\frac{-\alpha_n}{\beta_n - \alpha_n}$ are positive. Hence,

$$\frac{\beta_n}{\beta_n - \alpha_n}(L - \epsilon) < \frac{f(\beta_n) - f(0)}{\beta_n}\frac{\beta_n}{\beta_n - \alpha_n} < \frac{\beta_n}{\beta_n - \alpha_n}(L + \epsilon)$$
$$\frac{-\alpha_n}{\beta_n - \alpha_n}(L - \epsilon) < \frac{f(\alpha_n) - f(0)}{\alpha_n}\frac{-\alpha_n}{\beta_n - \alpha_n} < \frac{-\alpha_n}{\beta_n - \alpha_n}(L + \epsilon).$$

Combine two inequations,

$$L - \epsilon < D_n < L + \epsilon.$$

Hence, $\lim_{n\to\infty} D_n = f'(0)$.

2. We proceed as above proof, but note that $\frac{-\alpha_n}{\beta_n-\alpha_n} < 0$. Thus we only have the following inequations

$$\frac{\beta_n}{\beta_n-\alpha_n}(L-\epsilon) < \frac{f(\beta_n)-f(0)}{\beta_n}\frac{\beta_n}{\beta_n-\alpha_n} < \frac{\beta_n}{\beta_n-\alpha_n}(L+\epsilon)$$
$$\frac{-\alpha_n}{\beta_n-\alpha_n}(L+\epsilon) < \frac{f(\alpha_n)-f(0)}{\alpha_n}\frac{-\alpha_n}{\beta_n-\alpha_n} < \frac{-\alpha_n}{\beta_n-\alpha_n}(L-\epsilon).$$

Combining them, we get

$$L-\frac{\beta_n+\alpha_n}{\beta_n-\alpha_n}\epsilon < D_n < L+\frac{\beta_n+\alpha_n}{\beta_n-\alpha_n}\epsilon.$$

Note that $\left(\frac{\beta_n}{\beta_n-\alpha_n}\right)$ is bounded, i.e.,

$$\left|\frac{\beta_n}{\beta_n-\alpha_n}\right| \leq M$$

for some constant M. Thus

$$\left|\frac{\beta_n+\alpha_n}{\beta_n-\alpha_n}\right| = \left|\frac{2\beta_n}{\beta_n-\alpha_n}-1\right| \leq 2M+1.$$

Hence, $L-(2M+1)\epsilon < D_n < L+(2M+1)\epsilon$ Thus, $\lim_{n\to\infty} D_n = L = f'(0)$.

3. By using Mean Value Theorem, $D_n = f'(t_n)$ where t_n is between α_n and β_n. Note that

$$\min\{\alpha_n,\beta_n\} < t_n < \max\{\alpha_n,\beta_n\}$$

and

$$\max\{\alpha_n,\beta_n\} = \frac{1}{2}(\alpha_n+\beta_n+|\alpha_n-\beta_n|)$$
$$\text{and } \min\{\alpha_n,\beta_n\} = \frac{1}{2}(\alpha_n+\beta_n-|\alpha_n-\beta_n|)$$

Thus, $\max\{\alpha_n,\beta_n\} \to 0$ and $\min\{\alpha_n,\beta_n\} \to 0$ as $\alpha_n \to 0$ and $\beta_n \to 0$. By squeezing principle for limits, $t_n \to 0$, and by the continuity of f', we have $\lim_n D_n = \lim_n f'(t_n) = f'(\lim_n t_n) = f'(0)$.
Example: Let f be defined by

$$f(x) = \begin{cases} x^2\sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x \neq 0 \end{cases}$$

Thus $f'(x)$ is not continuous at $x=0$, and $f'(0)=0$. Let $\alpha_n = \frac{1}{\frac{\pi}{2}+2n\pi}$ and $\beta_n = \frac{1}{2n\pi}$. It is clear that $\alpha_n \to 0$ and $\beta_n \to 0$ as $n\to\infty$. Also

$$D_n = \frac{-4n\pi}{\pi(\pi/2+2n\pi)} \to -\frac{2}{\pi}$$

as $n\to\infty$. Thus, $\lim_n D_n = -2/\pi$ exists and is different from $f'(0)=0$. □

7.7.86 Problem. Suppose f is a real function on $(-1,1)$. Call x a fixed point of f if $f(x)=x$.

1. If f is differentiable and $f'(t)\neq 1$ for every real t, prove that f has at most one fixed point.
2. Show that the function f defined by $f(t)=t+(1+e^t)^{-1}$ has no fixed point, although $0<f'(t)<1$ for all real t.
3. However, if there is a constant $A<1$ such that $|f'(t)|<A$ for all real t, prove that a fixed point x of f exists, and that $x=\lim x_n$, where x_1 is an arbitrary real number and $x_{n+1}=f(x_n)$ for $n=1,2,3,...$
4. Show that the process describe in (3) can be visualized by the zigzag path $(x_1,x_2)\to(x_2,x_2)\to(x_2,x_3)\to(x_3,x_3)\to(x_3,x_4)\to....$

7.7.86.1 Solution.

1. If not, then there exists two distinct fixed points, say x and y, of f. Thus $f(x)=x$ and $f(y)=y$. Since f is differeniable, by applying Mean Value Theorem we have that
$$f(x)-f(y)=f'(t)(x-y)$$
where t is between x and y. Since $x\neq y, f'(t)=1$, a contradiction.

2. We show that $0<f'(t)<1$ for all real t, we see that
$$f'(t)=1+(-1)\left(1+e^t\right)^{-2}e^t=1-\frac{e^t}{(1+e^t)^2}$$
Since $e^t>0$ and $(1+e^t)^2=(1+e^t)(1+e^t)>1(1+e^t)=1+e^t>e^t>0$ for all real t, thus $0<(1+e^t)^{-2}e^t<1$ for all real t. Hence $0<f'(t)<1$ for all real t. Next, since $f(t)-t=(1-e^t)^{-1}>0$ for all real t, f has no fixed point.

3. Suppose $x_{n+1}\neq x_n\ \forall\ n$. (If $x_{n+1}=x_n$, then $x_n=x_{n+1}=...$ and x_n is a fixed point of f). By Mean Value Theorem, $f(x_{n+1})-f(x_n)=f'(t_n)(x_{n+1}-x_n)$ where t_n is betweem x_n and x_{n+1}. Thus,
$$|f(x_{n+1})-f(x_n)|=|f'(t_n)||(x_{n+1}-x_n)|$$
Note that $|f'(t_n)|$ is bounded by $A<1, f(x_n)=x_{n+1}$, and $f(x_{n+1})=x_{n+2}$. Thus $|x_{n+2}-x_{n+1}|\leq A|x_{n+1}-x_n|<CA^{n-1}$ where $C=|x_2-x_1|$. For two positive integers $p>q$,
$$\begin{aligned}|x_p-x_q|&\leq|x_p-x_{p-1}|+...+|x_{q+1}-x_q|=\\&=C\left(A^{q-1}+A^q+...+A^{p-2}\right)\\&\leq\frac{CA^{q-1}}{1-A}.\end{aligned}$$
Hence
$$|x_p-x_q|\leq\frac{CA^{q-1}}{1-A}.$$

Hence, for any $\epsilon > 0$, there exists

$$N = \left[\log_A \frac{\epsilon(1-A)}{C}\right] + 2$$

such that $|x_p - x_q| < \epsilon$ whenever $p > q \geq N$. By Cauchy criterion we know that x_n converges to x. Thus,

$$\lim_{n\to\infty} x_{n+1} = f\left(\lim_{n\to\infty} x_n\right)$$

since f is continuous. Thus, $x = f(x)$, i.e., x is a fixed point of f.

4. Since $x_{n+1} = f(x_n)$, it is trivial. □

7.7.87 Problem. Prove that $\sin x > x - (x^2/\pi)$ if $0 \leq x \leq \pi$.

7.7.87.1 Solution. We will show that

$$f(x) = \sin x - x + (x^2/\pi) \geq 0$$

on $[0, \pi]$. By observation of $f'(x) = \cos x - 1 + \frac{2x}{\pi}$, we find that $0, \pi/2 and \pi$ are critical points of $f(x)$. By noting that

$$f''(x) = -\sin x + 2/\pi$$

has exactly two zeros on $[0, \pi]$, we can apply Rolle's Theorem to verify that $0, \pi/2 and \pi$ are the only critical points of $f(x)$. (If f' had another zero, f'' would have a third zero.) Since $f(0) = f(\pi) = 0$ and $f(\pi/2) = 1 - \pi/4$, we can see that the minimum value of $f(x)$ on $[0, \pi]$ is 0. □

7.7.88 Problem. A function $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} 0, & \text{if } x = 0 \text{ or irrational} \\ \frac{1}{q^3} & \text{if } x = \frac{p}{q},\ p \in \mathbb{Z}, q \in \mathbb{N}; (p, q) = 1. \end{cases}$$

Show that f is differentiable at 0 and $f'(0) = 0$.

7.7.88.1 Solution. Hint. For $x \neq 0, 0 \leq \left|\dfrac{f(x)}{x}\right| \leq x^2$. □

7.7.89 Problem. A function $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is irrational} \\ x^2, & \text{if } x \text{ is rational.} \end{cases}$$

Show that f is differentiable at 0 and $f'(0) = 0$.

7.7.89.1 Solution. Hint. $0 \leq \frac{f(x)}{x} \leq x$, for $x > 0$ and $x \leq \frac{f(x)}{x} \leq 0$, for $x < 0$. □

7.7.90 Problem. A function $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} \sin x, & \text{if } x \text{ is irrational} \\ x, & \text{if } x \text{ is rational.} \end{cases}$$

Show that f is differentiable at 0 and $f'(0) = 1$.

7.7.90.1 Solution. Hint. $\cos x \le \frac{f(x)}{x} \le 1$, for $x \in \hat{B}(0;\pi/2)$. $\square$

7.7.91 Problem. Prove that $\frac{2x}{\pi} < \sin x$ for $0 < x < \frac{\pi}{2}$.

7.7.91.1 Solution. Let $f(x) = \dfrac{\sin x}{x}, 0 < x < \frac{\pi}{2}$. f is continuous on $[\delta, \frac{\pi}{2}]$, for some $\delta > 0$. $f'(x) = \dfrac{x \sin x - \cos x}{x^2}$ on $[\delta, \frac{\pi}{2}]$. Because $x < \tan x$ in $0 < x < \dfrac{\pi}{2}$, $f'(x) < 0$ in $\delta < x < \dfrac{\pi}{2}$. Therefore f is a strictly decreasing function on $(0, \frac{\pi}{2}]$. Because $f\left(\frac{\pi}{2}\right) = \frac{2}{\pi}$, it follows that $f(x) > \frac{2}{\pi}$ for $0 < x < \frac{\pi}{2}$, i.e., $\frac{2\pi}{x} < \sin x$ for $0 < x < \frac{\pi}{2}$. $\square$

7.7.91.2 Solution. (Geometrical viewpoint) The line joining the points (0,0) and $(\frac{\pi}{2}, 1)$ in the graph of the curve $y = \sin x$ is $y = \frac{2x}{\pi}$, the curve is above this line. Thus $\frac{2x}{\pi} < \sin x$ for $0 < x < \frac{\pi}{2}$. $\square$

7.7.92 Problem. Every function that has an antiderivative is Darboux continuous.

7.7.92.1 Solution. Let f be a differentiable function. We show that f' is Darboux continuous. Let $p, q \in [0,1], p < q$ and let $c \in \mathbb{R}$ be between $f'(p)$ and $f'(q)$. Without loss of generality, assume $f'(p) < c < f'(q)$. Now define $g : [p,q] \to \mathbb{R}$ by

$$g(x) = f(x) - cx.$$

Then g is differentiable. It attains a minimum value at r, say, $r \in [p,q]$. Since $g'(p) < 0$ and $g'(q) > 0$ we have $r \ne p$ and $r \ne q$, so $r \in (p,q)$. But then $g'(r)$ must be 0, i.e., $f'(r) = c$ and we are done. $\square$

7.7.93 Problem. Suppose that f is a function that satisfies

$$f''(x) + f'(x)g(x) - f(x) = 0$$

for some continuous function g. Prove that if f vanishes at a and at some $b > a$, then f is identically zero on $[a,b]$.

7.7.93.1 Solution. Choose $x_0 \in [a,b]$ such that $f(x_0)$ is the maximum value of f on $[a,b]$. Then $f'(x_0) = 0$ and we must also have $f''(x_0) < 0$. Therefore

$$0 = f(a) \le f(x_0) = f''(x_0) \le 0$$

so $f(x_0) = 0$. That is $f(x) \le 0 \ \forall \ x \in [a,b]$. A similar argument shows that $f(x) \ge 0$ on $[a,b]$ and hence f vanishes identically on $[a,b]$. Notice that this argument assumes that is in the interior of $[a,b]$, but the same conclusions can be drawn if $x_0 = a$ or b. $\square$

7.7.94 Problem. A function f is continuous on (a,b) and $f(x)$ is finite (a,b). If the line segment joining the points $A(a, f(a)), B(b, f(b))$ intersects the graph of f at some point P different from A, B. Prove that $f''(\xi) = 0$ for some $\xi \in (a,b)$.

7.7.94.1 Solution. The equation of the line joining the points $A(a, f(a))$ and $B(b, f(b))$ is

$$\frac{y - f(b)}{f(b) - f(a)} = \frac{x - b}{b - a}$$

since it passes through the point $P(c, f(c))$ we get

$$\frac{f(c) - f(b)}{f(b) - f(a)} = \frac{c - b}{b - a} \Rightarrow \frac{f(b) - f(c)}{b - c} = \frac{f(b) - f(a)}{b - a},$$

So, by mean value theorem $\exists\, p, q \in (a, b)$ such that

$$f'(p) = \frac{f(b) - f(c)}{b - c} = \frac{f(b) - f(a)}{b - a} = f'(q)$$

Again, by Rolle's theorem $\exists\, \xi \in (a, b)$ such that $f''(\xi) = 0$. ☐

7.7.95 Problem. Let f be twice-differentiable on $\mathbb{R}$ and suppose there are constants $A, C \in [0, \infty)$ such that for each $x \in \mathbb{R}, |f(x)| \leq A$ and $|f''(x)| \leq C$. Prove that there is a constant $B \in [0, \infty)$ such that for each $x \in \mathbb{R}, |f'(x)| \leq B$.

7.7.95.1 Solution. Using Taylor expansion, we get

$$\begin{aligned}
&f(x + 2h) = f(x) + (x + 2h - x)f'(x) + \frac{(x + 2h - x)^2}{2!} f''(\xi)\\
&f(x + 2h) = f(x) + 2hf'(x) + 2h^2 f''(\xi)\\
\Rightarrow &f'(x) = \frac{f(x + 2h) - f(x) - 2h^2 f''(\xi)}{2h}\\
\Rightarrow &|f'(x)| = \frac{1}{2h}\left|f(x + 2h) - f(x) - 2h^2 f''(\xi)\right| \leq \frac{1}{2h}|f(x + 2h) - f(x)| + |hf''(\xi)|\\
&< \frac{1}{2h} 2A + hC = \frac{A}{h} + C = B(say).
\end{aligned}$$

7.7.96 Problem. Let a function f be uniformly continuous on $[1, \infty)$. Prove that the function F defined by $F(x) = \dfrac{f(x)}{x}$ is bounded on $[1, \infty)$.

7.7.96.1 Solution. Since f is uniformly continuous, then $\delta > 0$ such that $\forall\, x, y \in [1, \infty)$, whenever $|x - y| < 2\delta$, we have that $|f(x) - f(y)| < 1$. Now, let $x \in [1, \infty)$ be arbitrary. Let $n \in \mathbb{N} \cup \{0\}$ be the largest integer so that $1 + n\delta < x$. Hence, we have the following

$$\begin{aligned}
&|f(x)| - |f(1)| \leq |f(x) - f(1)| =\\
&= |f(x) - f(n\delta + 1) + f(n\delta + 1) - f((n - 1)\delta + 1) + .. + f(\delta + 1) - f(1)|\\
&\leq |f(x) - f(n\delta + 1)| + |f(n\delta + 1) - f((n - 1)\delta + 1)| + .. + |f(\delta + 1) - f(1)|\\
&< n + 1
\end{aligned}$$

where by the choice of n, we have that $x - (n\delta + 1) < \delta$. Now, observe that

$$n\delta + 1 < x \Rightarrow n + 1 < \frac{x - 1}{\delta} + 1$$

Hence, we get that

$$|F(x)| = \left|\frac{f(x)}{x}\right| < \frac{n + 1 + |f(1)|}{x} = \frac{x - 1}{\delta x} + \frac{1 + |f(1)|}{x} \leq \frac{1}{\delta} + 1 + |f(1)|$$

where we used the fact that $x \geq 1$. Hence, F is bounded.

7.7.97 Problem. Let $f : [0, 1] \to \mathbb{R}$ be continuous. Suppose f is twice-differentiable on the open interval (0,1) and M is a real constant such that for each $x \in (0, 1)$, we have $|f''(x)| \leq M$. Let $a \in (0, 1)$. Prove that

$$|f'(a)| \leq |f(1) - f(0)| + \frac{M}{2}.$$

7.7.97.1 Solution. By Taylor's theorem, we can expand $f(x)$ at $x = a$. In particular, we have

$$f(0) = f(a) - af'(a) + \frac{a^2}{2} f''(\xi)$$
$$f(1) = f(a) + (1-a)f'(a) + \frac{(1-a)^2}{2} f''(\eta)$$

for some $0 < \xi < a$ and $a < \eta < 1$. Subtracting and rearranging, we get

$$f'(a) = f(0) - f(1) + \frac{a2}{2} f''(\xi) - \frac{(1-a)^2}{2} f''(\eta).$$

Note that the quadratic function $g(x) = 2x^2 - 2x + 1$ has axis at x=1/2 and has absolute maximum on [0,1] at $x = 0$ and 1, which is 1. Since $|f''(x)| < M \; \forall \, x \in (0,1)$, by triangle inequality we obtain

$$|f'(a)| \leq |f(0) - f(1)| + |a^2 + (1-a)^2| \frac{M}{2} \leq |f(0) - f(1)| + \frac{M}{2}.$$

as desired. □

7.7.98 Problem. Suppose that f maps the compact interval I into itself and that

$$|f(x) - f(y)| < |x - y| \; \forall \, x, y \in I, x \neq y.$$

Can one conclude that there is some constant $M < 1$ such that,

$$|f(x) - f(y)| < M|x - y| \; \forall \, x, y \in I?$$

7.7.98.1 Solution. Consider $f(x) = \sin x$. By the Mean Value Theorem we get

$$f(x) - f(y) = (\cos \xi)(x - y) \text{ for some } x, y \in (0,1)$$

and since $|\cos \xi| < 1$, so

$$|f(x) - f(y)| < |x - y| \text{ whenever } x \neq y.$$

However, if there be $M < 1$ such that

$$|f(x) - f(y)| < M|x - y| \text{ whenever } x \neq y.$$

then, putting $x = 0$ and letting $y \to 0$, we would get $|f'(0)| \leq M < 1$, which contradicts the fact that $f'(0) = 1$. □

7.7.99 Problem. Assume that $\lim_{x \to \infty} f(x) = 0$ and that f is differentiable. Does it follow that $\lim_{x \to \infty} f'(x) = 0$?

7.7.99.1 Solution. Hint. Not in general: $\frac{\sin x^2}{x}$.

7.7.100 Problem. Assume that $\lim_{x \to \infty} f'(x) = 0$. Does it imply that $\lim_{x \to \infty} f(x)$ exist?

7.7.100.1 Solution. Hint. Not in general: $\cos \log x$. □

7.7.101 Problem. Assume f is continuous on $[a, b]$ and has a finite second derivative f'' in the open interval (a, b). Assume that the line segment joining the points $A = (a, f(a))$ and $B = (b, f(b))$ intersects the graph of f in a third point P different from A and B. Prove that $f''(c) = 0$ for some c in (a, b).

7.7.101.1 Solution. Consider the equation,

$$g(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a).$$

Then $h(x) = f(x) - g(x)$. we get that there are three point $x = a, p$ and b such that $h(a) = h(p) = h(b) = 0$. So, by Mean Value Theorem on $[a, p]$ and on $[p, b]$, we get that there is a point $r \in (a, p)$ such that $h'(r) = 0$, and a point $s \in [p, b]$ such that $h'(s) = 0$. Now, on $[r, s]$ by Rolle's theorem, we get a point $c \in [r, s]$ such that $h''(c) = f''(c) = 0$, since g is a polynomial of degree 1. $\square$

7.7.102 Problem. Given n functions $f_1, ..., f_n$, each having nth order derivatives in (a, b). A function W, called the **Wronskian** of $f_1, ..., f_n$ is defined as follows: For each $x \in (a, b)$, $W(x)$ is the value of the determinant of order n whose element in the kth row and mth column is $f_m^{(k-1)}$ where $k = 1, 2, ..., n$ and $m = 1, 2, ..., n$. [The expression $f_m^{(0)}$ is written for $f_m(x)$.]

1. Show that $W'(x)$ can be obtained by replacing the last row of the determinant defining $W(x)$ by the nth derivatives $f_1^{(n)}, f_2^{(n)}, ..., f_n^{(n)}$.
2. Assuming the existence of n constants $c_1, ..., c_n$, not all zero, such that $c_1 f_1(x) + ... + c_n f_n(x) = 0 \ \forall\, x \in (a, b)$. show that $W(x) = 0$ for each $x \in (a, b)$.
   (**Note**: A set of functions satisfying such a relation is said to be a **linearly dependent set** on (a, b).)
3. The vanishing of the Wronskian throughout (a, b) is necessary, but not sufficient, for linear dependence of $f_1, ..., f_n$. Show that in the case of two functions, if the Wronskian vanishes throughout (a, b) and if one of the functions does not vanish in (a, b), then they form a linearly dependent set in (a, b).

7.7.102.1 Solution.

1. Write

$$W(x) = \begin{vmatrix} f_1 & f_2 & ... & f_n \\ f_1' & f_2' & ... & f_n' \\ ... & ... & ... & ... \\ f_1^{(n-1)} & f_2^{(n-1)} & ... & f_n^{(n-1)} \end{vmatrix}$$

and note that if any two rows are the same, its determinant is 0; we get, by the rule of expansion, that

$$W'(x) = \begin{vmatrix} f_1 & f_2 & ... & f_n \\ f_1' & f_2' & ... & f_n' \\ ... & ... & ... & ... \\ f_1^{(n-2)} & f_2^{(n-2)} & ... & f_n^{(n-2)} \\ f_1^{(n)} & f_2^{(n)} & ... & f_n^{(n)} \end{vmatrix}.$$

2. Since $c_1f_1(x) + ... + c_nf_n(x) = 0 \; \forall \; x \in (a, b)$, where $c_1, ..., c_n$, are not all zero. Without loss of generality, we may assume $c_1 \neq 0$, we know that $c_1f_1^{(k)}(x) + ... + c_nf_n^{(k)}(x) = 0$ for every $x \in (a, b)$, where $0 \leq k \leq n$. Hence, by eliminating $c_1, ..., c_n$, we have

$$\begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \cdots & \cdots & \cdots & \cdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix} = 0 \Rightarrow W(x) = 0.$$

since the first column is a linear combination of other columns.

3. Let f and g be continuous and differentiable on (a, b). Suppose that $f(x) \neq 0$ for all $x \in (a, b)$. Since the Wronskian of f and g is 0, for all $x \in (a, b)$, we have $f'g - fg' = 0$ for all $x \in (a, b)$. Since $f(x) \neq 0$ for all $x \in (a, b)$, we have by the above,

$$\frac{f'g - fg'}{f^2} = 0 \Rightarrow \left(\frac{f}{g}\right)' = 0$$

for all $x \in (a, b)$. Hence, there is a constant c such that $g = cf \; \forall \; x \in (a, b)$. Hence, f, g forms a linearly dependent set.
If $\{f_1, ..., f_n\}$ is linearly independent on I, It is NOT necessary that $W(t) = 0$ for some $t \in I$. For example, $f(t) = t^2|t|$, and $g(t) = t^3$. It is easy to check f, g is linearly independent on $(-1, 1)$. And $W(t) = 0 \; \forall \; t \in (-1, 1)$. □

7.7.103 Problem. Given a function defined and having a finite derivative in (a, b) and such that $\lim_{x\to b-} f(x) = \infty$. Prove that $\lim_{x\to b-} f'(x)$ either fails to exist or is infinite.

7.7.103.1 Solution. Suppose that, we have the existence of $\lim_{x\to b-} f'(x)$, denoted by L. So, given $\epsilon = 1$, there is a $\delta > 0$ such that for $x \in (b - \delta, b)$, we have $|f'(x)| < |L| + 1$. Consider $x, a \in (b - \delta, b)$ with $x > a$, then we have and Mean Value Theorem,

$$\begin{aligned} |f(x) - f(a)| &= |f'(\xi)(x - a)|, \xi \in (a, x) \\ &\leq (|L| + 1)|x - a| \end{aligned}$$

which implies that

$$|f(x)| \leq |f(a)| + (|L| + 1)\delta$$

which contradicts to $\lim_{x\to b-} f(x) = \infty$. □

7.7.104 Problem. Given a function f defined and having a finite derivative f' in the half-open interval $0 < x \leq 1$ and such that $|f'(x)| < 1$. Define $a_n = f(1/n)$ for $n \in \mathbb{N}$, and show that $\lim_n a_n$ exists.

7.7.104.1 Solution. Consider $n \geq m$, and by Mean Value Theorem,

$$|a_n - a_m| = |f(1/n) - f(1/m)| = |f'(p)| \left|\frac{1}{n} - \frac{1}{m}\right| \leq \left|\frac{1}{n} - \frac{1}{m}\right|,$$

then (a_n) is a Cauchy sequence since $1/n$ is a Cauchy sequence. Hence, $\lim_n a_n$ exists. □

7.7.105 Problem. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable, such that for some $\alpha \in \mathbb{R}$, $|f'(x)| \leq \alpha < 1 \ \forall\, x \in \mathbb{R}$. Let $a_1 \in \mathbb{R}$ and $a_{n+1} = f(a_n)$. Show that $\lim_n a_n$ exists.

7.7.105.1 Solution. Hint: We observe that

$$\begin{aligned}
|a_3 - a_2| &= |f(a_2) - f(a_1)| \\
&= f'(p)|a_2 - a_1| \leq \alpha|a_2 - a_1| \text{ where } p \text{ lies between } a_1 \text{ and } a_2 \\
\text{Similarly } |a_4 - a_3| &\leq \alpha|a_3 - a_2| \\
&\ldots\ldots\ldots \\
|a_{r+1} - a_r| &\leq \alpha|a_r - a_{r-1}| \\
\text{Thus } |a_{r+1} - a_r| &\leq \alpha^{r-1}|a_2 - a_1|.
\end{aligned}$$

Let $m > n$, then $m = n + p$ (say). Then,

$$\begin{aligned}
|a_m - a_n| &= |a_{n+p} - a_{n+p-1} + a_{n+p-1} - a_{n+p-2} + \ldots + a_{n+1} - a_n| \\
&\leq |a_{n+p} - a_{n+p-1}| + |a_{n+p-1} - a_{n+p-2}| + \ldots + |a_{n+1} - a_n| \\
&\leq \left(\alpha^{n+p-2} + \alpha^{n+p-3} + .. + \alpha^{n-1}\right)|a_2 - a_1| \\
&\leq \frac{\alpha^{n-1}}{1-\alpha}|a_2 - a_1|, \quad \square
\end{aligned}$$

7.7.106 Problem. Assume f is non-negative and has a finite third derivative f''' in the open interval (0,1). If $f(x) = 0$ for at least two values of x in (0,1), prove that $f'''(x) = 0$ for some c in (0,1).

7.7.106.1 Solution. Since $f(x) = 0$ for at least two values of x in (0,1), say $f(a) = f(b) = 0$, where $a, b \in (0, 1)$. By Rolle's Theorem, we have $f'(p) = 0$ where $p \in (a, b)$. Note that f is non-negative and differentiable on (0,1), so both $f(a)$ and $f(b)$ are local minima, where a and b are interior to (0,1). Hence, $f'(a) = f'(b) = 0$. Since $f'(a) = f'(p) = 0$, we have $f''(q_1) = 0$ where $q_1 \in (a, p)$ and since $f'(p) = f'(b) = 0$, we have $f''(q_2) = 0$ where $q_2 \in (p, b)$ by Rolle's Theorem. Since $f''(q_1) = f''(q_2) = 0$, we have $f'''(c) = 0$ where $c \in (q_1, q_2)$ by Rolle's Theorem. $\square$

7.7.107 Problem. Give an example of a pair of functions f and g having a finite derivatives in (0,1), such that

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = 0$$

but such that $\lim_{x\to 0} \frac{f'(x)}{g'(x)}$ does not exist, choosing g so that g' is never zero.

7.7.107.1 Solution. Hint. Let $f(x) = \sin(1/x)$ and $g(x) = 1/x$. $\square$

In this exercise, it tells us that the converse of L-Hospital Rule is NOT necessary true. Here is a good exercise very like L-Hospital Rule, but it does not! We write it as follows.

7.7.108 Problem. Suppose that $f'(a)$ and $g'(a)$ exist with $g'(a) \neq 0$, and $f(a) = g(a) = 0$. Prove that

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

7.7.108.1 Solution. Consider

$$\begin{aligned}\lim_{x\to a}\frac{f(x)}{g(x)} &= \lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)}\\ &= \lim_{x\to a}\frac{[f(x)-f(a)]/(x-a)}{[g(x)-g(a)]/(x-a)}\\ &= \frac{f'(a)}{g'(a)}. \quad \square\end{aligned}$$

It should be noticed that we CANNOT use L-Hospital Rule since the statement tells that f and g have a derivative at a, we do not make sure of the situation of other points.

7.7.109 Problem. Suppose that g is real function defined on $\mathbb{R}$, with bounded derivative, say $|g'| \le M$. Fix $\epsilon > 0$, and define $f(x) = x + \epsilon g(x)$. Show that f is 1-1 if ϵ is small enough. (It implies that f is strictly monotonic.)

7.7.109.1 Solution. Suppose that $f(x) = f(y)$, i.e., $x + \epsilon g(x) = y + \epsilon g(y)$ which implies that $|y - x| = \epsilon|g(y) - g(x)| \le M|y - x|$ by Mean Value Theorem, and hypothesis. So, as ϵ is small enough, we have $x = y$. That is, f is 1-1. $\square$

Definition:(**Convex Function**) Let f be defined on an interval I, and given $0 < \lambda < 1$, we say that f is a convex function if for any two points $x, y \in I$,

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda)f(y)$$

7.7.110 Problem. Let f be a differentiable real function defined on (a, b). Prove that f is convex if and only if f' is monotonically increasing.

7.7.110.1 Solution. Suppose f is convex, it suffices to consider $\lambda = 1/2$ and given $x < y$, we want to show that $f'(x) \le f'(y)$. Choose s and t such that $x < u < s < y$, then it is clear that we have

$$\frac{f(u)-f(x)}{u-x} \le \frac{f(s)-f(u)}{s-u} \le \frac{f(y)-f(s)}{y-s} \tag{7.4}$$

Let $s \to y-$, we have by (7.4)

$$\frac{f(u)-f(x)}{u-x} \le f'(y)$$

which implies that, let $u \to x+$, then $f'(x) \le f(y)$.
Conversely, Suppose that f' is monotonically increasing, if $x < y$, then

$$\begin{aligned}\frac{f(x)+f(y)}{2} - f\left(\frac{x+y}{2}\right) &= \frac{\left[f(x) - f\left(\frac{x+y}{2}\right)\right] + \left[f(y) - f\left(\frac{x+y}{2}\right)\right]}{2}\\ &= \frac{x-y}{4}[f'(\xi_1) - f'(\xi_2)],\ \xi_1 \le \xi_2\\ &\le 0\end{aligned}$$

Similarly for $x > y$, and there is nothing to prove $x = y$. Hence, f is convex. $\square$

7.7.111 Problem. Let f be convex on $[a, b]$, and let $c \in (a, b)$. Define

$$l(x) = f(a) + \frac{f(c) - f(a)}{c - a}(x - a),$$

then $f(x) \geq l(x)$ for all $x \in [c, b]$.

7.7.111.1 Solution. Consider $x \in [c, b]$, then

$$c = \frac{x - c}{x - a}a + \frac{c - a}{x - a}x \Rightarrow f(c) \leq \frac{x - c}{x - a}f(a) + \frac{c - a}{x - a}f(x),$$

which implies that

$$f(x) \geq f(a) + \frac{f(c) - f(a)}{c - a}(x - a) = l(x). \quad \square \tag{7.5}$$

7.7.112 Problem. Let f be a convex function defined on $[a, b]$. Let $a < s < t < u < b$, then we have

$$\frac{f(t) - f(s)}{t - s} \leq \frac{f(u) - f(s)}{u - s} \leq \frac{f(u) - f(t)}{u - t}.$$

7.7.112.1 Solution. By the definition of convex function, we know that

$$f(x) \leq f(s) + \frac{f(u) - f(s)}{u - s}(x - s) \tag{7.6}$$

and by the previous inequality

$$f(s) + \frac{f(t) - f(s)}{t - s}(x - s) \leq f(x),\ x \in [s, u]. \tag{7.7}$$

So, as $x \in [t, u]$, by (7.6) and (7.7), we finally have

$$\frac{f(t) - f(s)}{t - s} \leq \frac{f(u) - f(s)}{u - s}.$$

Similarly, we have

$$\frac{f(u) - f(s)}{u - s} \leq \frac{f(u) - f(t)}{u - t}.$$

Thus

$$\frac{f(t) - f(s)}{t - s} \leq \frac{f(u) - f(s)}{u - s} \leq \frac{f(u) - f(t)}{u - t}. \quad \square$$

7.7.113 Problem. Let $f(x)$ be convex on $[a, b]$, and assume that f is differentiable at $c \in (a, b)$, we have

$$l(x) = f(c) + f'(c)(x - c) \leq f(x)$$

That is, the equation of tangent line is below $f(x)$ if the equation of tangent line exists.

7.7.113.1 Solution. Since f is differentiable at $c \in [a, b]$, we write the equation of tangent line at c, as

$$l(x) = f(c) + f'(c)(x - c).$$

Define

$$m(s) = \frac{f(s) - f(c)}{s - c},\ a < s < c \text{ and } m(t) = \frac{f(t) - f(c)}{t - c},\ c < t < b$$

then it is clear that

$$m(s) \leq f'(c) \leq m(t)$$

which implies that

$$l(x) = f(c) + f'(c)(x - c) \leq f(x). \quad \square$$

7.7.114 Problem. Let $f : \mathbb{R} \to \mathbb{R}$ be convex. If f is bounded above, then f is a constant function.

7.7.114.1 Solution. Suppose that f is not constant, say $f(a) \neq f(b)$, where $a < b$. If $f(b) > f(a)$, we consider

$$\frac{f(x) - f(b)}{x - b} \geq \frac{f(b) - f(a)}{b - a},\ x > b$$

which implies that as $x > b$,

$$f(x) \geq \frac{f(b) - f(a)}{b - a}(x - b) + f(b) \to \infty \text{ as } x \to \infty.$$

If $f(b) < f(a)$, we consider

$$\frac{f(x) - f(a)}{x - a} \leq \frac{f(b) - f(a)}{b - a},\ x < a$$

which implies that as $x < a$,

$$f(x) \geq \frac{f(b) - f(a)}{b - a}(x - a) + f(a) \to \infty \text{ as } x \to \infty.$$

So, we obtain that f is not bouded above. Hence, f must be a constant function. $\square$

7.7.115 Problem. Note that e^x is convex on $\mathbb{R}$. Use this to show that $A.M. \geq G.M.$

7.7.115.1 Solution. Since $(e^x)'' = e^x \geq 0$, so e^x is convex. Hence,

$$e^{\frac{x_1 + x_2 + \ldots + x_n}{n}} \leq \frac{e^{x_1} + e^{x_2} + \ldots + e^{x_n}}{n}$$

where $x_i \in \mathbb{R}; i = 1, 2, , .., n$. So, let $e^{x_i} = y_i > 0$, for $i = 1, 2, ..., n$. Then

$$(y_1.y_2...y_n)^{\frac{1}{n}} \leq \frac{y_1 + y_2 + \ldots + y_n}{n}.$$

$\square$

7.7.116 Problem. Let us consider the following functions

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases} \text{ and } g(x) = \begin{cases} xe^{-\frac{1}{x^2}}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

Prove the following:

1. $f^{(n)}(0) = g^{(n)}(0) = 0.$
2. The function f has a minimum at $x = 0$, but the function g has no extrema at $x = 0$.

7.7.116.1 Solution.

1. It is clear that

$$f'(x) = \frac{2}{x^3}e^{-\frac{1}{x^2}}, f''(x) = \left(\frac{4}{x^6} - \frac{6}{x^4}\right)e^{-\frac{1}{x^2}}, .., f^{(n)}(x) = R_{3n}\left(\frac{1}{x}\right)e^{-\frac{1}{x^2}}$$

where $R_{3n}\left(\frac{1}{x}\right)$ is a polynomial of $\frac{1}{x}$ and of order $3n$. Applying the L'Hôpital's rule k-times, we get

$$\lim_{x \to 0} \frac{e^{-\frac{1}{x^2}}}{x^k} = 0 \; k \in \mathbb{N}.$$

So we obtain that $f'(0) = 0$. Analogously, we can show that $f'(0) = 0$.

2. The function f has a minimum at $x = 0$, because it holds $f(x) > 0, x \neq 0$. The function g has no extrema at $x = 0$, because we have

$$g(x) > 0, x > 0, \text{ and } g(x) < 0, x < 0.$$

7.7.117 Problem. Determine the biggest term of the sequence given by

1. $x_n = \frac{\sqrt{n}}{n+2020}$
2. $x_n = n^{\frac{1}{n}}$

7.7.117.1 Solution. Hint: Find maxima for the functions $f(x) = \frac{\sqrt{x}}{x+2020}$, $g(x) = x^{\frac{1}{x}}$. □

7.7.118 Problem. Let $f(x) = a_1 \sin x + a_2 \sin 2x + ... + a_n \sin nx$, where $a_1, a_2, ..., a_n$ are real numbers and where n is a positive integer. Given that $|f(x)| \leq |\sin x| \; \forall \; x \in \mathbb{R}$, prove that

$$|a_1 + 2a_2 + ... + na_n| \leq 1.$$

7.7.118.1 Solution. Note that

$$\begin{aligned} &f'(x) = a_1 \cos x + 2a_2 \cos 2x + ... + na_n \cos nx \\ \Rightarrow &|f'(0)| = |a_1 + 2a_2 + ... + na_n|. \end{aligned}$$

Now,

$$\begin{aligned}|f'(0)| &= \left|\lim_{x\to 0}\frac{f(x)-f(0)}{x}\right| \\ &= \left|\lim_{x\to 0}\frac{f(x)}{\sin x}\frac{\sin x}{x}\right| \\ &= \left|\lim_{x\to 0}\frac{f(x)}{\sin x}\right| \le 1.\end{aligned}$$

Thus $|a_1 + 2a_2 + ... + na_n| \le 1$. □

7.7.119 Problem. Let V be the collection of all quadratic polynomials P with real coefficients such that $|P(x)| \le 1 \; \forall \, x \in [0,1]$. Determine

$$\sup\{|P'(0)|; P \in V\}.$$

7.7.119.1 Solution. Let $f(x) = ax^2 + bx + c$ be an arbitrary quadratic polynomial. Then

$$\begin{aligned} f(0) &= c, \\ f\left(\frac{1}{2}\right) &= \frac{1}{4}a + \frac{1}{2}b + c, \\ \text{and } f(1) &= a + b + c. \\ f'(0) = b = 4f\left(\frac{1}{2}\right) &- 3f(0) - f(1).\end{aligned}$$

Using the given conditions,

$$|P'(0)| \le 4\left|P\left(\frac{1}{2}\right)\right| + 3|P(0)| + |P(1)| \le 8.$$

Furthermore, $P(x) = 8x^2 - 8x + 1$ satisfies the given conditions and has $|P'(0)| = 8$. □

7.7.120 Problem. Let $f(x)$ be defined for $a \le x \le b$. Assuming appropriate properties of continuity and derivability, prove for $a < x < b$ that

$$\frac{\frac{f(x)-f(a)}{x-a} - \frac{f(b)-f(a)}{b-a}}{x-b} = \frac{1}{2}f''(\xi)$$

where ξ is some number between a and b.

7.7.120.1 Solution. Assume f is continuous on $[a,b]$ and has a second derivative at each point of (a,b). Let x be fixed with $a < x < b$. Let

$$g(t) = \begin{vmatrix} f(t) & t^2 & t & 1 \\ f(x) & x^2 & x & 1 \\ f(a) & a^2 & a & 1 \\ f(b) & b^2 & b & 1 \end{vmatrix}$$

Then $g(a) = g(x) = g(b) = 0$. By the mean value theorem there exist numbers α and β such that $a < \alpha < x < \beta < b$ and $g'(\alpha) = g'(\beta) = 0$. Applying the mean value theorem once again, we see that there is a number ξ such that $\alpha < \xi < \beta$ and $g''(\xi) = 0$. Evidently, ξ is between a and b. Now

$$g''(\xi) = \begin{vmatrix} f''(\xi) & 2 & 0 & 0 \\ f(x) & x^2 & x & 1 \\ f(a) & a^2 & a & 1 \\ f(b) & b^2 & b & 1 \end{vmatrix} = 0.$$

Thus expanding the determinant, we find

$$f''(\xi) \begin{vmatrix} x^2 & x & 1 \\ a^2 & a & 1 \\ b^2 & b & 1 \end{vmatrix} - 2 \begin{vmatrix} f(x) & x & 1 \\ f(a) & a & 1 \\ f(b) & b & 1 \end{vmatrix} = 0,$$

and the result follows. □

7.7.120.2 Solution. Assume f is continuous on $[a, b]$ and has a second derivative at each point of (a, b). For a fixed x with $a < x < b$, let

$$\lambda = \frac{\frac{f(x)-f(a)}{x-a} - \frac{f(b)-f(a)}{b-a}}{x-b}$$

Then

$$f(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a) + \lambda(x - a)(x - b)$$

Define

$$h(t) = f(t) - \left(f(a) + \frac{f(b) - f(a)}{b - a}(t - a) + \lambda(t - a)(t - b) \right)$$

Then $h(a) = h(x) = h(b) = 0$, so by the mean value theorem there are numbers α and β such that $a < \alpha < x < \beta < b$ and $h'(\alpha) = h'(\beta) = 0$. Then h'' vanishes for some number $\xi \in (\alpha, \beta)$ and hence between a and b. We have $h''(\xi) = f''(\xi) - 2\lambda = 0$, and therefore $\lambda = \frac{1}{2} f''(\xi)$ as required. □

7.7.121 Problem. Let $f : [-1, 1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 0, & \text{if } x \in [-1, 0] \\ 1, & \text{if } x \in (0, 1]. \end{cases}$$

Does there exist a functien g such that $g'(x) = f(x), x \in [-1, 1]$?

7.7.121.1 Solution. If possible, let there exist a function $g : [-1, 1] to \mathbb{R}$ such that $g'(x) = f(x), x \in [-1, 1]$. Then g is differentiable on $[-1, 1]$ and

$$g'(x) = \begin{cases} 0, & \text{if } x \in [-1, 0] \\ 1, & \text{if } x \in (0, 1]. \end{cases}$$

Since g is differentiable on $[-1, 1]$ and $g'(-1) \neq g'(1)$, by Darboux's theorem must assume every real number lying between $g'(-1)$ and $g'(1)$, i.e., between 0 and 1 on $[-1, 1]$. But this is impossible and therefore g does not exist. □

7.7.122 Problem. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x^2}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

Show that f is differentiable on $\mathbb{R}$ but f' is not continuous on $\mathbb{R}$.

7.7.122.1 Solution. For $x \neq 0$,

$$f'(x) = 2x \sin \frac{1}{x^2} - \frac{2}{x} \cos \frac{1}{x^2},$$

Now $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{x^2 \sin \frac{1}{x^2}}{x} = \lim_{x\to 0} x \sin \frac{1}{x^2} = 0$. Therefore $f'(0) = 0$. Hence f is differentiable on $\mathbb{R}$ and the derived function f' is defined by

$$f'(x) = \begin{cases} 2x \sin \frac{1}{x^2} - \frac{2}{x} \cos \frac{1}{x^2}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

Now $\lim_{x\to 0} f'(x)$ does not exist. This proves that f' is not continuous at 0. f' is unbounded on any neighbourhood of 0 and 0 is a point of infinite discontinuity of f'. □

7.7.123 Problem. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -x & \text{if } x < 0. \end{cases}$$

Show that $\exists$ a differentiable function g on $\mathbb{R}$ such that $g' = f$ on $\mathbb{R}$.

7.7.123.1 Solution. Hint: Consider $g(x) = \frac{1}{2}x|x|$. □

7.7.124 Problem. If f is differentiable in both right-hand and left-hand neighborhoods of a point a and

$$\lim_{x\to a+} f'(x) = \lim_{x\to a-} f'(x)$$

then f is differentiable at a. True or false?

7.7.124.1 Solution. For any point $a \neq 0$ the function $f(x) = sgn x$ has zero derivative: for every $x < 0, f(x) = -1$ and so $f'(x) = 0$; for every $x > 0, f(x) = 1$ and then $f'(x) = 0$. Therefore,

$$\lim_{x\to 0+} f'(x) = \lim_{x\to 0-} f'(x) = 0.$$

However, f is not differentiable at $a = 0$ because it is not even continuous at this point. □

7.7.125 Problem. If

$$\lim_{h\to 0} \frac{f(x+h) - f(x-h)}{2h} = A$$

at a point x, then $f'(x) = A$. True or false?

7.7.125.1 Solution. The function $f(x) = |x|$ satisfies the above condition at $x = 0$, but it does not differentiable at this point. □

7.7.126 Problem. If f is continuous in a neighborhood of a point a and is differentiable at a, then there is a neighborhood of a where f is differentiable. True or false?

7.7.126.1 Solution. The function

$$f(x) = \begin{cases} x^2 \left|\sin \frac{\pi}{x}\right| & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is continuous at $x \neq 0$ according to the composition and arithmetic properties of continuous functions, and at $x = 0$ one gets $\lim_{x\to 0} f(x) = 0 = f(0)$ Therefore, f is continuous on $\mathbb{R}$. The definition of this function can be rewritten as follows:

$$f(x) = \begin{cases} x^2 \sin \frac{\pi}{x} & \text{if } \frac{1}{2n+1} < x < \frac{1}{2n},\ n \in \mathbb{Z} \\ -x^2 \sin \frac{\pi}{x} & \text{if } \frac{1}{2n} < x < \frac{1}{2n-1},\ n \in \mathbb{Z} \\ 0 & \text{if } x = 0, \frac{1}{n},\ n \in \mathbb{Z} \setminus \{0\}. \end{cases}$$

Then for $x \neq 0$ and $x \neq \frac{1}{n}$ the derivative can be found by applying the arithmetic and chain rules:

$$f(x) = \begin{cases} 2x \sin \frac{\pi}{x} - \pi \cos \frac{\pi}{x} & \text{if } \frac{1}{2n+1} < x < \frac{1}{2n},\ n \in \mathbb{Z} \\ -2x \sin \frac{\pi}{x} + \pi \cos \frac{\pi}{x} & \text{if } \frac{1}{2n} < x < \frac{1}{2n-1},\ n \in \mathbb{Z}. \end{cases}$$

Let us show that the derivative does not exist at any point $x_n = \frac{1}{2n}, n \in \mathbb{Z} \setminus \{0\}$. In fact,

$$\lim_{x\to x_n-} f'(x) = \lim_{x\to x_n-} \left(2x \sin \frac{\pi}{x} - \pi \cos \frac{\pi}{x}\right) = -\pi, \text{ while}$$
$$\lim_{x\to x_n+} f'(x) = \lim_{x\to x_n+} \left(-2x \sin \frac{\pi}{x} + \pi \cos \frac{\pi}{x}\right) = \pi.$$

Note that $x_n = \frac{1}{2n} \to 0$ as $n \to \pm\infty$, but, at the same time the function is differentiable at $x = 0$:

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} x \left|\sin \frac{\pi}{x}\right| = 0. \quad \square$$

7.7.127 Problem. If f is not differentiable at a point a, and g is not differentiable at the point $f(a)$, then $g \circ f$ is not differentiable at a. True or false?

7.7.127.1 Solution. For the functions f, g defined by $f(x) = g(x) = \begin{cases} \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ is not differentiable at $a = 0$ (it has a discontinuity of the second kind at 0), but the function $g(f(x)) = x$ is differentiable at 0. □

7.7.128 Problem. If f is not differentiable at x_0, then either the inverse function does not exist in a neighborhood of the point $y_0 = f(x_0)$ or (if it exists) it is not differentiable at y_0. True or false?

7.7.128.1 Solution. The function $y = f(x) = \sqrt[3]{x}$ is strictly monotone and continuous in a neighborhood of $x_0 = 0$, but its derivative does not exists at the origin. However, the inverse function $x = f^{-1}(y) = y^3$ is defined and differentiable on $\mathbb{R}$, including at $y_0 = 0$.
Remark: This example shows that the conditions of the Inverse Function Theorem are just sufficient.

7.7.129 Problem. If f is differentiable on $\mathbb{R}$, f' is bounded in a neighborhood of a point x_0 and $f'(x_0) \neq 0$, then in a neighborhood of the point $y_0 = f(x_0)$, there exists the inverse function $x = f^{-1}(y)$. True or false?

7.7.129.1 Solution. The function f defined by

$$f(x) = \begin{cases} x + 2x^2 \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

is differentiable at every real point. In fact, for any $x \neq 0$ the derivative can be calculated by using the arithmetic and chain rules: $f'(x) = 1 + 4x \sin \frac{1}{x} - 2 \cos \frac{1}{x}$. For $x = 0$, we should appeal to definition:

$$f'(0) = \lim_{x \to 0} \frac{x + 2x^2 \sin 1/x}{x} = \lim_{x \to 0} \left(1 + 2x \sin \frac{1}{x}\right) = 1$$

Hence,

$$f'(x) = \begin{cases} 1 + 4x \sin \frac{1}{x} - 2 \cos \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \text{ and } f'(0) \neq 0$$

Besides, f' is bounded on $[-1, 1]$:

$$\left|1 + 4x \sin \frac{1}{x} - 2 \cos \frac{1}{x}\right| \leq 1 + 4|x| + 2 \leq 7.$$

Thus, all the conditions of the statement are satisfied. Nevertheless, one can show that there is no neighborhood of $x_0 = 0$ on which f is one-to-one. Let us consider, for simplicity, a right-hand neighborhood of 0. On such a neighborhood $\sin \frac{1}{x}$ takes all the values in $[-1, 1]$, and therefore, f oscillates between the parabolas $x - 2x^2$ and $x + 2x^2$. The apex of the parabola $x - 2x^2$ is at the point $x = 1/4$. Consider $0 < \delta < 1/4$ and $\delta_1 = \frac{\delta}{\pi} < \delta$. Let us find a point x_1 such that $f(x_1) = x_1 - 2x_1^2$ and $x_1 < \delta_1$. From the first condition, $\sin \frac{1}{x_1} = -1$, that gives $x_k = \frac{1}{2k\pi - \pi/2}, k \in \mathbb{N}$, and the second condition imposes restriction on $k : 2k - 1/2 > 1/\delta$, that is, $k > 1/4 + 1/2\delta$. Let us choose some fixed k_1, which satisfies the last condition, and the corresponding $x_1 = \frac{1}{2k_1\pi - \pi/2} \in (0, \delta_1) \subseteq (0, \delta)$. Note that the functional value at the auxiliary point $x_2 = \frac{1}{2k_1\pi + \pi/2}$ is larger than at x_1 : a little calculation shows that $f(x_2) - f(x_1) > 0$.
Consider one more point $x_3 = \frac{1}{(2k_1+1)\pi - \pi/2} < x_1 < 1/4$. Since $2x - x^2$ is increasing on $\left(0, \frac{1}{4}\right)$, it follows that $f(x_3) < f(x_1)$. Hence, we have the three points $x_3 < x_2 < x_1 < \delta$ such that $f(x_3) < f(x_1) < f(x_2)$. Applying the Intermediate Value Theorem to the continuous function f on the interval $[x_3, x_2]$, we conclude that for all A such that $f(x_3) < A < f(x_2)$, in particular, for $A = f(x_1)$, there exists a point $x_A \in [x_3, x_2]$ at which $f(x_A) = A = f(x_1)$. Thus, f takes the same value $f(x_A) = f(x_1)$ at two different points of the interval $(0, \delta)$, that is, f is not injective on $(0, \delta)$. Since it holds for all $\delta > 0$, the function f is not one-to-one in any neighborhood of $x_0 = 0$ and an inverse does not exist in a neighborhood of the origin.
Remark: This false statement is a weakened version of the Inverse Function Theorem where the condition of the continuity of the derivative is replaced by the weaker condition of its boundedness. In fact, the derivative of f is discontinuous at 0: for $x_n = \frac{1}{2n\pi + \pi} \to 0$ as $n \to \infty$, one has $f'(x_n) = 1 + 2 = 3 \neq 1 = f'(0)$.

7.7.130 Problem. If both functions f and g are differentiable on $\mathbb{R}$ and $f(x) > g(x)$ on $\mathbb{R}$, then there exists at least one point in $\mathbb{R}$ where $f'(x) > g'(x)$. True or false?

7.7.130.1 Solution. Evidently, the functions $f(x) = -e^{-x}$ and $f(x) = e^{-x}$ are differentiable on $\mathbb{R}$ and $f(x) > g(x)$ on $\mathbb{R}$, but $f'(x) = -e^{-x} < e^{-x} = g'(x)$ on $\mathbb{R}$. □

7.7.131 Problem. If a function is differentiable and monotone on $\mathbb{R}$, then its derivative is also monotone on $\mathbb{R}$. True or false?

7.7.131.1 Solution. The function $f(x) = 2x + \sin x$ is differentiable and strictly increasing on $\mathbb{R}$ $(f'(x) = 2 + \cos x > 0, x \in \mathbb{R})$, but its derivative is not monotone on $\mathbb{R}$. □

Remark: The following variation of the statement is also false: if a function is differentiable and non-monotone on $\mathbb{R}$ then its derivative is also non-monotone on $\mathbb{R}$. It can be disproved by the counterexample $f(x) = x^2$.

7.7.132 Problem. If f is differentiable on $[a, b]$ then f' is bounded on $[a, b]$. True or false?

7.7.132.1 Solution. Consider the function f defined by

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \quad \text{has the derivative}$$

$$f'(x) = \begin{cases} 2x \sin \frac{1}{x^2} - \frac{2}{x} \cos \frac{1}{x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

However, f is unbounded on $[-1, 1]$, because $\lim_{x \to 0} 2x \sin \frac{1}{x^2} = 0$,
and for $x_k = \frac{1}{\sqrt{2k\pi}}$ $k \in \mathbb{N}$, it follows

$$\lim_{x_k \to 0} \frac{2}{x_k} \cos \frac{1}{x_k^2} = \lim_{k \to \infty} 2\sqrt{2k\pi} \cos 2k\pi = \infty,$$

so $\lim_{x_k \to 0} f'(x_k) = -\infty$. □

7.7.133 Problem. "If f is continuous on $[a, b]$, has a tangent line at each point of its graph on (a, b) and $f(a) = f(b)$, then at least one of these tangent lines is horizontal. True or false?

7.7.133.1 Solution. The function $f(x) = \sqrt[3]{x^2}$ is continuous on $[-a, a]$, $a > 0$, satisfies the condition $f(-a) = f(a)$, and has the derivative $f'(x) = \frac{2}{3\sqrt[3]{x}} \neq 0$ for every $x \neq 0$. Therefore, a tangent line at any point $(x, f(x)), x \in [a, 0) \cup (0, a]$, exists, but it is not horizontal. At the point $x = 0$ the derivative does not exist, since $\lim_{x \to 0} \frac{\sqrt[3]{x^2}}{x} = \infty$, but the tangent line at this point exists, although it is vertical and not horizontal. □
Remark: This statement is a wrong and weaken geometric version of Rolle's theorem: if f is continuous on $[a, b]$, differentiable on (a, b) and $f(a) = f(b)$ then there is a point $c \in (a, b)$ such that $f'(c) = 0$. (It implies that the tangent line at the point $(c, f(c))$ is horizontal.)

7.7.134 Problem. If f is bounded on $[a, b]$, differentiable on (a, b) and $f(a) = f(b) = 0$, then its derivative is zero at some point in (a, b). True or false?

7.7.134.1 Solution. The function $f(x) = x - [x]$ is bounded on $[0, 1]$, differentiable on $(0, 1)$ ($f(x) = x$ on $(0, 1)$) and $f(0) = f(1) = 0$, but $f'(x) = 1$ for every $x \in (0, 1)$. $\square$
Remark This is a wrong version of Rolle's theorem with dropped condition of the continuity of f on $[a, b]$. $\square$

7.7.135 Problem. If f is continuous on $[a, b]$ and differentiable on (a, b), then for any $c \in (a, b)$ there are two points $x_1, x_2 \in (a, b)$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

True or false?

7.7.135.1 Solution. The function $f(x) = x^3$ is continuous on $[-1, 1]$ and differentiable on $(-1, 1)$. However, for $c = 0$ there is no pair $x_1, x_2 \in (-1, 1)$ such that $f'(0) = \frac{f(x_2)-f(x_1)}{x_2-x_1}$. because $f(x) = x^3$ is strictly increasing on $[-1, 1]$. $\square$
Remark 1: This is a wrong formulation of the Mean Value theorem.
Remark 2: The above statement can be also considered as a wrong extension of the following theorem: if f is convex on $[a, b]$ and differentiable on (a, b), then for any $c \in (a, b)$ there are two points $x_1, x_2 \in (a, b)$ such that $f'(c) = \frac{f(x_2)-f(x_1)}{x_2-x_1}$.

7.7.136 Problem. If a function f is differentiable on (a, b) and f' is unbounded on (a, b), then f is also unbounded on (a, b). True or false?

7.7.136.1 Solution. The function $f(x) = \sqrt[3]{x}$ is differentiable on $(0, 1)$ and its derivative $f'(x) = \frac{1}{3\sqrt[3]{x^2}}$ is unbounded on $(0, 1)$. $\square$
Remark: This is the false converse to the following correct statement:
if a function f is differentiable on a bounded interval (a, b) and f is unbounded on (a, b), then f' is also unbounded on (a, b). If (a, b) is unbounded, then the last statement is false. The corresponding counterexample is: $f(x) = x$ on $\mathbb{R}$.

7.7.137 Problem. If a function f is differentiable on a finite interval (a, b) and $\lim_{x\to a+} f(x) = \infty$, then $\lim_{x\to a+} f'(x) = \infty$. True or false?

7.7.137.1 Solution. The function $f(x) = \frac{1}{x} + \cos\frac{1}{x}$ is differentiable on (0,1) and However $\lim_{x\to 0+} f(x) = \infty$, the derivative $f'(x) = -\frac{1}{x^2} + \frac{1}{x^2}\sin\frac{1}{x}$ does not have a limit (finite or infinite) when x approaches 0. Indeed, for $x_n = 2/(4n+1)\pi, n \in \mathbb{N}$ we have $\lim_{x_n\to 0+} f'(x) = \lim_{n\to\infty} 0 = 0$, but for $x_n = 2/(4n-1)\pi, n \in \mathbb{N}$,it follows that $\lim_{x_n\to 0+} f'(x_n) = \lim_{n\to\infty}\left(-\frac{2}{x_n^2}\right) = -\infty$. $\square$

7.7.138 Problem. If f is differentiable on (a, ∞) and both the function and its derivative are bounded on (a, ∞), then the existence of $\lim_{x\to\infty} f(x)$ implies the existence of $\lim_{x\to\infty} f'(x)$, and vice-versa. True or false?

7.7.138.1 Solution. First, consider the function f defined by $f(x) = \frac{\sin x^2}{x}$ on $(1, \infty)$. Since $\left|\frac{\sin x^2}{x}\right| \le \frac{1}{x}$ for $x > 1$ it follows that $\lim_{x\to\infty}\frac{\sin x^2}{x} = 0$. However, $f'(x) = 2\cos x^2 - \frac{\sin x^2}{x^2}$ has no limit at infinity. It is sufficient to compare two partial limits: the first corresponding to the sequence $x_n = \sqrt{2n\pi}$, $n \in \mathbb{N}$ gives

$$\lim_{x_n\to\infty} f'(x_n) = \lim_{n\to\infty} f'(\sqrt{2n\pi}) = 2$$

and the second related to $y_n = \sqrt{\pi + 2n\pi},\ n \in \mathbb{N}$

$$\lim_{y_n\to\infty} f'(y_n) = \lim_{n\to\infty} f'(\sqrt{\pi+2n\pi}) = -2$$

Since two partial limits are different, the limit $\lim_{x\to a+} f'(x)$ does not exist.
Second, consider $f(x) = \cos(\ln x)$ on $(1, \infty)$. The derivative $f'(x) = -\frac{\sin(\ln x)}{x}$ has zero limit at infinity. However, the function has no limit at infinity, since for the partial limit with the sequencess $x_n = e^{2n\pi}, y_n = e^{\pi+2n\pi},\ n \in \mathbb{N}$ we have

$$f(x_n) \to 1, f(y_n) \to -1 \text{ as } n \to \infty$$

Hence $\lim_{x\to\infty} f(x)$ does not exist. □

7.7.139 Problem. If $f'(x) = 0$ on a set S then $f(x)$=const on this set. True or false?

7.7.139.1 Solution. Let $f(x) = \begin{cases} 2, & \text{if } x \in (1,2) \\ 3, & \text{if } x \in (2,3). \end{cases}$ where $S = (1,2) \cup (2,3)$. □

Remark: The statement will be correct if the set S is connected (that is, S is an interval).

7.7.140 Problem. If f is strictly increasing and differentiable on (a, b), then $f'(x) > 0$ on (a, b). True or false?

7.7.140.1 Solution. The function $f(x) = x^3$ is strictly increasing and differentiable on $(-1, 1)$, but $f'(0) = 0$. □

Remark 1: This is the wrong converse to the following statement: if $f'(x) > 0$ on (a, b), then $f(x)$ is strictly increasing on (a, b). This is also the wrong extension of the statement: if f is increasing and differentiable on (a, b), then $f'(x) \geq 0$ on (a, b).

7.7.141 Problem. If $f'(c) > 0$, then f is increasing in a neighborhood of the point c. True or false?

7.7.141.1 Solution. The function $f(x) = \begin{cases} x + 2x^2 \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ is differentiable at every real point, and its derivative is $f'(x) = \begin{cases} 1 + 4x\sin\frac{1}{x} - 2\cos\frac{1}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$ and $f'(0) = 1 > 0$. However, in every neighborhood of zero $f'(x)$ has both positive and negative values: for the points $x_k = \frac{2}{(4n+1)\pi}, k \in \mathbb{Z}$ we obtain $f'(x_k) = 1 + \frac{8}{(4n+1)\pi} > 0$; while for the points $x_n = \frac{1}{2n\pi},\ n \in \mathbb{Z} \setminus \{0\}$, we obtain $f'(x_n) = 1 - 2 < 0$. It means that f cannot be increasing in any neighborhood of the point $c = 0$. □

Remark: If one adds the condition of continuity of $f'(x)$ at the point c, then the statement becomes true.

7.7.142 Problem. If $f'(c) = 0$, then the point c is a local extremum of the function f. True or false?

7.7.142.1 Solution. The function $f(x) = x^3$ is differentiable at any real point and $f'(0) = 0$, but $c = 0$ is not a local extremum, because $f(x) = x^3$ is a strictly increasing function on $\mathbb{R}$. □

7.7.143 Problem. If c is a local extremum of f, then $f'(c) = 0$. True or false?

7.7.143.1 Solution. The function $f(x) = |x|$ is not differentiable at $c = 0$, but this point is a local (and global) minimum of this function. $\square$

Remark: This example is a wrong modification of the necessary condition for a local extremum: if f is differentiable at a point c and c is a local extremum, then $f'(c) = 0$.

7.7.144 Problem. If c is a local extremum of f and f is twice differentiable at c, then $f''(c) \neq 0$. True or false?

7.7.144.1 Solution. The function $f(x) = x^4$ is twice differentiable at $c = 0$ and the origin is a local (and global) minimum of this function, but $f''(0) = 0$. $\square$

Remark: This is the false converse to the Second Derivative test: if f is twice differentiable at a stationary point c and $f''(c) \neq 0$, then c is a local extremum of f.

7.7.145 Problem. If the derivative f' is defined in a neighborhood of a point c, and it does not change the sign passing through this point, then c is not a local extremum. True or false?

7.7.145.1 Solution. The function f defined by $f(x) = \begin{cases} 2x^2 + x^2 \cos \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$ is differentiable at every real point. In fact, for any $x \neq 0$ the derivative can be calculated by using the arithmetic and chain rules: $f'(x) = 4x + 2x \cos \frac{1}{x} + \sin \frac{1}{x}$. For $x = 0$ applying the definition we obtain

$$f'(0) = \lim_{x \to 0} \frac{2x^2 + x^2 \cos \frac{1}{x}}{x} = \lim_{x \to 0} \left(2x + x \cos \frac{1}{x} \right) = 0.$$

Hence,

$$f'(x) = \begin{cases} 4x + 2x \cos \frac{1}{x} + \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Now, using the sequences $(x_n), (y_n)$ defined by $x_n = \frac{2}{(4n+1)\pi}, y_n = \frac{2}{(4n-1)\pi}, n \in \mathbb{Z}$, we obtain

$$f'(x_n) = \frac{8}{(4n+1)\pi} + 1 > 0, \ f'(y_n) = \frac{8}{(4n-1)\pi} - 1 < 0$$

Note that in any neighborhood of the point c=0 there exist both points x_n and y_n (for n sufficiently large), that is, in any neighborhood of zero there are both positive and negative values of $f'(x)$. Therefore, the derivative does not preserve the sign in one-sided neighborhoods of zero, so f' does not change its sign passing through the point $c = 0$. On the other hand, the following evaluation is satisfied: $f(x) = 2x^2 + x^2 \cos \frac{1}{x} > 0 = f(0)$. Therefore, by the definition, $c = 0$ is a strict local (and global) minimum of f. $\square$

7.7.146 Problem. If f is strictly concave upward and twice differentiable on (a, b), then $f''(x) > 0$ on (a, b). True or false?

7.7.146.1 Solution. The function $f(x) = x^4$ is strictly concave upward and twice differentiable on $(-1, 1)$, but $f''(0) = 0$. $\square$

Remark: This is the wrong converse to the following statement: if $f''(x) > 0$ on (a, b), then f is strictly concave upward on (a, b). This is also the wrong extension of the statement: if f is concave upward and twice differentiable on (a, b), then $f''(x) \geq 0$ on (a, b).

7.7.147 Problem. If f is twice differentiable in a neighborhood of a point c and $f''(c) = 0$, then the point c is a point of inflection. True or false?

7.7.147.1 Solution. The function $f(x) = x^4$ is twice differentiable on $\mathbb{R}$ and $f''(0) = 0$, but $c = 0$ is not a point of inflection, since f is concave upward on $\mathbb{R}$ ($f''(x) = 12x^2 > 0 \,\forall\, x \neq 0$). □

7.7.148 Problem. If c is a point of inflection of f, then $f''(c) = 0$. True or false?

7.7.148.1 Solution. The function f defined by

$$f(x) = \begin{cases} -x^2 & \text{if } x < 0 \\ x^4 & \text{if } x \geq 0 \end{cases}$$

is not twice differentiable at $c = 0$. ($f''(x) = -2, \forall\, x < 0, f''(x) = 12x^2 \,\forall\, x > 0$, so $\lim_{x\to 0-} f''(x) = -2 \neq 0 = \lim_{x\to 0+} f''(x)$), but this point is a point inflection for f, because the function has a downward concavity at negative points ($f''(x) = -2 < 0 \,\forall\, x < 0$) and an upward concavity at positive points ($f''(x) = 12x^2 > 0 \forall\, x > 0$). □

Remark: Each of these Examples is the wrong converse to the necessary condition of a point inflection: if f is twice differentiable at a point c and the point c is a point inflection, then $f''(c) = 0$.

7.7.149 Problem. If f is continuous on (a, b) and is not differentiable at a point $c \in (a, b)$, then c is not a point inflection of the graph of f. True or false?

7.7.149.1 Solution. The function f defined by

$$f(x) = \begin{cases} x^2 + x & \text{if } x < 0 \\ 2x - x^2 & \text{if } x \geq 0 \end{cases}$$

is continuous on $\mathbb{R}$ and does not have a derivative at $c = 0$, since the one-sided derivatives are different:

$$f'(0-) = \lim_{x\to 0-} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0-} (x + 1) = 1$$
$$f'(0+) = \lim_{x\to 0+} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0+} (2 - x) = 2.$$

At the same time, $f''(x) = 2$ for $x < 0$, and $f''(x) = -2$ for $x > 0$, that is, the function is concave upward on $(-\infty, 0)$ and downward on $(0, \infty)$. Therefore, $c = 0$ is a point inflection. □

Remark: Let us consider the statement with the opposite conclusion: if f is continuous on (a, b) and is not differentiable at a point $c \in (a, b)$, then c is an inflection point of the graph of f. This statement is also false and a simple counterexample is

$$f(x) = \begin{cases} x^2 + x & \text{if } x < 0 \\ x^2 - x & \text{if } x \geq 0 \end{cases}$$

at $c = 0$.

7.7.150 Problem. "If $y = c$ is a tangent line to the graph of f then $y = c$ cannot be an asymptote to the same graph. True or false?

7.7.150.1 Solution. The function f defined by

$$f(x)=\begin{cases}\sin \pi x & \text{if } x<1\\ \frac{x+1}{x-1} & \text{if } x\geq 1\end{cases}$$

has the tangent line $y=1$ at the point $x=\frac{1}{2}$, since $f'\left(\frac{1}{2}\right)=\pi\cos\left(\frac{\pi}{2}\right)=0$, and the same line is the horizontal asymptote, because $\lim_{x\to\infty} f(x)=\lim_{x\to\infty}\frac{x+1}{x-1}=1$. $\square$

Remark 1: The statement is also false for vertical tangents and asymptotes. A simple counterexample is $f(x)=\begin{cases}\frac{1}{x} & \text{if } x<0\\ \sqrt[3]{x} & \text{if } x\geq 0\end{cases}$ at the point $x=0$.

Remark 2: The statement is true for vertical tangents and asymptotes of continuous functions.

7.7.151 Problem. The graph of a function cannot cross or touch its asymptote. True or false?

7.7.151.1 Solution. The function $f(x)=\frac{\sin x}{x}$ has the horizontal asymptote $y=0$ (since $\lim_{x\to\infty} f(x)=\lim_{x\to\infty}\frac{\sin x}{x}=0$) and the graph of f crosses this asymptote infinitely many times at the points $x_n=n\pi, n\in\mathbb{N}$. Another function $f(x)=e^{-x}(1-\cos x)$ has the horizontal asymptote $y=0$ (since $\lim_{x\to\infty} f(x)=\lim_{x\to\infty} e^{-x}(1-\cos x)=0$) and the graph of f touches this asymptote infinitely many times at the points $x_n=2n\pi,\ n\in\mathbb{N}$ (at all these points, $y=0$ is the tangent line to the graph of f). $\square$

7.7.152 Problem. If f and g are continuous in a neighborhood of a point a, and $g(a)=0$, then the function $\frac{f}{g}$ has a vertical asymptote at the point a. True or false?

7.7.152.1 Solution. Although the functions $f(x)=\sin x$ and $g(x)=x$ are continuous on $\mathbb{R}$, and $g(x)=x$ equals 0 at the point $x=0$, the function $h(x)=\frac{f(x)}{g(x)}=\frac{\sin x}{x}$ has a finite limit at 0. Hence, $h(x)$ has no asymptote at $x=0$. $\square$

Remark: The statement becomes true under the additional condition that $f(a)\neq 0$. Notice, however, that without the continuity of f and g, the last condition does not guarantee the existence of a vertical asymptote. A simple counterexample is:

$$f(x)=x+1, g(x)=\begin{cases}x+1, & \text{if } x\neq 0\\ 0, & \text{if } x=0\end{cases}$$

which at the point $x=0$ gives $\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{x+1}{x+1}=1$.

7.7.153 Problem. If $y=Ax+B$ is a slant asymptote to f, then the slope of the graph of f approaches A, as x approaches infinity. True or false?

7.7.153.1 Solution. The function $f(x)=\frac{\sin x^3}{x}$ has the horizontal asymptote $y=0$, since $0\leq\left|\frac{\sin x^3}{x}\right|\leq\frac{1}{|x|}\to 0$ as $x\to\infty$. However, its derivative $f'(x)=3x\cos x^3-\frac{\sin x^3}{x^2}$ does not approach 0 at infinity, because the second term tends to 0 (just like the function f), but the first term is unbounded at infinity: if one chooses $x_n=\sqrt[3]{2n\pi},\ n\in\mathbb{N}$, then $\lim_{x_n\to\infty} 3x_n\cos x_n^3=\lim_{x_n\to\infty} 3\sqrt[3]{2n\pi}=\infty$. $\square$

Remark: To obtain a slant asymptote with non-zero slope, just use a similar function $f(x)=x+\frac{\sin x^3}{x}$.

7.7.154 Problem. If f and g are differentiable in a deleted neighborhood of a point $c, g'(x) \neq 0$ in this neighborhood and $\lim_{x\to c} \frac{f'(x)}{g'(x)}$ exists, then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}.$$

True or false?

7.7.154.1 Solution. The functions $f(x) = x+2$ and $g(x) = x+1$ are differentiable on $\mathbb{R}$, and $g'(x) = 1 \neq 0$ on $\mathbb{R}$. However, $\lim_{x\to c} \frac{f(x)}{g(x)} = 2$, while $\lim_{x\to c} \frac{f'(x)}{g'(x)} = 1$ □

Remark 1. It happens because the important condition $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$ in L'Hospital's rule is missing here.

Remark 2. For an indeterminate form $\frac{\infty}{\infty}$ a similar false modification of L'Hospital's rule can be constructed and the same functions $f(x)$ and $g(x)$ can be used for a counterexample.

Remark 3. For both indeterminate forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$ the point c can also be infinite. For instance, for the latter form the following counterexample can be given: if $f(x) = 3 - \frac{1}{x^2}$ and $g(x) = 1 - \frac{1}{x}$ on $(0,\infty)$, then $f(x) = \frac{2}{x^3}, g(x) = \frac{1}{x^2} \neq 0$. It happens because $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 3 \neq 0 = \lim_{x\to\infty} \frac{f(x)}{g(x)}$. It happens because the important condition $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$ L'Hospital's rule is missing here.

7.7.155 Problem. If f and g are differentiable in a deleted neighborhood of a point $c, g'(x) \neq 0$ in this neighborhood and $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$, then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}.$$

True or false?

7.7.155.1 Solution. The functions $f(x) = x^2 \cos \frac{1}{x}$ and $g(x) = \sin x$ are differentiable in $\mathbb{R} \setminus \{0\}, g'(x) = \cos x \neq 0$ in $(-1,1)$ and $\lim_{x\to 0} x^2 \cos \frac{1}{x} = \lim_{x\to 0} \sin x = 0$ (for the first limit we can apply the evaluation $\left|x^2 \cos \frac{1}{x}\right| \leq x^2$. Nevertheless, $\lim_{x\to 0} \frac{f(x)}{g(x)} \neq \lim_{x\to 0} \frac{f'(x)}{g'(x)}$. In fact, since $\lim_{x\to 0} \frac{x}{\sin x} = 1$ and $\lim_{x\to 0} x \cos \frac{1}{x} = 0$, we can calculate the original limit just applying the arithmetic rules:

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{x^2 \cos \frac{1}{x}}{\sin x} = \lim_{x\to 0} \frac{x}{\sin x} x \cos \frac{1}{x} = 0$$

At the same time,

$$\lim_{x\to 0} \frac{f'(x)}{g'(x)} = \lim_{x\to 0} \frac{2x \cos \frac{1}{x} + \sin \frac{1}{x}}{\cos x}$$

does not exist. □

7.7.4 Remark. This statement is a wrongly weakened version of L'Hospital's rule for an indeterminate form $\frac{0}{0}$ where the condition of the existence of $\lim_{x\to 0} \frac{f'(x)}{g'(x)}$ (finite or infinite) is omitted.

7.7.156 Problem. Let the function f be defined as follows

$$f(x) = \begin{cases} x \sin \frac{1}{x}, & \text{if } 0 < x \leq 1 \\ 0, & \text{if } x = 0. \end{cases}$$

1. Show that f is not Lipschitz on any interval $(0, \delta)$, but it is Lipschitz on any $(\delta, 1)$, if $\delta > 0$.
2. Show that $f'(0)$ does not exist.
3. Show that f is continuous on [0,1].
4. Show directly that f is uniformly continuous on [0,1].
5. Given $x \in [0, 1]$, show that there is a constant $K > 0$ such that if $y \in [0, 1]$, then $|f(x)-f(y)| \leq K|x - y|$ (we say that f is pointwise Lipschitz on [0,1].)

7.7.156.1 Solution.

1. For $x \neq 0$, we have
$$f'(x) = \sin \frac{1}{x} - \frac{1}{x} \sin \frac{1}{x},$$
and this function is bounded on any $(\delta, 1)$. Use then the Mean Value Theorem to see that f is Lipschitz on $[\delta, 1]$. Note that f' is not bounded on any interval $(0, \delta)$ (consider $h_n = (\pi/2 + 2n\pi)^{-1}$ and $k_n = (2n\pi)^{-1}$), so f cannot be Lipschitz there (use the definition of the derivative).
2. $$f'(0) = \lim_{h \to 0} \frac{f(h)}{h} = \lim_{h \to 0} \sin \frac{1}{h}$$
does not exist.
3. f is continuous (from the right) at 0: Indeed, given $\epsilon > 0$ put $\delta = \epsilon$. If $|x| < \delta$, then $|f(x)| \leq |x| \leq \epsilon$. The function f is continuous at any nonzero point as it has a derivative there (see (1) above).
4. Let $\epsilon > 0$ be given. Since f is Lipschitz and thus uniformly continuous on $[\epsilon, 1]$, choose $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever $x, y \in [\epsilon, 1]$ are such that $|x - y| \leq \delta$. Note that $|f(x) - f(y)| \leq |f(x)| + |f(y)| \leq 2\epsilon$, if $x, y \in [0, \epsilon]$. If $x \in [0, \epsilon]$ and $y \in [\epsilon, 1]$, let z be in the segment $[x, y]$ with $z = \epsilon$. Then $|f(x) - f(y)| \leq |f(x) - f(z)| + |f(z) - f(y)| \leq 3\epsilon$. Thus, altogether, $|f(x) - f(y)| \leq 3\epsilon$ whenever $x, y \in [0, 1]$ are such that $|x - y| \leq \delta$.
5. First, if $x = 0, |f(y) - f(0)| = |f(y)| \leq 1$, so we can take $K = 1$. Let $x \in (0, 1]$. Since f is K-Lipschitz on $[x/2, 1]$ by 1, we have $|f(x)-f(y)| \leq K|x-y|$ whenever $y \in [x/2, 1]$. Let $L > 4/x$. If $y \in [0, x/2]$, then $|x-y| \geq x/2$. Thus $|f(y)-f(x)| \leq |f(x)|+|f(x)| \leq 2 < Lx/2 \leq L|x-y|$. Therefore, for any $x \in [0, 1]$, there is a constant $C > 0$ such that $|f(x)-f(y)| \leq C|x-y|$ (take $C = \max\{K, L\}$). □

7.7.157 Problem. Assume that a real-valued function f is bounded on an interval I , and that it has a (finite) derivative at some point $a \in I$. Then f is Lipschitz at a.

7.7.157.1 Solution. Hint. Assume that $\lim_{x \to a} \frac{f(x)-f(a)}{x-a} = L$. Then, there is $\delta > 0$ such that $|f(x) - f(a)| \leq (|L| + 1)|x - a|$ whenever $x \in I, |x - a| < \delta$. Let a constant $D > 0$ be such that $|f(x)| \leq D$ for each $x \in I$. Let $\alpha > 0$ be such that $D < \alpha\delta$. Then, if $|x - a| \geq \delta$ we get $|f(x) - f(a)| \leq 2D < 2\alpha\delta \leq 2\alpha|x - a|$. Therefore, $\max\{|L| + 1, 2\alpha\}$ can be used to see that f is Lipschitz at a. □

7.7.158 Problem. Show that the function

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x^2}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

is pointwise Lipschitz but not Lipschitz on [0,1].

7.7.158.1 Solution. Hint. $f(x)$ exists as a real number at each $x \in [0,1]$ (at 0, note that $\lim_{x\to 0} x \sin \frac{1}{x^2} = 0$. Thus, one can see that f is pointwise Lipschitz on [0,1]. If $x \neq 0$, then

$$f'(x) = 2x \sin \frac{1}{x^2} - \frac{2}{x} \cos \frac{1}{x^2}$$

which is not a bounded function on (0,1]. Thus, f cannot be Lipschitz on [0,1]. The function f is used to give an example of a function having a derivative not Riemann integrable. □

7.7.159 Problem. Assume that f and g are two bounded real-valued Lipschitz functions on a general interval I. Show that fg is a Lipschitz function on I.

7.7.159.1 Solution. Hint.

$$\begin{aligned} |f(x)g(x) - f(y)g(y)| &= |(f(x) - f(y))g(x) + f(y)(g(x)g(y))| \\ &\leq |f(x) - f(y)|.|g(x)| + |f(y)|.|g(x) - g(y)|. \quad \square \end{aligned}$$

7.7.160 Problem. Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function. Suppose $f(0) = 0$. Prove that there exists $\xi \in (-\pi/\pi/2)$ such that

$$f''(\xi) = f(\xi)(1 + 2\tan^2 \xi).$$

7.7.160.1 Solution. Let $g(x) = f(x)\cos x$. Since $g(\pi/2) = g(0) = g(\pi/2) = 0$; by Rolle's theorem there exist some $\xi_1 \in (-\pi/2, 0)$ and $\xi_2 \in (0, \pi/2)$ such that

$$g'(\xi_1) = g'(\xi_2) = 0.$$

Now consider the function

$$h(x) = \frac{g'(x)}{\cos^2 x} = \frac{f'(x)\cos x - f(x)\sin x}{\cos^2 x}.$$

We have $h(\xi_1) = h(\xi_2) = 0$; so by Rolle's theorem there exist $\xi \in (\xi_1, \xi_2)$ for which

$$\begin{aligned} 0 = h'(\xi) &= \frac{g''(\xi)\cos^2 \xi + 2\cos\xi \sin\xi g'(\xi)}{\cos^4 \xi} \\ &= \frac{(f''(\xi)\cos\xi - 2f'(\xi)\sin\xi - f(\xi)\cos\xi) + 2\sin\xi(f'(\xi)\cos\xi - f(\xi)\sin\xi)}{\cos^3 \xi} \\ &= \frac{f''(\xi)\cos^2\xi - f(\xi)(\cos^2\xi + 2\sin^2\xi)}{\cos^3\xi} = \frac{1}{\cos\xi}\left(f''(\xi) - f(\xi)\right)(1 + 2\tan^2\xi). \end{aligned}$$

This gives the desired equality. □

7.7.161 Problem. Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Suppose that f has infinitely many zeros, but there is no $x \in (a,b)$ with $f(x) = f'(x) = 0$.

1. Prove that $f(a)f(b) = 0$.
2. Give an example of such a function on [0,1].

7.7.161.1 Solution.

1. Choose a convergent sequence (x_n) of zeros of $f(x)$ and let $c = \lim x_n \in [a,b]$. By the continuity of f we obtain $f(c) = 0$. We want to show that either $c = a$ or $c = b$, so $f(a) = 0$ or $f(b) = 0$; then the statement follows. If c was an interior point then we would have $f(c) = 0$ and $f'(c) = \lim_n \frac{f(x_n)-f(c)}{x_n - c} = \frac{0-0}{x_n-0} = 0$ simultaneously, contradicting the conditions. Hence, $c = a$ or $c = b$.
2. Let
$$f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } 0 < x \leq 1 \\ 0 & \text{if } x = 0. \end{cases}$$
This function has zeros at the points $1/k\pi$ for $k = 0, 1, 2, ...$, and it is continuous at 0 as well. In (0,1) we have
$$f'(x) = \sin \frac{1}{x} - \frac{1}{x} \cos \frac{1}{x}$$
Since $\sin \frac{1}{x}$ and $\cos \frac{1}{x}$ cannot vanish at the same point, we have either $f(x) \neq 0$ or $f'(x) \neq 0$ everywhere in (0,1). □

7.7.162 Problem. Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous functions such that g is differentiable. Assume that $(f(0) - g'(0))\,(g'(1) - f(1)) > 0$. Show that there exists a point $c \in (0,1)$ such that $f(c) = g'(c)$.

7.7.162.1 Solution. Since every continuous function has an antiderivative such that $F'(x) = f(x)$, so let $h(x) = F(x) - g(x)$. By the continuouity of f we have $F' = f$, so $h' = f - g'$. The assumption can be rewritten as $h'(0)\,(-h'(1)) > 0$, so $h'(0)$ and $h'(1)$ have opposite signs. Then, by the Mean Value Theorem For derivatives (Darboux property of derivatives) it follows that there is a point c between 0 and 1 where $h'(c) = 0$, so $f(c) = g'(c)$. □

7.7.163 Problem. Let $f \in C^1(a,b), \lim_{x\to a+} f(x) = \infty, \lim_{x \to b-} f(x) = -\infty$ and $f'(x) + f^2(x) \geq -1$ for $x \in (a,b)$. Prove that $b - a \geq \pi$ and give an example where $b - a = \pi$.

7.7.163.1 Solution. From the inequality we get,
$$\frac{d}{dx}\left(\tan^{-1} f(x) + x\right) = \frac{f'(x)}{1 + f^2(x)} + 1 \geq 0$$
for $x \in (a,b)$. Thus $\tan^{-1} f(x) + x$ is non-decreasing in the interval and using the limits we get
$$\frac{\pi}{2} + a \leq -\frac{\pi}{2} + b$$
Hence $b - a \geq \pi$. Consider the function $f(x) = \cot x, a = 0, b = \pi$. □

7.7.164 Problem. Prove or disprove the following statements:

1. There exists a monotone function $f : [0,1] \to [0,1]$ such that for each $y \in [0,1]$ the equation $f(x) = y$ has uncountably many solutions x.

2. There exists a continuously differentiable function $f : [0, 1] \to [0, 1]$ such that for each $y \in [0, 1]$ the equation $f(x) = y$ has uncountably many solutions x.

7.7.164.1 Solution.

1. It does not exist. For each y the set $\{x; y = f(x)\}$ is either empty or consists of 1 point or is an interval. These sets are pairwise disjoint, so there are at most countably many of the third type.

2. Let f be such a map. Then for each value y of this map there is an x_0 such that $y = f(x)$ and $f'(x) = 0$, because an uncountable set $\{x; y = f(x)\}$ contains an accumulation point x_0 and clearly $f'(x_0) = 0$. For every $\epsilon > 0$ and every x_0 such that $f'(x_0) = 0$ there exists an open interval I_{x_0} such that if $x \in I_{x_0}$ then $|f'(x)| < \epsilon$. The union of all these intervals I_{x_0} may be written as a union of pairwise disjoint open intervals J_n. The image of each J_n is an interval (or a point) of length less than ϵ.length(J_n) due to Lagrange Mean Value Theorem. Thus the image of the interval [0,1] may be covered with the intervals such that the sum of their lengths is $1\epsilon = \epsilon$. This is not possible for $\epsilon < 1$. □

7.7.165 Problem. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable at $x = a$. Prove that

$$\lim_{n\to\infty} \left\{ f\left(a + \frac{1}{n^2}\right) + f\left(a + \frac{2}{n^2}\right) + ... + f\left(a + \frac{n}{n^2}\right) - nf(a) \right\} = \frac{1}{2} f'(a).$$

7.7.165.1 Solution. Hint: Write

$$\begin{aligned} & f\left(a + \frac{1}{n^2}\right) + f\left(a + \frac{2}{n^2}\right) + ... + f\left(a + \frac{n}{n^2}\right) - nf(a) \\ = & \left[f\left(a + \frac{1}{n^2}\right) - f(a) \right] + \left[f\left(a + \frac{2}{n^2}\right) - f(a) \right] + ... + \left[f\left(a + \frac{n}{n^2}\right) - f(a) \right], \end{aligned}$$

Now, divide and multiply each term by $\frac{1}{n^2}$ then after adding take the limit $n \to \infty$.

7.8 Additional Exercises on Chapter 7.

7.8.1 Exercise. Find the number of real solutions of the following equation

$$x^{18} + e^{-x} + 5x^2 - 2\cos x = 0.$$

7.8.2 Exercise. Verify if the following statement is true: if for some fixed $x, f'(x) = A$. Then

$$\lim_{h\to 0} \frac{f(x + 2h) + f(x) - 2f(x - h)}{4h} = A.$$

What about the converse statement? Provide a counterexample to the false statement.
Formulate similar statements for

$$\lim_{h\to 0} \frac{4f(x + h) - 3f(x) - f(x + 2h)}{2h} = A$$

in the form of counterexample?

7.8.3 Exercise. Verify if the following statement is true: if for some fixed x, $f''(x) = A$. Then

$$\lim_{h\to 0} \frac{f(x+2h) - 2f(x+h) + f(x)}{4h} = A.$$

What about the converse statement? Provide a counterexample to the false statement.
Formulate similar statements for

$$\lim_{h\to 0} \frac{f(x+2h) - f(x+h) - f(x) + f(x-h)}{2h^2} = A$$

in the form of counterexample?

7.8.4 Exercise. Provide a counterexample to the following statement: if $f'(x)$ exists and is bounded on (a, b), then $f'(x)$ cannot have infinitely many points of discontinuity.
Hint: Consider the function f on $(-1, 1)$ defined by

$$f(x) = \begin{cases} x^2 g_n(-x), & \text{if } x \in \left(-\frac{1}{n}, -\frac{1}{n+1}\right] \\ 0, & \text{if } x = 0 \\ x^2 g_n(x), & \text{if } x \in \left[\frac{1}{n+1}, \frac{1}{n}\right) \end{cases}$$

where

$$g_n(x) = g\left(\frac{x - \frac{1}{n+1}}{n(n+1)}\right)$$

$$g(x) = x(x-1)^2 \sin\frac{\pi}{(x-1)^2}.$$

7.8.5 Exercise. It is well-known that the derivative of a differentiable periodic function is also periodic with the same period. Show that the converse is false. Formulate the result as a counterexample.

7.8.6 Exercise. Provide a counterexample to the statement "if f is differentiable and periodic with a fundamental period T, then $f'(x)$ is also periodic with the fundamental period T".

7.8.7 Exercise. Give a counterexample to the statement: "if f is differentiable on an interval, then it is uniformly continuous on the same interval". What about the converse? Consider separately open and closed intervals.

7.8.8 Exercise. It is well-known that a function f that has a bounded derivative on a (finite or infinite) interval is uniformly continuous on that interval. Show that the uniform continuity and differentiability on an interval do not guarantee boundedness of derivative. (Hint: take $f(x) = \sqrt{x}$ on (0,1).)

7.8.9 Exercise. Recall that a function f is Lipschitz-continuous on an interval I if there exists a constant $C \geq 0$ such that $|f(x) - f(y)| \leq C|x - y|\ \forall\, x, y \in I$. Show that Lipschitz-continuity does not imply differentibility. What about the converse? What if f is infinitely differentiable on I?

7.8.10 Exercise. There are two apparently natural and similar definitions of a increasing at a point a function. The first says that f is increasing at a point a if there exists a neighborhood of a where f is increasing; and the second determines that f is increasing at a point a if there exists a neighborhood of a such that $f(x) \leq f(a)\ \forall x < a$ and $f(x) \geq f(a)\ \forall\, x > a$. (Analogous definitions can be specified for a strictly increasing function.) Show that these definitions are not equivalent.

7.8.11 Exercise. The theorem states that if $\lim_{x\to\infty} f(x) = A$ and $\lim_{x\to\infty}(f(x) - xf'(x)) = B$, then $y = Ax + B$ is a slant asymptote of f. Show that the converse is not true.

7.8.12 Exercise. Show that if $f : (a, b) \to \mathbb{R}$ has a local minimum at $c \in (a, b)$, and if f is twice differentiable at c, then $f''(c) \geq 0$.
Suppose that F satisfies the differential equation

$$F''(x) + F'(x)g(x) - F(x) = 0$$

for some function g. Prove that if F is 0 at two points, then F is 0 on the interval between them.

7.8.13 Exercise. Assume next that $f''(x)$ exists for every $x \in [a, b]$, and prove that f is convex if and only if for all $x \in [a, b]$.

7.8.14 Exercise. Let

$$f_a(x) = \begin{cases} x^a \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

1. Find $f_a'(x)$ for $x \neq 0$.
2. Find $f'(0)$ using the definition of derivative for $a > 1$.
3. Find $f'(0)$ using the definition of derivative for $a \leq 1$.
4. Does $\lim_{x\to 0} f'(x) = f'(0)$?

7.8.15 Exercise. Let $f : [a, \infty) \to \mathbb{R}$ and f' exists in (a, ∞) and $\lim_{x\to\infty} f(x)$ is finite. Prove that $\exists$ a sequence (ξ_n) in $[a, \infty)$ such that $\lim_{n\to\infty} \xi_n = \infty$ and $\lim_{n\to\infty} f'(\xi_n) = 0$.

7.8.16 Exercise. Let $f : [a, b] \to \mathbb{R}$ be continuous and f'' exists in (a, b) and $f(a) = f(b) = 0$, prove that if $f(c) > 0$ for some $c \in (a, b)$ then $\exists\, \xi \in (a, b)$ such that $f''(\xi) < 0$.

7.8.17 Exercise. Prove that if M is any real number, f is a real valued continuous function on $[a, b]$ which has a derivative $f'(x) \leq M$ at each $x \in (a, b)$, and $f(b) - f(a) = M(b - a)$, then $f(x) = f(a) + M(b - a)\ \forall\, x \in [a, b]$.

7.8.18 Exercise. The real-valued functions f, g have domain $\mathbb{R}$, satisfy the condition that $f(x) < g(x)\ \forall\, x \in \mathbb{R}$, and have derivatives $f'(x), g'(x)$ at each $x \in \mathbb{R}$. Is it true that $f'(x) < g'(x)\ \forall\, x$?

7.8.19 Exercise. The real-valued functions f and its first derivative f' are continuous on the interval $[0, \infty)$, and has positive derivative at each interior point of the interval. Prove that

1. if $\lim_{x\to\infty}\{xf'(x) - f(x)\} \leq 0$, then the function $x \mapsto f(x)/x$ is strictly decreasing on $(0, \infty)$.
2. if $f'(0) \leq 0$, then the function $x \mapsto f(x)/x$ is strictly increasing on $(0, \infty)$.
3. if $f'(0) = 0$, then the function $x \mapsto \begin{cases} f(x)/x & \text{if } x > 0 \\ f'(0) & \text{if } x = 0 \end{cases}$ is strictly increasing on $[0, \infty)$.

7.8.20 Exercise. The function f has a derivative at each point of (a, ∞), and
$\lim_{x\to\infty} f'(x) = \infty$. Prove that $\lim_{x\to\infty}(f(x)/x) = \infty$.

7.8.21 Exercise. The function f has a third derivative at x_0 and

$$\frac{1}{2h}\{f(x_0+h)-f(x_0-h)\} = f'(x_0)$$

for all sufficiently small positive h. Prove that $f'''(x_0) = 0$. The function g has domain $\mathbb{R}$ and has a third derivative at $x \in \mathbb{R}$. Prove that the relation

$$g(b)-g(a) = (b-a)g'\left(\frac{a+b}{2}\right)$$

holds for all real a, b iff g is quadratic.

7.8.22 Exercise. The function f is given by $f(x) = x - \sin x$. Prove that f has an inverse f^{-1}, and show that

$$\lim_{y\to 0}\left(\frac{f^{-1}(y)}{\sqrt[3]{y}}\right) = \sqrt[3]{6}.$$

7.8.23 Exercise. The function f is defined on some nbhd. of x_0, and has the property that there exists numbers $\alpha_1, \alpha_2, ..., \alpha_n$ such that

$$\lim_{x\to x_0}\frac{f(x)-f(x_0)-\alpha_1(x-x_0)-...-\alpha_n(x-x_0)^n}{(x-x_0)^n} = 0,$$

for all $n \geq 2$. Prove that f has a first derivative at x_0 equal to α_1, and that for $m = 2, 3, ..., n$

$$\alpha_m = \lim_{x\to x_0}\frac{f(x)-f(x_0)-\alpha_1(x-x_0)-...-\alpha_{m-1}(x-x_0)^{m-1}}{(x-x_0)^m}$$

(so that the α_m are uniquely determined). Does the condition imply that f has a second derivative at x_0?

7.8.24 Exercise. The real-valued functions f which has a third derivative f''' at x_0, let $f'(x_0) = 0$ and $f'''(x_0) \neq 0$, and let $\phi(x) = f(x) - f(x_0) - (x - x_0)f'(x_0)$. Prove that $\exists\, \delta > 0$ such that either $\phi(x)$ has the same sign as $x - x_0$ whenever $0 < |x - x_0| < \delta$ or $\phi(x)$ has the opposite sign as $x - x_0$ whenever $0 < |x - x_0| < \delta$. (This result implies that the curve with the equation $y = f(x)$ crosses its tangent line at x_0, i.e. that x_0 is the **point of inflexion** of the curve.)

7.8.25 Exercise. The real-valued functions g has a non-zero derivative at each point of an interval I. Prove that g is monotonic on I. Hence show that if f is continuous on the interval $[a, \infty)$ and has a derivative at each interior point of this interval and if $f(a) = \lim_{x\to\infty} f(x)$, then $\exists\, \xi \in (a, \infty)$ such that $f'(\xi) = 0$.

7.8.26 Exercise. The real-valued functions f has a derivative at each point of an interval $[a, b]$, and $f'(a) = f'(b)$. Prove that $\exists\, \xi \in (a, b)$ such that

$$\frac{f(\xi)-f(a)}{\xi - a} = f'(\xi).$$

[In geometric terms this result states that $\exists\, \xi \in (a, b)$ such that the tangent line at ξ passes through $(a, f(a))$.]

7.8.27 Exercise. Let $f, g : [a, b] \to \mathbb{R}$ be continuous and f', g' exist in (a, b), then prove that $\exists\, \xi \in (a, b)$ such that
$$(f(b) - f(a))g'(\xi) = (g(b) - g(a))f'(\xi).$$

7.8.28 Exercise. The real-valued functions f, g have domain $[a, b]$ satisfy the condition that $f'(x) \le g'(x)\ \forall\, x \in \mathbb{R}$, and have derivatives $f'(x), g'(x)$ at each $x \in [a, b]$,
$$|f(b) - f(a)| \le g(b) - g(a).$$

7.8.29 Exercise. Let $f : [a, b] \to \mathbb{R}$ be continuous and f', f'' exists in (a, b) and let $a < c < b$. By considering the function F given by
$$F(x) = f(x) - \alpha - \beta x - \gamma x^2,$$
then prove that $\xi \in (a, b)$ such that
$$\begin{vmatrix} 1 & a & f(a) \\ 1 & c & f(c) \\ 1 & b & f(b) \end{vmatrix} \div \begin{vmatrix} 1 & a & a^2 \\ 1 & c & c^2 \\ 1 & b & b^2 \end{vmatrix} = \frac{1}{2} f''(\xi),$$
and deduce that
$$f(c) - f(a) - \left(\frac{c-a}{b-a}\right)(f(b) - f(a)) = -\frac{1}{2}(b - c)(c - a)f''(\xi).$$

7.8.30 Exercise. Prove that if f is odd, then f' is even, and if f is even, then f' is odd. Can f' be odd or even when f is neither odd or even?

7.8.31 Exercise. Prove that if f is periodic, then f' is periodic, and if f has a period μ, then is it necessary that f' has a period μ?

7.8.32 Exercise. If f is convex on an interval I and α is any non-negative real number, then αf is convex on I, and that if f, g are convex on I, so is $f + g$. Prove also that if (f_n) is a sequence of functions each convex on I and $\lim_{n\to\infty} f_n(x) = f(x)$ for each $x \in I$, then f is convex on I.

7.8.33 Exercise. Prove also that if $\{f_\alpha; \alpha \in A\}$ is a family of functions each convex on I and such that the set $\{f_\alpha; \alpha \in A\}$ is bounded above for each $x \in I$, then the function $x \mapsto \sup_{\alpha\in A} f_\alpha(x)$ is convex on I.

7.8.34 Exercise. Let f be increasing and convex on I and g be a function convex on an interval J such that $g(J) \subseteq I$. Prove that $f \circ g$ is convex on J. Deduce that for a function h, if $\log h$ is convex on an interval J, then h is convex on J.

7.8.35 Exercise. Prove that if f is convex on I and has a derivative at an interior point x_0 of I, then for all $x \in I$, we have
$$f(x) \ge f(x_0) + (x - x_0)f'(x_0).$$
[The inequality states that the graph of f lies above the tangent line to this graph at x_0.]

7.8.36 Exercise. The real-valued function f has the domain $[a, b]$ satisfies the condition that $f(x) > 0\ \forall\, x \in I$, and the derivative $f''(x)$ exists at each $x \in [a, b]$, Prove that $\log f$ is convex on $[a, b]$ iff $f(x)f''(x) \ge (f'(x))^2\ \forall\, x \in [a, b]$.

7.8.37 Exercise. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is convex, $f(5) = 12$ and $\alpha = \lim_{x\to\infty} f(x)$. What are the possible values of α?

7.8.38 Exercise. Consider the set S of all Cauchy sequences $(a_n) \in \mathbb{N}^{\mathbb{N}}$. Is the set S countable? Jusify.

7.8.39 Exercise. Let A be a compact set of $\mathbb{R} \setminus \{0\}$ and B be a closed subset of $\mathbb{R}$. Prove that the set $\{a.b; a \in A, b \in B\}$ is closed in $\mathbb{R}$.

7.8.40 Exercise. Prove that there exists a constant $c > 0$ such that $\forall\, x \in [1, \infty)$,

$$\sum_{n\geq x} \frac{1}{n^2} \leq \frac{c}{x}.$$

7.8.41 Exercise. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ such that $f'''(x)$ exists for all $x \in \mathbb{R}$ but is discontinuous at $x = 0$.

7.8.42 Exercise. Show that the function

$$f(x) = \begin{cases} x, & \text{if } x \in \mathbb{Q} \\ -x, & \text{if } x \in \mathbb{Q}^C \end{cases}$$

is nowhere differentiable. Show, however, that $(f \circ f)(x) = x \;\forall\, x \in \mathbb{R}$.

7.8.43 Exercise. Suppose that $f(x) = xg(x)$ where g is continuous at $x = 0$. Show that f is differentiable at $x = 0$ and find $f'(0)$.

7.8.44 Exercise. A function f is defined on some nbhd. $N(0; r)$ of 0 by

$$f(x) = \begin{cases} \dfrac{x}{1 + e^{\frac{1}{x}}}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

Find $Lf'(0)$ and $Rf'(0)$. Show that f is not differentiable at 0.

7.8.45 Exercise. A function f is defined by

$$f(x) = \begin{cases} x\left(\dfrac{e^{\frac{1}{x}} - e^{-\frac{1}{x}}}{e^{\frac{1}{x}} + e^{-\frac{1}{x}}}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

Show that f is continuous at 0 but not differentiable at 0.

7.8.46 Exercise. Let $f(x) = \begin{cases} x^2 \sin\frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$ and $g(x) = x,\ \forall\, x \in \mathbb{R}$. Show that $\lim_{x\to 0} \frac{f'(x)}{g'(x)}$ does not exist, but $\lim_{x\to 0} \frac{f(x)}{g(x)} = \frac{f'(0)}{g'(0)}$.

7.8.47 Exercise. Let I be an interval and $f : I \to \mathbb{R}$, suppose $a \in I$. Consider the following statement: the function f has a relative maximum value at a if there exists an interval $J \subseteq I$ such that $a \in J$ and $f : J \to \mathbb{R}$ has maximum value at a. Show that this statement is false. Then determine how to modify the statement so that it is true.

7.8.48 Exercise. Give an example of a function $f : (0, 2) \to \mathbb{R}$ such that f has relative maximum value at 1, but not bounded above on $(0, 2)$.

7.8.49 Exercise. Find the derivatives of the functions$f$ and g defined by

$$f(x) = \begin{cases} \frac{x}{x+1}, & \text{if } x \geq 0 \\ \frac{x}{1-x}, & \text{if } x < 0. \end{cases}$$

and $g(x) = \frac{x}{|x|-1}$, if $x \neq \pm 1$.

7.8.50 Exercise. Suppose f is a real valued function on an open interval I, and differentiable at every $x \in I$. If $[a, b] \subset I$ and $f'(a) < 0 < f'(b)$, then show that there exists $c \in (a, b)$ such that

$$f(c) = \inf_{x \in [a,b]} f(x).$$

7.8.51 Exercise. Let $f : [0, 1] \to \mathbb{R}$ is continuous, and differentiable on (0,1) with $f'(x) - f(x) \geq 0 \;\forall\, x \in [0, 1]$ and $f(0) = 0$, prove that $f(x) \geq 0 \;\forall\, x \in [0, 1]$.

7.8.52 Exercise. Suppose that f is continuously differentiable on (a, b). Prove that f is uniformly differentiable on $[c, d] \subseteq (a, b)$ in the sense that for $n \in \mathbb{N}, \exists\, m \in \mathbb{N}$ such that for $p \in [c, d]$,

$$|f(x) - f(p) - f'(p)(x - p)| \leq \frac{|x - p|}{n}$$

whenever $|x - p| < \dfrac{1}{m}$.

7.8.53 Exercise. Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiabile function. Suppose that $\delta = \inf f'(x) > 0$, prove that $f(a) = 0$ for some $a \in \mathbb{R}$.

7.8.54 Exercise. Let $f : [0, 1] \to \mathbb{R}$ be differentiable, $|f'(x)| < 1 \;\forall\, x \in [0, 1]$. Show that there exists at most one $c \in [0, 1]$ such that $f(c) = c$.

7.8.55 Exercise. If in some nbhd. of x,

$$\left| \frac{f(x + h) - f(x)}{h} - f'(x) \right| < \epsilon$$

for $|h| < \eta(\epsilon)$ independent of x, we say that $f(x)$ is differentiable uniformly in the nbhd. of x. Prove that uniform differentiability of $f(x)$ over an interval is the necessary and sufficient for the continuity of $f'(x)$ throughout the interval.

7.8.56 Exercise. If $f(x) \to 0$ and $f(x)g(x) \to L$ as $x \to 0$, show that

$$\lim_{x \to 0} [1 + f(x)]^{g(x)} = e^L.$$

7.8.57 Exercise. If $\lim_{x\to\infty} f(x) = \infty$, show that $\lim_{x\to\infty} f'(x) = L$ secures

$$\lim_{x\to\infty} \{f(x + 1) - f(x)\} = L$$

which again secures $\lim_{x\to\infty} \frac{f(x)}{x} = L$ but we cannot argue in the reverse direction, even if $f(x)$ be monotonic.

7.8.58 Exercise. Let $f : (0, \infty) \to \mathbb{R}$ satisfy $f(xy) = f(x)+f(y)$ for all $x, y \in \mathbb{R}$. If f is differentiable at $x = 1$, show that f is differentiable at every $c \in (0, \infty)$ and $cf'(c) = f'(1)$. Show that f is in fact infinitely differentiable.

7.8.59 Exercise. Let $f : \mathbb{R} \to \mathbb{R}$ satisfy $f(x + y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$. If f is differentiable at $x = 1$, show that f is differentiable at every $c \in (0, \infty)$ and $f'(c) = f'(0)f(c)$. Show that f is in fact infinitely differentiable.

7.8.60 Exercise. (Straddle Lemma.) Let $f : I \to \mathbb{R}$ be differentiable at $c \in I$. Given $\epsilon > 0$ show that there exists $\delta > 0$ such that if u, v satisfy $c - \delta < u \le c \le v < c + \delta$ then we have

$$|f(v) - f(u) - (v - u)f'(c)| \le \epsilon|v - u|.$$

7.8.61 Exercise. Let $f : [a, b) \to \mathbb{R}$ be differentiable on (a, b), continuous on $[a, b)$, and let the limit $\lim_{x\to a+} f'(x) = L$ exist. Prove that the right derivative $f'_+(a)$ exists and that $f'_+(a) = L$. Formulate for the left derivative $f'_-(b)$ and then prove.

7.8.62 Exercise. Use the MVT to show that if a function $f : (a, b) \to \mathbb{R}$ is twice differentiable in (a, b) with $f''(x) > 0$, then f is strictly convex in (a, b). (f is strictly convex in (a, b) if $f(tx + (1 - t)y) < tf(x) + (1 - t)f(y)$ for all $x, y \in (a, b)$ and $0 < t < 1$.)

7.8.63 Exercise. Prove that $\frac{x}{1+x} < \log(1 + x) < x$ for all $x > 0$. Deduce that

$$\log \frac{2n + 1}{n + 1} < \frac{1}{n + 1} + \frac{1}{n + 2} + ... + \frac{1}{2n} < \log 2.$$

n being a positive integer.

7.8.64 Exercise. If $y = \frac{\log x}{x}$, then prove that

$$y_n = \frac{(-1)^n n!}{x^{n+1}} \left[\log x - 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - .. - \frac{1}{n}\right].$$

7.8.65 Exercise. If $y = x \log \frac{x-1}{x+1}$, then prove that

$$y_n = (-1)^n (n - 2)! \left[\frac{x - n}{(x - 1)^n} - \frac{x + n}{(x + 1)^n}\right].$$

7.8.66 Exercise. If $y = x^{n-1} \log x$, then prove that $y_n = \frac{(n-1)!}{x}$.

7.8.67 Exercise. If $y = x^n \log x$, then prove that

$$y_n = n! \left[\log x + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + .. + \frac{1}{n}\right].$$

7.8.68 Exercise. If $y = x^{n-1}e^{\frac{1}{x}}$, prove that $y_n = \left[(-1)^n e^{\frac{1}{x}}\right] / x^{n+1}$.

7.8.69 Exercise. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable, $\exists\ \alpha \in (0, 1)$ such that $|f'(x)| \le \alpha\ \forall\ x \in \mathbb{R}$. Let $a_1 \in \mathbb{R}$ and $a_{n+1} = f(a_n)\ \forall\ n \in \mathbb{N}$. Prove that (a_n) converges.

7.8.70 Exercise. If $f : \mathbb{R} \to \mathbb{R}$ is continuous on each compact subset of $\mathbb{R}$, then f is continuous on $\mathbb{R}$.

7.8.71 Exercise. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable, $|f'(x)| < 1 \; \forall \; x \in \mathbb{R}$. Show that $f(x) \neq x \; \forall \; x \neq 0$.

7.8.72 Exercise. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable at c and $f(c) = 0$. Prove that $|f|$ is differentiable at c iff $f'(c) = 0$.

7.8.73 Exercise. If $f : (a, b) \to \mathbb{R}$ is differentiable on (a, b) and $a < c < b$, $f'(c) > 0$, then $\{x \in (a, b); f'(x) > 0\}$ is infinite.

7.8.74 Exercise. Let $c \in (a, b)$, and let $f : (a, b) \to \mathbb{R}$ be such that f is

1. differentiable on $(a, b) \setminus \{c\}$,

2. continuous on (a, b), and

3. $\lim_{x \to c} f'(x)$ exists.

Then f is differentiable at c, and $f'(c) = \lim_{x \to c} f'(x)$.

7.8.75 Exercise. Let D be an open set and $f : D \to \mathbb{R}$ be differentiable on D and $\exists \; a \in D$ such that $\lim_{x \to a} f'(x)$ exists. Prove that,

$$\lim_{x \to a} f'(x) = f'(a),$$

that is, f' is continuous at a.

7.8.76 Exercise. Let $f : [-1, 1] \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} 0, & \text{if } x \in [-1, 0] \\ 1, & \text{if } x \in (0, 1], \end{cases}$$

prove that there does not exist a differentiable function $F : [-1, 1] \to \mathbb{R}$ such that $F' = f$.

7.8.77 Exercise. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function. Suppose $\delta = \inf_{x \in \mathbb{R}} f'(x) > 0$. Prove that $f(a) = 0$ for some $a \in \mathbb{R}$.

7.8.78 Exercise. Let $a < b, c < d$. Define

$$f(x) = \begin{cases} ax \sin^2 \frac{1}{x} + bx \cos^2 \frac{1}{x} & \text{if } x > 0 \\ 0; & \text{if } x = 0 \\ cx \sin^2 \frac{1}{x} + dx \cos^2 \frac{1}{x} & \text{if } x < 0. \end{cases}$$

Calculate $D^+ f, \; D^- f, \; D_+ f, \; D_- f$ at $x = 0$.

7.8.79 Exercise. Let $f(x) = x^2 \sin(x^{-2})$ for $x \in [-1, 1], x \neq 0$ and $f(0) = 0$. Show that f is differentiable on $[-1, 1]$ but f' is unbounded on $[-1, 1]$.

7.8.80 Exercise. If f is continuous and has a finite derivative at every point in [0,1], can f' have a removable discontinuity? can f' have a discontinuity of the second kind? Illustrate your conclusion in each case.

7.8.81 Exercise. Let f have a finite derivatiive in (a, b) and assume that $c \in (a, b)$. Consider the following condition: For every $\epsilon > 0$, there exists a ball $B(c; \delta)$; where δ depends only on $\epsilon > 0$ and not on c, such that if $x \in B(c; \delta)$; , and $x \neq c$, then

$$\left| \frac{f(x) - f(c)}{x - c} - f'(c) \right| < \epsilon.$$

Show that f' is continuous on (a, b) if this condition holds throughout (a, b).

7.8.82 Exercise. Let f be a twice differentiable function on (0,1). It is given that for all $x \in (0, 1)$, $|f''(x)| \leq M$ where M is a non-negative real number. Prove that f is uniformly continuous on (0,1).

7.8.83 Exercise.

1. Prove that f is convex if and only if the set S of points above its graph is convex in $\mathbb{R}^2$. The set $S = \{(x, y) : f(x) \leq y\}$.

2. Prove that a convex function is continuous.

3. Suppose that f is convex and $a < x < u < b$. The slope σ of the line through $(x, f(x))$ and $(u, f(u))$ depends on x and $u, \sigma = \sigma(x, u)$. Prove that σ increases when x increases, and σ increases when u increases.

4. Suppose that f is second-order differentiable. Prove that f is convex if and only if $f''(x) \geq 0 \; \forall \; x \in (a, b)$.

7.8.84 Exercise. Let I be an interval, let x_0 be an interior point of I, and let $n \geq 2$. Suppose that the derivatives $f', f'', .., f^{(n)}$ exist and are continuous in a neighborhood of x_0 and that $f'(x_0) = ... = f^{(n-1)}(x_0) = 0$, but $f^{(n)}(x_0) \neq 0$. Prove that

1. If n is even and $f^{(n)}(x_0) > 0$, then f has a relative minimum at x_0.

2. If n is even and $f^{(n)}(x_0) < 0$, then f has a relative maximum at x_0.

3. If n is odd, then f has neither a relative maximum nor a relative minimum at x_0.

7.8.85 Exercise. Let $f : [a, b] \to \mathbb{R}$ be differentiable and $a \geq 0$. Using Cauchy mean value theorem, show that there exist $c_1, c_2 \in (a, b)$ such that $\frac{f'(c_1)}{a+b} = \frac{f'(c_2)}{2c_2}$.

7.8.86 Exercise. Let $f : [0, 1] \to \mathbb{R}$ be differentiable and $f(0) = 0$. Suppose that $|f'(x)| \leq |f(x)| \; \forall \in [0, 1]$. Show that $f(x) = 0 \; \forall \in [0, 1]$.

7.8.87 Exercise. Let $f : [0, 1] \to \mathbb{R}$ be twice differentiable. Suppose that the line segment joining the points $(0, f(0))$ and $(1, f(1))$ intersect the graph of f at a point $(a, f(a))$ where $0 < a < 1$. Show that there exists a point $c \in [0, 1]$ such that $f''(c) = 0$.

7.8.88 Exercise. Let $a \in \mathbb{R}$. If

$$f(x) = \begin{cases} (x + a)^2 & \text{if } a \leq 0 \\ (x + a)^3 & \text{if } a > 0 \end{cases}$$

then which of the following is true?

1. $\frac{d^2f}{dx^2}$ does not exist at $x = 0$ for any value of a.
2. $\frac{d^2f}{dx^2}$ exists at $x = 0$ for exactly one value of a.
3. $\frac{d^2f}{dx^2}$ exists at $x = 0$ for two values of a.
4. $\frac{d^2f}{dx^2}$ exists at $x = 0$ for infinitely many values of a.

7.8.89 Exercise. Let $f(x) = (\log x)^2$; $x > 0$. Then which of the following is true?

1. $\lim_{x\to\infty} \frac{f(x)}{x}$ does not exist.
2. $\lim_{x\to\infty} f'(x) = 2$.
3. $\lim_{x\to\infty}(f(x+1) - f(x)) = 0$.
4. $\lim_{x\to\infty}(f(x+1) - f(x))$ does not exist.

7.8.90 Exercise. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $f'(x) > f(x)$ for all $x \in \mathbb{R}$, and $f(0) = 1$. Then which of the following is true?

1. $f(1) \in (0, e^{-1})$.
2. $f(1) \in (e^{-1}, \sqrt{e})$.
3. $f(1) \in (\sqrt{e}, e)$.
4. $f(1) \in (e, \infty)$.

Chapter 8

Compactness and Connectedness

The only thing I know is that I don't know anything:
- Socrates

8.1 Introduction

In analysis it is often important to know when a property that is valid at each point of a space is in some sense uniformly valid over the space. For instance, the question of when a function that is continuous at each point of a space is in fact uniformly continuous on the space. The condition under which such uniformity tends to occur for sets in $\mathbb{R}$ or $\mathbb{R}^n$ is that the set to be closed and bounded. However, this condition of being closed and bounded, for sets in arbitrary metric space, does not guarantee that the set has the properties we desire. There is another condition, called **compactness**, that does guarantee that the set has these properties. There is a certain class of properties that finite sets have trivially, that are retained by compactness. For instance, it will be seen that every continuous real valued function on a compact space is bounded and assumes its maximum and minimum values. In the theory of metric spaces, the notion of compactness is a substitute for the notion of "**finiteness**" in pure set theory; and the notion of precompactness in the metric space is, so to speak, "**approximately finite**." Note that, from the definition, it will follow that compactness is a topological notion, but precompactness is not.
Our aim in this section is to define the notion of compactness and to prove some important characterization of some subsets of $\mathbb{R}$.

8.2 Completeness

8.2.1 Definition. A subset $A \subseteq \mathbb{R}$ is "**complete**" if every Cauchy sequence (x_n) of elements $x_n \in A$ has a limit $x \in A$.
A space (set) can be "**completed**" means that by collecting all "*missing*" limit points we can make an incomplete space complete.

8.3 Compact Sets

8.3.1 Definition. Let $S \subseteq \mathbb{R}$, then a family $\mathcal{C} = \{A_\alpha; \alpha \in \Lambda\}$ of subsets of $\mathbb{R}$ is said to be a **cover** of a set S iff $S \subseteq \bigcup_{\alpha\in\Lambda} A_\alpha$. A subfamily $\mathcal{B} \subseteq \mathcal{C}$ which also covers S is called a **subcover** of S. If every member of the cover $\mathcal{C}$ is an open set, then $\mathcal{C}$ is said to be an **open** cover of S.

8.3.2 Definition. A subset K of $\mathbb{R}$ is compact iff it satisfies the following equivalent conditions:

1. Every sequence (x_n) in K has at least one cluster point in K.
2. Every open cover has a finite subcover. (also called the **Borel-Lebesgue** property).(**Open-cover compact**:)
3. Every sequence has a convergent subsequence. (**Sequentially compact**)
4. Every countable open cover has a finite subcover. (**Countably compact**)
5. Every infinite subset K has a limit point in K. (also called **Limit-point compact**, **Fréchet compact** or the **Bolzano-Weierstrass property**).

8.3.3 Remark (Topological characterization)**.** A subset $S \subseteq \mathbb{R}$ is said to be a **compact** subset of $\mathbb{R}$ iff every open cover of S has a finite subcover. In other words, S is compact subset of $\mathbb{R}$ if, whenever $\mathcal{A} = \{A_\alpha; \alpha \in \Lambda\}$ is an open cover of S, there exists a finite subfamily $\mathcal{B} = \{A_{\alpha_i} \in \mathcal{A}; i = 1, 2, .., n\}$ of $\mathcal{A}$, such that

$$S \subseteq \bigcup_{i=1}^{n} A_{\alpha_i}.$$

We can show that the conditions are equivalent.

8.3.4 Example. A finite set is compact. Let $A = \{a_1, a_2, ..., a_n\} \subseteq \mathbb{R}$, and $\mathcal{A} = \{A_\alpha; \alpha \in \Lambda\}$ be any open cover of A. Then select A_{α_i} in which $a_i \in A_{\alpha_i}$ for each $i = 1, 2, .., n$. Hence $\{A_{\alpha_i}; i = 1, 2, .., n\}$ is a finite subcover of $\mathcal{A}$.

8.3.5 Example. $\mathbb{R}$ is not compact.
Consider the cover $\mathcal{A} = \{A_n = (-n, n); n = 1, 2, ..\}$ of $\mathbb{R}$. We observe that

$$A_1 \subseteq A_2 \subseteq .... \subseteq A_n \subseteq A_{n+1} \subseteq .....$$

Now, suppose that there exist a finite subcover $\{A_{n_i}; n_i \in \mathbb{N}, i = 1, 2, .., k\}$ of $\mathbb{R}$. Let $m = \max\{n_i; i = 1, 2, .., k\}$, then $A_m = \bigcup_{i=1}^{k} A_{n_i} = \mathbb{R}$. i.e. $(-m, m) = \mathbb{R}$, a contradiction. Hence $\mathbb{R}$ is not compact.

8.4 Finite Intersection Property:

8.4.1 Definition. A family $\mathcal{F}$ of sets is said to have the **finite intersection property (FIP)** iff the intersection of the members of each finite subfamily of $\mathcal{F}$ is nonempty.

8.4.2 Definition (Nested sequences of sets)**.** A sequence (A_n) of sets is said to be a **nested sequence** if $A_1 \supseteq A_2 \supseteq ... \supseteq A_n \supseteq A_{n+1} \supseteq .....$

8.5 Problems and Solutions on Compactness.

Warning! *Never Never Ever read this solution unless you tried problems quite long time and gave up. Doing so may impair your ability to think and solve problems.*

Our aim is to describe all compact subsets of $\mathbb{R}$.

8.5.1 Problem. Every closed subset C of a compact set S is compact.

8.5.1.1 Solution. Let $\mathcal{A} = \{A_\alpha; \alpha \in \Lambda\}$ be any open cover of C. Then $\mathcal{A} \cup \{C^C\}$ is an open cover of S, as C is closed. Since S is compact, so there exists $\alpha_1, \alpha_2, ..., \alpha_n$ such that $S \subseteq \bigcup_{i=1}^n A_{\alpha_i} \cup C^C$. Hence $C \subseteq \bigcup_{i=1}^n A_{\alpha_i}$ shows that C is compact. □

8.5.2 Problem. Every continuous image of a compact set is compact.

8.5.2.1 Solution. Let C be a compact set in $\mathbb{R}$ and $f : C \to \mathbb{R}$. Suppose that $\mathcal{A} = \{A_\alpha; \alpha \in \Lambda\}$ be any open cover of $f(C)$. Since f is continuous, $f^{-1}(A_\alpha)$ is open for each $\alpha \in \Lambda$, so $\{f^{-1}(A_\alpha); \alpha \in \Lambda\}$ is an open cover of C. Since C is compact, so there exists $\alpha_1, \alpha_2, ..., \alpha_n \in \Lambda$ such that $C \subseteq \cup_{i=1}^n f^{-1}(A_{\alpha_i})$. Hence $f(C) \subseteq f(\cup_{i=1}^n f^{-1}(A_{\alpha_i})) \subseteq \cup_{i=1}^n A_{\alpha_i}$. Thus $f(C)$ is compact. □

8.5.3 Problem. Let A be a subset of $\mathbb{R}$. Then the following three assertions are equivalent:

1. A is closed.
2. A contains all its points of accumulation.
3. If (a_n) is a sequence in A, converges to $a \in \mathbb{R}$, then $a \in A$.

8.5.3.1 Solution. Left to the reader. □

8.5.4 Problem. (M. Fréchet) If K is a subset of $\mathbb{R}$, then the following assertions are equivalent:

1. K is closed and bounded subset;
2. Every sequence of points of K contains a subsequence convergent to a point also belonging to K.

8.5.4.1 Solution.

1. (1) implies (2) follows from the problem above and the Bolzano-Weierstrass Theorem.
2. (2) implies (1). The fact that K is closed also follows from the problem above. If K were not bounded, then no finite union of balls of radius 1, centered at the points of K can include K. In other words, for every finite subset F of K,

$$K \setminus \bigcup_{x \in F} B(x; 1) \neq \emptyset.$$

Fix arbitrarily $x_1 \in K$ and choose $x_2 \in K \setminus B(x_1; 1)$ that is, $|x_1 - x_2| \geq 1$. Repeating the argument above, we choose a point x_3 in $K \setminus B(x_1; 1) \cup B(x_2; 1)$, which assures $|x_1 - x_3| \geq 1$ and $|x_2 - x_3| \geq 1$. By mathematical induction, we conclude the existence of a sequence (x_n) of elements of K such that

$$|x_i - x_j| \geq 1 \ \forall \ i < j.$$

Such sequence cannot contain any Cauchy subsequence, a fact that contradicts our hypothesis. Therefore K is a bounded set. □

8.5.5 Problem.

1. Every closed subset $A \subseteq \mathbb{R}$ is complete.

2. If $A \subseteq \mathbb{R}$ is complete, then A is closed.

8.5.5.1 Solution.

1. Let (x_n) be a Cauchy sequence in A. Then (x_n) converges to x in $\mathbb{R}$, and since A is closed, x belongs to A. Thus A is complete.

2. Let (x_n) be a sequence in A converging to an element x of $\mathbb{R}$. Thus (x_n) is a Cauchy sequence in $\mathbb{R}$ and in A, and since A is complete, (x_n) converges to an element x belonging to A. Thus A is closed. □

8.5.6 Problem. Let (A_n) be a decreasing sequence of nonempty closed subsets A_n of $\mathbb{R}$ such that

$$\lim_n \delta(A_n) = 0$$

where $\delta(A_n)$, is the diameter of A_n. Then

$$\bigcap_{n=1}^{\infty} A_n = \{\xi\}$$

for some $\xi \in \mathbb{R}$.

8.5.6.1 Solution. Suppose that (A_n) is a decreasing sequence of closed sets (i.e. $A_n \supset A_{n+1}; n \in \mathbb{N}$) with $\delta(A_n) \to 0$. Note that $\delta(A) = \sup\{|x-y|; x, y \in A\}$. Let $a_n \in A_n$. Since $A_n \subset A_p \ \forall \ n \geq p$, so $a_n \in A_n$ implies $a_n \in A_p \ \forall \ n \geq p$. Now, the sequence (a_n) is a Cauchy sequence in $\mathbb{R}$: For since $\delta(A_n)$ tends to 0, so $\forall \ \epsilon > 0 \ \exists \ n_0 \in \mathbb{N}$ such that $n \geq n_0 \Rightarrow \delta(A_n) \leq \delta(A_{n_0}) < \epsilon$. Thus if $m, n \geq n_0$ then $|a_m - a_n| \leq \delta(A_{n_0}) < \epsilon$. Since $\mathbb{R}$ is complete, the sequence (a_n) converges to an element $a \in \mathbb{R}$. Moreover, for each fixed n, the sequence a_{n+p} converges to a as p goes to infinity. Since $a_{n+p} \in A_n$ for every p and since A is closed, we conclude that $a \in \overline{A_n} = A_n$. Thus $a \in \bigcap_n A_n = A(\text{say})$. Since $\delta(A) \leq \delta(A_n)$ for $A \subseteq A_n \ \forall \ n$ and since $\delta(A_n)$ tends to 0, then $\delta(A) = 0$, which implies that A contains exactly one point. □

8.5.7 Problem. The following are equivalent:

1. A subset $A \subseteq \mathbb{R}$ is said to be **totally bounded** if, given any $\epsilon > 0$, there exist finitely many points $x_1, x_2, ..x_n \in \mathbb{R}$ such that $A \subseteq \bigcup_{i=1}^{n} B(x_i, \epsilon)$.

2. A subset $A \subseteq \mathbb{R}$ is said to be **totally bounded** if, given any $\epsilon > 0$, there exist finitely many points $y_1, y_2, ..y_m \in A$ such that $A \subseteq \bigcup_{i=1}^{m} B(y_i, \epsilon)$.

8.5.7.1 Solution. Exercise. □

8.5.8 Problem. Let $K \subseteq \mathbb{R}$. Suppose that for all continuous functions $f : K \to \mathbb{R}$, $f(K)$ is bounded. Then K is compact.

8.5.8.1 Solution. Assume that K is not compact. Then we will show that there is a continuous function $f : K \to \mathbb{R}$ such that $f(K)$ is unbounded. Since K is not compact, it is true that K is not closed or it is not bounded (perhaps both).

1. K is not closed. Then there is some sequence (x_n) in K that converges to a point c not in K. Then the function $f(x) = \frac{1}{x-c}$, which is continuous, satisfies the condition that $f(K)$ is unbounded.

2. K is not bounded. Then the function $f(x) = x$ is continuous and $f(K)$ is unbounded. □

8.5.9 Problem. Are there infinite compact subsets of $\mathbb{Q}$? Prove your assertion.

8.5.9.1 Solution. Yes, $\left\{\frac{1}{n}; n \in \mathbb{N}\right\} \cup \{0\}$ is an infinite compact subset of $\mathbb{Q}$. Clearly this set is bounded by 1, and it contains its only limit point of 0, hence it is closed. Thus it is compact. □

8.5.10 Problem. Prove that the interval $(-1, 1)$ in $\mathbb{R}$ is homeomorphic to $\mathbb{R}$. This shows that neither boundedness nor completeness is a topological property.

8.5.10.1 Solution. $f(x) = \frac{2x}{1-x^2}$. □

8.5.11 Problem. Let $f : \mathbb{R} \to \mathbb{R}$ then for all bounded subsets $A \subseteq \mathbb{R}$, $f(A)$ is bounded.

8.5.11.1 Solution. Let $A \subseteq \mathbb{R}$ be bounded, then $\exists$ a closed interval $[a, b]$ containing A, since $[a, b]$ is compact, so $f[a, b]$ is also compact. Thus $f(A) \subseteq f[a, b]$ is bounded. □

8.5.12 Problem. Prove that f is continuous on S iff f is continuous on every compact subset of S.

8.5.12.1 Solution. Hint. If $x_n \to p \in S$, the set $\{p, x_1, x_2, ...\}$ is compact. □

8.5.13 Problem. A set $S \subseteq \mathbb{R}$ is compact iff every family of closed sets in S which has the finite intersection property has a nonempty intersection.

8.5.13.1 Solution. Let $\mathcal{C} = \{A_i; i \in \Lambda\}$ be a family of closed sets having FIP. We show that $\bigcap_{i\in\Lambda} A_i \neq \emptyset$. Again, $\{A_i^C; i \in \Lambda\}$ is the family of open sets. Suppose that

$$\bigcap_{i\in\Lambda} A_i = \emptyset \Rightarrow \bigcup_{i\in\Lambda} A_i^C = \mathbb{R}.$$

So $\mathcal{A} = \{A_i^C; i \in \Lambda\}$ is a cover of $\mathbb{R}$ and hence is a cover of S, since S is compact so $\exists$ a finite subfamily $\mathcal{B} = \{A_j^C; j = 1, 2, .., n\}$ of $\mathcal{A}$ such that

$$\begin{aligned} S \subseteq \bigcup_{j=1}^{n} A_j^C &\Rightarrow S^C \supseteq \bigcap_{j=1}^{n} A_j \\ \Rightarrow S \cap S^C \supseteq S \cap \bigcap_{j=1}^{n} A_j &= \bigcap_{j=1}^{n} A_j \Rightarrow \bigcap_{j=1}^{n} A_j = \emptyset, \end{aligned}$$

a contradiction that $\mathcal{C}$ does not have FIP. □

8.5.14 Problem. Any countably compact subset $S \subseteq \mathbb{R}$ is Fréchet compact; i.e., every infinite subset $K \subseteq S$ has a limit point in S.

8.5.14.1 Solution. Since every infinite set contains a countably infinite subset, we may assume that K is countably infinite. So let $K = \{x_1, x_2, ...\}$. If no $x \in K$ is a limit point of S; then each x_n is an isolated point of K and K is closed. (Why?) For each $n \in \mathbb{N}$, let B_n be an open ball with $K \cap B_n = \{x_n\}$ The collection $\{B_n\} \cup K^C$ is then a countable open cover of S with no finite subcover, contradicting the countable compactness of S. □

8.5.15 Problem. (Uniform Continuity and Compactness) Let S be a compact set in $\mathbb{R}$. If $f : S \to \mathbb{R}$ is continuous, then f is also uniformly continuous on S.

8.5.15.1 Solution. Let $\epsilon > 0$. Since f is continuous at each $a \in S$, so there exists $r > 0$ depending on a such that $f(N(a;r) \cap S) \subseteq N(f(a);\epsilon/2)$. i.e. Now, consider the collection $\mathcal{C}$ of all nbhds $N(a;r/2)$ for each $a \in S$. Then $\mathcal{C}$ is a cover of S, since S is compact, so, there exists a finite subcover, say $\mathcal{B} = \{N(a_1;r_1/2), ..., N(a_k;r_k/2)\}$ of S i.e.

$$S \subseteq N(a_1;r_1/2) \cup ... \cup N(a_k;r_k/2).$$

Let $2\delta = \min\{r_1, r_2, .., r_k\}$, and $x, y \in S$ with $|x - y| < \delta$. Now, $x \in N(a_i, r_i)$ for some $1 \leq i \leq k$. Then $|f(x) - f(a_i)| < \epsilon/2$, and we have

$$\begin{aligned} |y - a_i| &= |y - x + x - a_i| \\ &\leq |y - x| + |x - a_i| \\ &< \delta + r_i/2 \\ &\leq r_i/2 + r_i/2 = r_i \Rightarrow y \in N(a_i;r_i) \cap S. \end{aligned}$$

Hence $|f(x) - f(y)| \leq |f(x) - f(a_i)| + |f(a_i) - f(y)| < \epsilon/2 + \epsilon/2 = \epsilon$. Thus the result follows. □

8.5.15.2 Solution. If f is not uniformly continuous on S; then $\exists\ \epsilon > 0$ and two sequences $(x_n), (y_n) \in S^{\mathbb{N}}$ such that $\lim_n |x_n - y_n| = 0$ and $|f(x_n) - f(y_n)| \geq \epsilon\ \forall\ n \in \mathbb{N}$. (Why?) Since the compact space S is sequentially compact, there is a subsequence (x_{n_k}) of (x_n) such that $\lim x_{n_k} = x_0$ for some $x_0 \in S$. But then $\lim_n |x_{n_k} - y_{n_k}| = 0$ implies that we also have $\lim x_{n_k} = x_0$. Therefore, by the continuity of f at x_0,

$$\lim_{k\to\infty} f(x_{n_k}) = \lim_{k\to\infty} f(y_{n_k}) = f(x_0).$$

This, however, is impossible since $|f(x_n) - f(y_n)| \geq \epsilon\ \forall\ n \in \mathbb{N}$. □

8.5.16 Problem. Every compact subset of $\mathbb{R}$ is closed and bounded.

8.5.16.1 Solution. Suppose that A is a compact subset of $\mathbb{R}$ and that p is a limit of A. There is a sequence (a_n) in A converging to p. By compactness, some subsequence (a_{n_k}) converges to some $q \in A$, but every subsequence of a convergent sequence converges to the same limit as does the sequence (a_n), so $q = p$ and $p \in A$. Thus, A is closed. To see that A is bounded, choose and fix any point $p \in \mathbb{R}$. Either A is bounded or else for each $n \in \mathbb{N}$ there is a point $a_n \in A$ such that $|p - a_n| > n$. Compactness implies that some subsequence (a_{n_k}) converges. Convergent sequences are bounded, which contradicts the fact that $|p - a_{n_k}| \to \infty$ as $k \to \infty$. Therefore (a_n) can not exist and A is bounded. □

8.5.16.2 Solution. Let $K \subseteq \mathbb{R}$ be compact. Then the open cover $\{(-n, n); n \in \mathbb{R}\}$ has a finite subcover, say $\{(-n_1, n_1), (-n_2, n_2), ..., (-n_k, n_k)\}$. If $N = \max\{n_1, n_2, ..., n_k\}$ then $K \subset [-N, N]$ and hence is bounded. Next, if K has no limit points, then it is closed. If, to get a contradiction, we assume that $\xi \notin K$ is a limit point of K; so for all $n \in \mathbb{N}$, $\left(\xi - \frac{1}{n}, \xi + \frac{1}{n}\right)$ contains no point of K then the family $\mathcal{A} = \left\{A_n = \left(-\infty, \xi - \frac{1}{n}\right) \bigcup \left(\xi + \frac{1}{n}, \infty\right); n \in \mathbb{N}\right\}$ is an open cover of K. We show that $\mathcal{A}$ has no finite subcover. From the construction of A_n, we observe that $A_1 \subseteq A_2 \subseteq ... \subseteq A_n \subseteq A_{n+1} \subseteq ...$ Now a finite subcover $\{A_{n_1}, A_{n_2}, .., A_{n_k}\}$ of $\mathcal{A}$ implies $\bigcup_{i=1}^k A_{n_i} = A_M$ where $M = \max\{n_1, n_2, .., n_k\}$. Hence $K \subseteq A_M = \left(-\infty, \xi - \frac{1}{M}\right) \bigcup \left(\xi + \frac{1}{M}, \infty\right)$and $\left(\xi - \frac{1}{M}, \xi + \frac{1}{M}\right) \cap K = \emptyset$ implies ξ not a limit point, a contradiction. Thus $\xi \in K$. □

8.5.17 Problem. For a set $K \subseteq \mathbb{R}$ the following statements are equivalent:

1. K is compact.
2. K is closed and bounded.
3. Every infinite subset of K has a limit point in K.
4. Every sequence in K has a subsequence that converges (to an element of K).

8.5.17.1 Solution. The equivalence (1) $\Leftrightarrow$ (2) is, of course, the Heine–Borel Theorem. Let us then prove the implications (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (2)

1. (2) $\Rightarrow$ (3): Suppose that K is closed and bounded, and let $S \subseteq K$ be an infinite subset. Then S is bounded (because K is) and hence, by the Bolzano-Weierstrass theorem, it has a limit point, say ξ. Since K is closed we must have $\xi \in K$ and (3) follows.
2. (3) $\Rightarrow$ (4): Next, suppose that (3) is satisfied and let (x_n) be a sequence in K. If $\{x_n; n \in \mathbb{N}\}$ is finite, then we can find integers n_0 and $n_k; k \in \mathbb{N}$ such that $n_1 < n_2 < n_3 < ..$ and $x_{n_k} = x_{n_0}\ \forall\ k \in \mathbb{N}$(Why?). Thus $\lim_{k\to\infty} x_{n_k} = x_{n_0}$ as desired. If, on the other hand, $\{x_n; n \in \mathbb{N}\}$ is infinite, then [by (3)] it has a limit point $\xi \in K$. By the very definition of limit point, for each $k \in \mathbb{N}$, we can find increasing $n_k \in \mathbb{N}$ such that $|x_{n_k} - \xi| < 1/k$. It is then obvious that $\lim_{k\to\infty} x_{n_k} = x_{n_0}$ and (4) is satisfied.
3. (4) $\Rightarrow$ (2): Finally, suppose that (4) is satisfied, and let ξ be a limit point of K. We can find (using the definition of limit point) a sequence (x_n) in K such that $\lim(x_n) = \xi$. This implies that all subsequences of (x_n) also converge to ξ and hence [by (4)] we must have $\xi \in K$. Thus, K contains all its limit points and is therefore closed. If K is unbounded, then we can find a sequence (x_n) in K such that $|x_n| > n\ \forall\ n \in \mathbb{N}$. But then no subsequence of (x_n) converges, contradicting (4). □

8.5.18 Problem. (Nested Interval Theorem) Consider a sequence (I_k) of closed bounded intervals in $\mathbb{R}$, say $I_k = [a_k, b_k], k\mathbb{N}$, so that $I_{k+1} \subseteq I_k\ \forall\ k \in \mathbb{N}$. Assume that the lengths of the intervals I_k approach zero, which means that for every $\epsilon > 0$ there exists $k_\epsilon \in \mathbb{N}$ so that $b_{k_\epsilon} - a_{k_\epsilon} < \epsilon$. Then we have

$$I = \bigcap_{i=1}^{\infty} I_k = \{\xi\} \text{ for some } \xi \in \mathbb{R}.$$

8.5.18.1 Solution. Define the sets

$$A = \{a_k; k \in \mathbb{N}\} \text{ and } B = \{b_k; k \in \mathbb{N}\}.$$

Observe that $a_1 \leq a_2 \leq .... \leq b_2 \leq b_1$. Thus, the quantities $\alpha = \sup(A)$ and $\beta = \sup(B)$ do exist in $\in \mathbb{R}$. Observe,also, that $a_k \leq \alpha \leq \beta \leq b_k$ for each $k \in \mathbb{N}$. Now, note that $[\alpha, \beta] = \bigcap_{i=1}^{\infty} I_k$. Indeed, if $x \in [\alpha, \beta]$ then, by the previous observation, $a_k \leq x \leq b_k\ \forall\ k \in \mathbb{N}$, meaning that $x \in I_k\ \forall\ k \in \mathbb{N}$. On the other hand, if $x \in I_k\ \forall\ k \in \mathbb{N}$, then $a_k \leq x \leq b_k\ \forall\ k \in \mathbb{N}$, and so $\alpha \leq x \leq \beta$. Assume that $\epsilon = \beta - \alpha > 0$. We can find k_ϵ such that $b_{k_\epsilon} - a_{k_\epsilon} < \epsilon$. Since $[\alpha, \beta] \subseteq [a_{k_\epsilon}, b_{k_\epsilon}]$, we have $\epsilon = b_{k_\epsilon} - a_{k_\epsilon} < \epsilon$, acontradiction. This shows that $\alpha = \beta$, hence $I = \{\xi\}$ for some $\xi \in \mathbb{R}$, where $\xi = \alpha = \beta$. □

8.5.19 Problem. Every infinite compact subset $A \subseteq \mathbb{R}$ has at least one accumulation point that belongs to A.

8.5.19.1 Solution. Since A is bounded, we can find an interval $[a,b]$ such that $A \subseteq [a,b]$. Halve this interval to obtain two adjacent closed intervals. At least one of them contains an infinite number of elements of A. Call this interval I_2. Halve this interval again. At least one of the resulting intervals contains an infinite number of elements of A. Call this interval I_3. Continue in this way to obtain (I_n). Use Cantor's theorem to conclude that $\cap_{n=1}^{\infty} I_n = \{x_0\}$ for some $x_0 \in \mathbb{R}$. Note that $x_0 \in \overline{A}$ since the lengths of the intervals I_n approach zero, and any of them contains infinitely many points of A; due to the fact that A is closed, we get $x_0 \in A$. It is clear that x_0 is an accumulation point of A. □

8.5.1 Definition. A function $f : S \to \mathbb{R}$ has a **maximizer** x^* in S if $f(x^*) = \sup\{f(x); x \in S\}$, and has a **minimizer** x_* in S if $f(x_*) = \inf\{f(x); x \in S\}$.

8.5.2 Theorem (Weierstrass's Theorem)**.** If K is a nonempty compact set and $f : K \to \mathbb{R}$ is continuous, then f has both a maximizer and a minimizer in K.

8.5.20 Problem. If K is a nonempty compact set and $f : K \to \mathbb{R}$ is upper semicontinuous, then f has a maximizer in K. If K is a nonempty compact set and $f : K \to \mathbb{R}$ is lower semicontinuous, then f has a minimizer in K.

8.5.20.1 Solution. Outline of proof : Assume K is a nonempty compact set and $f : K \to \mathbb{R}$ is upper semicontinuous. Then $F_\alpha = \{x; f(x) \geq \alpha, \alpha \in range f\}$ is a family of closed sets having the finite intersection property. The set of maximizers is the nonempty set $\bigcap F_\alpha$. □

8.5.21 Problem. The closed interval $[a,b] \subseteq \mathbb{R}$ is compact.

8.5.21.1 Solution. Let (x_n) be a sequence in $[a,b]$. Let

$$C = \{x \in [a,b]; x_n < x \text{ only finitely often}\}.$$

Since $a \in C, C \neq \emptyset$. Clearly b is an upper bound for C. By the least upper bound property of $\mathbb{R}, c = \sup C$ exists, $c \in [a,b]$. We claim that a subsequence of (x_n) converges to c. Suppose not, i.e., no subsequence of (x_n) converges to c. Then for some $\epsilon > 0$, $x_n \in (c-\epsilon, c+\epsilon)$ only finitely often, which implies that $c+\epsilon \in C$, contrary to c being an upper bound for C. Hence some subsequence of (x_n) does converge to c, and $[a,b]$ is compact. □

8.5.22 Problem. A closed subset of a compact set is compact.

8.5.22.1 Solution. If A is a closed subset of the compact set C and if (a_n) is a sequence of points in A, then clearly (a_n) is also a sequence of points in C, so by compactness of C, there is a subsequence (a_{n_k}) converging to a limit $p \in C$. Since A is closed, p lies in A, which proves that A is compact. □

8.5.22.2 Solution. Suppose that A is a closed subset of the compact set C and let $\mathcal{A}$ be an open cover of A, then $\mathcal{B} = \mathcal{A} \cup \{A^C\}$ is an open cover of C. Since C is compact, so there exist a finite subcover $\{A_1 A_2, ..., A_k, A^C\}$ of $\mathcal{B}$. Hence

$$\begin{aligned}
&C \subseteq A_1 \cup A_2 \cup ... \cup A_k \cup A^C \\
\Rightarrow &A \subseteq C \subseteq A_1 \cup A_2 \cup ... \cup A_k \cup A^C \\
\Rightarrow &A \subseteq A_1 \cup A_2 \cup ... \cup A_k.
\end{aligned}$$

Thus A is compact. □

8.5.23 Problem. The intersection of a nested sequence of compact non-empty sets is compact and non-empty.

8.5.23.1 Solution. Let (A_n) be such a sequence. So, A_n is closed. The intersection of closed sets is always closed. Thus, $\bigcap_{n=1}^{\infty} A_n$ is a closed subset of the compact set A_1, and is therefore compact. It remains to show that the intersection is non-empty. Since A_n is non-empty, so for each $n \in \mathbb{N}$ we can choose $a_n \in A_n$. The sequence (a_n) lies in A_1, since the sets are nested. Compactness of A_1 implies that (a_n) has a subsequence (a_{n_k}) converging to some point $p \in A_1$. The limit p also lies in the set A_2 since except possibly for the first term, the subsequence (a_{n_k}) lies in A_2 and A_2 is a closed set. The same is true for A_3 and for all the sets in the nested sequence. Thus, $p \in \bigcap_{n=1}^{\infty} A_n$ and $\bigcap_{n=1}^{\infty} A_n$ is shown to be non-empty. □

8.5.3 Remark. A nested sequence of non-empty noncompact sets can have empty intersection. For example, $\bigcap_{n=1}^{\infty} A_n = \emptyset$ where $A_n = (1/n, 0)$, or $A_n = \{n, n+1, n+2, ...\}$.

8.5.24 Problem. If K is a compact set in $\mathbb{R}$ and p is a point in $\mathbb{R} \setminus K$, then there exist disjoint open sets U and V of $\mathbb{R}$ such that $K \subseteq U$ and $p \in V$.

8.5.24.1 Solution. For each point $a \in K$, there exist disjoint open sets U_a and V_a of $\mathbb{R}$ such that $a \in U_a$ and $p \in V_a$. Since $\mathbb{R}$ is a Hausdorff space and $a \neq p$ then the family $\mathcal{F} = \{U_a; a \in K\}$ is a cover of K by open sets of $\mathbb{R}$. Since K is compact, so it has a finite subcover of $\mathcal{F}$. Hence, there are a finite number of points $a_1, a_2, ...a_n$ of K such that K is contained in the union

$$U = U_{a_1} \cup U_{a_2} \cup .... \cup U_{a_n}.$$

On the other hand, let

$$V = U_{a_1} \cap U_{a_2} \cap .... \cap U_{a_n}.$$

Then U and V are open sets of $\mathbb{R}$ such that $K \subseteq U$ and $p \in V$ with $U \cap V = \emptyset$. □

8.5.25 Problem. Any interval $[a, b]$, where a and b are real numbers, and $a \leq b$, is compact.

8.5.25.1 Solution. Assume, on the contrary, that there exists an open cover $\mathcal{C}$ of $[a, b]$ that admits no finite subcover. Split the interval $[a, b]$ into two consecutive closed subintervals, each of length $(b-a)/2$ i.e., $[a, (a+b)/2]$ and $[(a+b)/2, b]$. Certainly, one of these subintervals, say I_1, cannot be covered by any finite subcover of the cover $\mathcal{C}$. Split the interval I_1 into two consecutive closed subintervals each of length $(b-a)/4$; again, one of these subintervals, say I_2, cannot be covered by any finite subcover of the cover $\mathcal{C}$. Note that $I_2 \subseteq I_1$. We continue this algorithm and produce a sequence of closed intervals $\{I_1, I_2, ..\}$, where $I_{k+1} \subseteq I_k \; \forall \; k \in \mathbb{N}$ and none of them can be covered by a finite subcover of $\mathcal{C}$. For each $k \in \mathbb{N}$, the length of the interval I_k is $(b-a)/2^k$. The intersection of these intervals is a single point $x \in [a, b]$, due to Cantor's intersection theorem. Consider a member A in the family $\mathcal{C}$ for which $x \in A$. Note that A is open, and thus we can choose an open interval I containing x, say $(x-\epsilon, x+\epsilon)$ for some $\epsilon > 0$, so that $I \subseteq A$. Choose $k_0 \in \mathbb{N}$ such that $(b-a)/2^{k_0} < \epsilon$. Since $x \in I_{k_0}$ as $\{x\} = \bigcap_{k=1}^{\infty} I_k$, for every $y \in I_{k_0}$ we have $|x-y| < \epsilon$. Thus $y \in I \subseteq A$ for every $y \in I_{k_0}$, hence $I_{k_0} \subseteq A$ and the subfamily of $\mathcal{C}$ consisting of the single set A is a finite subcover of the cover $\mathbb{C}$ for the interval I_{k_0}, a contradiction. □

8.5.26 Problem. Using the following definition, a subset $K \subseteq \mathbb{R}$ is said to be **compact** if every open cover has a finite subcover, show that

1. Let A and B be two subsets of $\mathbb{R}$ such that A is closed and B is compact, then $A \cap B$ is compact.

2. the intersection of two compact subsets of $\mathbb{R}$ is compact.

3. $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{N}$ are not compact.

4. $(a,b), [a,b)$ and $K = \{1, 1/2, 1/3, .., 1/n, ..\}$ are not compact,
but $K_1 = \{0, 1, 1/2, 1/3, .., 1/n, ..\}$ is compact.

5. the union of two compact subsets of $\mathbb{R}$ is compact.

8.5.26.1 Solution.

1. Let A and B be two subsets of $\mathbb{R}$ such that A is closed and B is compact. let $\mathcal{A}$ be an open cover of $A \cap B$. As A is closed, then consider the family $\mathcal{B} = \{A^C\} \cup \{C \in \mathcal{A}\}$ which covers $\mathbb{R}$, it also covers B. Since B is compact, there exists a finite subfamily $\{A^C, C_1, C_2, .., C_k\}$ which covers B also covers $A \cap B$. For $A \cap B \subseteq B \subseteq A^C \cup C_1 \cup C_2 \cup .... \cup C_k \Rightarrow A \cap B \subseteq C_1 \cup C_2 \cup .... \cup C_k$. Thus $A \cap B$ is compact.

2. Let A and B be two compact subsets of $\mathbb{R}$. Then A is closed. Proceed as above.

3. For $\mathbb{R}$:
Consider an open cover $\mathcal{A} = \{A_n = (-n, n); n \in \mathbb{N}\}$ of $\mathbb{R}$. We observe that $A_n \subseteq A_{n+1}\ \forall\ n \in \mathbb{N}$. Suppose that $\mathcal{B} = \{A_{n_i} \in \mathcal{A}; i = 1, 2, ..., k\}$ be a finite subcover of $\mathbb{R}$, then let $n_t = \max\{n_1, n_2, .., n_k\}$, hence $\mathbb{R} = \cup_{i=1}^{k} A_{n_i} = (-n_t, n_t)$, which is impossible.
For $\mathbb{Z}$:
Consider an open cover $\mathcal{A} = \{A_n = (n-1, n+1); n \in \mathbb{Z}\}$ of $\mathbb{Z}$. Since union of any finite number of members of $\mathcal{A}$ cotains a finite number of members of $\mathbb{Z}$, so this family has no finite subcover.
For $\mathbb{N}$:
Take $\mathcal{A} = \{A_n = (n-1, n+1); n \in \mathbb{N}\}$ and proceed as the same way.

4. $\mathcal{A} = \{A_n = (a + \frac{1}{n}, b); n \in \mathbb{N}\}$ for (a,b).

   $\mathcal{A} = \{A_n = (a - \frac{1}{n}, b - \frac{1}{n}); n \in \mathbb{N}\}$ for $[a,b)$.

   $\mathcal{A} = \{A_1 = (\frac{1}{2}, \frac{3}{2}); A_n = (\frac{1}{n+1}, \frac{1}{n-1}); n \geq 2\}$ for $K = \{1, 1/2, 1/3, .., 1/n, ..\}$

   has no finite subcover.
   For K_1 :
   Let $\mathcal{A}$ be an open cover of K_1, then take $A_0 \in \mathcal{A}$ containing 0. Since 0 is the limit point of K_1, so at most finite number of members of K_1, say $\{\frac{1}{n_1}, \frac{1}{n_2}, ..., \frac{1}{n_p}\}$ will be outside A_0. Hence the finite subfamily $\{A_0, A_1 .., A_p\}$ of $\mathcal{A}$ such that $\frac{1}{n_i} \in A_i;\ i = 1, 2, ..p$ is a cover of K_1. Thus K_1 is compact.

5. Let A and B be two compact subsets of $\mathbb{R}$. Let $\mathcal{A}$ be an open cover of $A \cup B$. Since it is a cover of both A and B, then there exists a finite subfamily $\{A_i; i = 1, 2, .., k\}$ which covers A and there exists a finite subfamily $\{A'_i; i = 1, 2, .., p\}$ which covers B. Hence the finite subfamily $\{A_i; i = 1, 2, .., k\} \cup \{A'_i; i = 1, 2, .., p\}$ is a cover of $A \cup B$.

8.5.27 Problem. Find an infinite collection $\{K_n; n \in \mathbb{N}\}$ of compact sets in $\mathbb{R}$ such that the union $\bigcup_{n=1}^{\infty} K_n$ is not compact.

8.5.27.1 Solution. Consider the family $\mathcal{A} = \left\{ K_n = \left[\frac{1}{n}, 1 - \frac{1}{n}\right]; n \in \mathbb{N} \right\}$, then $\bigcup_{n=1}^{\infty} K_n = (0,1)$ which is not compact. $\square$

8.5.28 Problem. Let (K_n) be a sequence of nonempty compact sets in $\mathbb{R}$ such that $K_1 \supseteq K_2 \supseteq .... \supseteq K_n \supseteq K_{n+1} \supseteq ......$ Prove that there exists at least one point $x \in \mathbb{R}$ such that $x \in K_n$ for all $n \in \mathbb{N}$; that is, the intersection $\bigcap_{n=1}^{\infty} K_n$ is not empty, and give an example showing this is not necessarily true if the K_n were merely nonempty and closed in $\mathbb{R}$.

We give two solutions, one based on the sequential definition of compactness and the other based on the covering definition. The sequential one is easier to follow.

8.5.28.1 Solution. Since each K_n is nonempty, we can pick a point from each one: say for each n we pick $p_n \in K_n$. Now, the sequence (p_n) is, in particular, a sequence in K_1 since each K_n is contained in K_1 for $n > 1$. Since K_1 is compact, this has a subsequence (p_{n_k}) converging to some $p \in K_1$.
Now, except for maybe the first term, the terms in (p_{n_k}) also belong to K_2 since $K_n \subseteq K_2$ for $n > 2$. Since K_2 is compact, this has a convergent subsequence $(p_{n_{k_l}})$ in K_2; since we already know this subsequence must converge to p since the larger sequence (p_{n_k}) does, we have that $p \in K_2$.
Similarly, $(p_{n_{k_l}})$ is, except for possibly the first term, a sequence in K_3, so by looking at a convergent subsequence we conclude that $p \in K_3$. Continuing in this manner shows that $p \in K_n$ for all n, so that $p \in \cap_n K_n$ and hence $\cap_n K_n$ is non-empty.
Here is another way to show that $p \in K_n \ \forall \ n$. As above, the subsequence (p_{n_k}) constructed in the first step is, except for maybe the first term, a sequence in K_2. Since $K2$ is compact, it is closed in $\mathbb{R}$ so the limit p of (p_{n_k}) is in K_2. Similarly, except for maybe the first two terms, (p_{n_k}) is a sequence in K_3 and closed-ness of K_3 in $\mathbb{R}$ shows that $p \in K_3$. Continuing in this manner shows that $p \in K_n$ for all n as required. $\square$

8.5.28.2 Solution. Suppose that $\bigcap_{n=1}^{\infty} K_n = \emptyset$. Now, given any $p \in K_1$, there is some K_n such that $p \notin K_n$ since the intersection of all the K_n is empty. Thus, given $p \in K_1$, there is some n so that $p \in K_n^C$. Since each complement K_n^C is open in $\mathbb{R}$ since K_n being compact implies K_n is closed in $\mathbb{R}$. this says that the collection $\{K_n^C\}$ is an open cover of K_1. Since K_1 is compact, this open cover has a finite subcover; say

$$\{K_{n_1}^C, .., K_{n_k}^C\}$$

are the sets in this finite subcover and are ordered so that n_k is the largest index among the n_i. So, K_1 is contained in the union $K_{n_1}^C \cup .. \cup K_{n_k}^C$. But any set in the original nested sequence beyond the n_k-th one, say K_{n_k+1}, is contained in all the previous sets, and so in particular contained in $K_{n_1} \cap .. \cap K_{n_k}$. This is not possible unless K_{n_k+1} is empty since K_{n_k+1} is also supposed to be contained in $K_1 \subseteq K_{n_1}^C \cup .. \cup K_{n_k}^C$, contradicting the assumption that all the K_n were empty. (To elaborate, a set cannot be contained in both $K_{n_1}^C \cup .. \cup K_{n_k}^C$ and $K_{n_1} \cap .. \cap K_{n_k}$ since these two sets are precisely complements of one another.) We conclude that $\bigcap_{n=1}^{\infty} K_n$ must be non-empty.
Now, for each $n \in \mathbb{N}$, let $K_n = [n, \infty)$. These are all nonempty and closed in $\mathbb{R}$, but their intersection is empty since there is no real number x with the property that $n < x$ for all n by the Archimedian Property of $\mathbb{R}$. There are many other examples which work, such as $K_n = \mathbb{N} \setminus \{1, 2, .., n-1\}$. $\square$

8.5.29 Problem. (**Generalized Cantor's Intersection Theorem**)
Let $\mathcal{D} = \{D \subseteq \mathbb{R}; D \text{ is bounded}\}$. Define $\alpha : \mathcal{D} \to \mathbb{R}$ by

$$\alpha(D) = \inf \left\{ r > 0; D \subseteq \bigcup_{i=1}^{n} A_i; diam(A_i) \leq r \right\}.$$

Show that

1. If D is compact, then $\alpha(D) = 0$.
2. $D_1 \subseteq D_2 \Rightarrow \alpha(D_1) \leq \alpha(D_2)$; α is monotone.
3. $\alpha(\overline{D}) = \alpha(D)$. (invariant when given the closure).
4. If (F_n) is a decreasing sequence of nonempty closed and bounded subsets of $\mathbb{R}$ and if $\lim_{n\to\infty} \alpha(F_n) = 0$, then the intersection of all the F_n is nonempty and compact.

8.5.29.1 Solution.

1. Follows from the definition.
2. If $D_1 \subseteq D_2$ and $D_2 \subseteq \bigcup_{i=1}^n A_i$ with $diam(A_i) \leq r$, then covering for D_2 also covers D_1 and hence $\alpha(D_1) \leq \alpha(D_2)$.
3. Since $D \subseteq \overline{D}$ by the above part we have $\alpha(D) \leq \alpha(\overline{D})$. Conversely, if $D \subseteq \bigcup_{i=1}^n A_i$, then $\overline{D} \subseteq \bigcup_{i=1}^n \overline{A_i}$; however, $diam(A_i) = diam(\overline{A_i}) \leq r$ implies $\alpha(\overline{D}) \leq \alpha(D)$.
4. $\lim_{n\to\infty} \alpha(F_n) = 0$ implies that for each n, F_n is compact, by the nested intervals property of the compact non-empty sets we have $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$. □

8.5.30 Problem. Let $A \neq \emptyset$ be a compact set in $\mathbb{R}$. Show that $\inf A$ and $\sup A$ exist and belong to A.

8.5.30.1 Solution. Suppose that $\sup A = a \notin A$, then for any $\epsilon > 0\ \exists\, a' \in A$ such that $a' > a - \epsilon$, which means that a is a limit point of A. Since A is compact, A is bounded and closed, hence $a \in A$. Similarly $\inf A \in A$. □

8.5.31 Problem. Given a point $x \in \mathbb{R}$ and a set $A(\neq \emptyset) \subseteq \mathbb{R}$, define the distance $\rho(x; A)$ from x to A by $\rho(x; A) = \inf\{|x - a|; a \in A\}$.

1. Prove that $x \in \overline{A} \Leftrightarrow \rho(x; A) = 0$.
2. Let $\alpha \geq 0$. Prove that $\{x \in \mathbb{R}; \rho(x; A) \geq \alpha\}$ is a closed set.
3. If A is compact and $c \in \mathbb{R}$, then prove that there exist points $a, b \in A$ such that $|c - a| = \inf\{|c - x|; x \in A\}$ and $|c - b| = \sup\{|c - x|; x \in A\}$.
4. Define the distance between the sets $A, B \subseteq \mathbb{R}$ by
$$\rho(A, B) = \inf\{|a - b|; a \in A, b \in B\},$$
then if A and B are disjoint nonempty compact sets, show that there exist $p \in A, q \in B$ such that
$$0 < |p - q| = \rho(A, B) = \inf\{|a - b|; a \in A, b \in B\}.$$
and $\rho(A, B) > 0$.

8.5.31.1 Solution.

1. Left to the reader.

2. Left to the reader.
3. Let $f : A \to \mathbb{R}$ defined by $f(x) = |c - x|; x \in A$. Now, let $p \in A$, then

$$\begin{aligned} f(x) &= |c-x| \leq |c-p| + |p-x| = f(p) + |p-x| \\ \text{and } f(p) &= |c-p| \leq |c-x| + |x-p| = f(x) + |x-p| \\ \Rightarrow & |f(x) - f(p)| \leq |x-p| \end{aligned}$$

Let $\epsilon > 0$, then $|f(x) - f(p)| < \epsilon$ if $|x - p| < \epsilon = \delta$, therefore f is continuous on A. Since A is compact, so there exist $a, b \in A$ such that

$$f(a) = |c - a| = \inf f(A) = \inf\{|c - x|; x \in A\}$$

and

$$f(b) = |c - b| = \sup f(A) = \sup\{|c - x|; x \in A\}.$$

4. Here $A \cap B = \emptyset$ and A, B are compact. Now, for $b \in B$ define $f : A \to \mathbb{R}$ by $f(x) = |b - x|$. By (3), $\exists\, p \in A$ such that

$$f(p) = |b - p| = \inf f(A) = \inf\{|b - x|; x \in A\}.$$

Again, for $p \in A$, let $g : B \to \mathbb{R}$ defined by $g(x) = |p - x|$. So, $\exists\, q \in B$ such that

$$g(q) = |p - q| = \inf g(B) = \inf\{|p - x|; x \in B\}$$

and thus $\rho(A, B) > 0$. □

8.5.32 Problem. Show that completeness, boundedness and total boundedness are not topological properties.

8.5.32.1 Solution. Consider $X = (0, 1], Y = [1, \infty)$ with usual metric X is not complete, bounded and totally bounded, but Y is complete but neither bounded nor totally bounded. On the other hand $f : X \to Y$ defined by $f(x) = 1/x$ is a homeomorphism. □

8.5.33 Problem. Let A and B be disjoint compact subsets of $\mathbb{R}$. Show that there exist disjoint open sets U and V in $\mathbb{R}$ such that $A \subseteq U$ and $B \subseteq V$.

8.5.33.1 Solution. Left to the reader. □

8.5.34 Problem. Construct a compact set of real numbers whose limit points form a countable set.

8.5.34.1 Solution. Let

$$\left\{ \frac{1}{2^m} \left(1 - \frac{1}{n} \right); m, n \in \mathbb{N} \right\}.$$

Consider the points of the form $p = \frac{1}{2^m}$ with $m \in \mathbb{N}$. Any neighborhood of one of these points of radius $r > 0$ will also contain the point $q = \frac{1}{2^m}\left(1 - \frac{1}{n}\right)$ where we choose the positive integer n such that $\frac{1}{n} < 2^m r$, so that $|p - q| = \frac{1}{2^m} - \frac{1}{2^m}\left(1 - \frac{1}{n}\right) = \frac{1}{2^m n} < \frac{1}{r}$. Since $q \neq p$ and $q \in E$, that means p is a limit point, and thus E has at least a countably infinite number of limit points. The fact that E is compact for it closed (since it contains its limit points) and bounded (since each point of E is contained in $[0, 1/2]$). □

8.5.35 Problem.

1. Give an example to show that the distance between two sets can be 0 even if the two sets are disjoint.

2. Give an example to show that the distance between two closed sets can be 0 even if the two sets are disjoint.

8.5.35.1 Solution.

1. Let $A = \left\{\frac{1}{n}; n \in \mathbb{N}\right\}$ and $B = \left\{-\frac{1}{n}; n \in \mathbb{N}\right\}$.

2. Let $A = \left\{n + \frac{1}{n}; n \in \mathbb{N}\right\}$ and $B = \left\{n - \frac{1}{n}; n \in \mathbb{N}\right\}$. □

8.5.36 Problem. Give an example of a set of points in $\mathbb{R}$ that form a compact set and whose limit points form a countable set.

8.5.36.1 Solution. Consider the set $\{0\} \cup \left\{\frac{1}{m} + \frac{1}{n}; m, n \in \mathbb{N}\right\}$. □

8.5.37 Problem. Prove that $K = \{(x, 0); x \in \mathbb{R}\}$ is not sequentially compact.

8.5.37.1 Solution. Left to the reader. □

8.5.38 Problem. Prove that a set with finitely many points is a sequentially compact set.

8.5.38.1 Solution. Hint: We apply the Pigeonhole Principle: if there are only N holes and $N+1$ pigeons, then one hole must have at least to pigeons. If there are infinitely many pigeons, one hole must have infinitely many pigeons. We can think of the holes as the points in the set and the pigeons as the terms in the sequence. □

8.5.39 Problem. Prove that a compact set is sequentially compact.

8.5.39.1 Solution. Assume that K is compact and nonempty, and let (x_n) be a sequence in K. (If K is empty, the theorem is vacuously true.) Define $F_n = \overline{\{x_n, x_{n+1}, ...\}}$, the closure of the set of values of the n-tail of (x_n). Then $\{F_n; n \in \mathbb{N}\}$ is a family of closed subsets of K having the finite intersection property. Since K is compact $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$. Now let y belong to this intersection. That is, $y \in \overline{\{x_n, x_{n+1}, ...\}} \subseteq K$ for every n. Construct the strictly increasing function $k \mapsto n_k$ from $\mathbb{N} \to \mathbb{N}$ inductively as follows. Let $n_1 = 1$. Given $n_1 < ... < n_k$, pick n_{k+1} to satisfy $n_{k+1} \geq n_k + 1$ and $|x_{n_{k+1}} - y| < 1/(k+1)$. Since $y \in F_{n_k+1}$, there is always such an $x_{n_{k+1}}$ in $\{x_{n_k+1}, x_{n_k+2}, ..\}$.Then $x_{n_k} \to y$ is the desired subsequence. □

8.5.40 Problem. Let E be the set of all points x such that either $x = 0$ or there exist $m, n \in \mathbb{N}$ such that $x = \frac{m+n}{mn}$. Prove that E is a compact set.

8.5.40.1 Solution. We see that $E = \left\{0, \frac{1}{m} + \frac{1}{n}; m, n \in \mathbb{R}\right\}$. Since E is bounded and closed, hence E is compact. □

8.5.41 Problem. Assume that K is a compact subset of a metric space, and that f is an isometry of K into K. Show that $f(K) = K$.

8.5.41.1 Solution. Assume the contrary. Let $x \in K$ be so that $d(x, f(K)) = \epsilon > 0$. Form the sequence $\{x, f(x), f(f(x)), ...\}$ Since the distance of x to any $f(f(..f(x)...))$ is greater then ϵ, by applying the fact that f is an isometry we get that the distance of $f(x)$ to any $f(f(...f(x)...))$ is greater than ϵ. Proceed recursively to show that the distance between two different elements in the sequence $\{x, f(x), f(f(x)), ...\}$ is greater than ϵ. This contradicts the compactness of K. □

8.5.42 Problem. Does there exist an infinite compact $C \subseteq \mathbb{R}$, such that every real valued function on it is continuous?

8.5.42.1 Solution. No: Take a sequence (x_n) in $\mathbb{R}$ such that $x_n \to x$, and $x_n \neq x$ for each $n \in \mathbb{N}$. Define a function f on $\{x_n; n \in \mathbb{N}\} \cup \{x\}$ by $f(x_n) = 0$ for $n \in \mathbb{N}$ and $f(x) = 1$. Extend f by taking it be 0 on other points. Then f is discontinuous at x. $\square$

8.5.42.2 Solution. (Aritra Bera - semester 6) Let there exist an infinite compact set $C \subseteq \mathbb{R}$, then $\forall\, x \in C$ define $f_x : C \to \mathbb{R}$ such that

$$f_x(y) = \begin{cases} 0, & \text{if } y = x \\ 1, & \text{if } y \neq x. \end{cases}$$

Since f_x is continuous $f_x^{-1}(0) = \{x\}$ is an open set (as $\{0\}$ is open in $\{0,1\}$). Thus $\bigcup_{x \in C} \{x\}$ in an open cover of C which cannot have any finite subcover. Hence C cannot be compact. $\square$

8.5.43 Problem. Let (K_n) be a decreasing sequence of compact sets in a metric space M. Let

$$K = \bigcap_{n=1}^{\infty} K_n$$

Prove that for each $\epsilon > 0$ there is $n_0 \in \mathbb{N}$ such that

$$n \geq n_0 \Rightarrow K_n \subseteq B(K;\epsilon) = \bigcup_{x \in K} B(x, \epsilon),$$

where $B(x;\epsilon)$ is the open ball centered at x and having radius ϵ.

8.5.43.1 Solution. Assume that there exists a subsequence (K_{n_k}) and $x_{n_k} \in K_{n_k}$ for each $k \in \mathbb{N}$, such that $d(x_{n_k}, K) \geq \epsilon$. Without loss of generality assume $x_{n_k} \to x_0$ for some $x_0 \in K_1$ (note that K_1 is compact). Then x_0 belongs to all K_i , thus it belongs to K. However, since the distance function is continuous and $d(x_{n_k}, K) \geq \epsilon$, we get $d(x_{n_k}, K) \geq \epsilon$, which is a contradiction. $\square$

8.5.44 Problem. (Cantor) Let $\{K_n; n \in \mathbb{N}\}$ be a family of nonempty, compact subsets of $\mathbb{R}$. Show that, if $K_{n+1} \subseteq K_n$ for all $n \in \mathbb{N}$; then $\bigcap_{n \in \mathbb{N}} K_n \neq \emptyset$.

8.5.44.1 Solution. Left to the reader. $\square$

8.5.45 Problem. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as follows. For $x \in (0,1]$

$$f(x) = \begin{cases} \frac{1}{q}, & \text{if } x = \frac{p}{q} \;\; (p, q \in \mathbb{N},\ \gcd(p,q) = 1) \\ 0, & \text{if } x \in \mathbb{Q}^C \cap (0,1) \end{cases}$$

and for $x \notin (0,1]$, we define $f(x)$ by periodicity, i.e. by $f(x) = f(x-n)$ where $n \in \mathbb{Z}$ is the unique integer with $x \in (n, n+1]$. Show that f is continuous at each $x \in \mathbb{Q}^C$ and discontinuous at each $x \in \mathbb{Q}^C$.

8.5.45.1 Solution. Hint: If p_n/q_n is a sequence of (reduced) rationals in (0,1) with $\lim_{n\to\infty}(p_n/q_n) = x \in \mathbb{Q}^C \cap (0,1)$, show that $\lim_{n\to\infty} q_n = \infty$. $\square$

8.5.46 Problem. Let X be a compact subset of $\mathbb{R}$, and let $f : X \to X$ be continuous. Define $X_1 = X$ and $X_{n+1} = f(X_n)$, and let $A = \bigcap_{i=1}^{n} X_n$. Prove that A is a nonempty subset of X and $f(A) = A$.

8.5.46.1 Solution. Each of the subsets X_n is compact by an inductive argument, and since X is Hausdorff each one is also closed. Since each set in the sequence contains the next one, the intersection of finitely many sets $X_{k(1)}, .., X_{k(n)}$ in the collection is the set $X_{k(m)}$ where $k(m)$ is the maximum of the $k(i)$. Since X is compact, the Finite Intersection property implies that the intersection A of these sets is nonempty. We need to prove that $f(A) = A$. By construction A is the set of all points that lie in the image of the k-fold composition $f \circ f \circ .. \circ f = f^k$ of f with itself. To see that f maps this set into itself note that if $a = f^k(x_k)$ for each positive integer k then $f(a) = f^k \circ (f(x_k))$ for each k. To see that f maps this set onto itself, note that $a = f^k(x_k)$ for each positive integer k implies that

$$a = f(f^k(x_{k+1}))$$

for each k. $\square$

8.5.47 Problem.

1. Define the ϵ-neighborhood $U(A; \epsilon)$ to be the set of all u such that $d(u; A) < \epsilon$. Show that this is the union of the neighborhoods $N(a; \epsilon)\ \forall\ a \in A$.

2. Suppose that A is compact and that U is an open set containing A. Prove that there is an $\epsilon > 0$ such that $A \subseteq U(A; \epsilon)$.

8.5.47.1 Solution.

1. The union is contained in $U(A; \epsilon)$ because $d(x, a) < \epsilon$ implies $d(x; A) < \epsilon$. To prove the reverse inclusion suppose that y is a point such that $\delta = d(y; A) < \epsilon$. It then follows that there is some point $a \in A$ such that $d(y, a) < \epsilon$ because the greater than the greatest lower bound of all possible distances. The reverse inclusion is an immediate consequence of the existence of such a point a.

2. Let $F = X \setminus U$ and consider the function $g(a) = d(a; F)$ for $a \in A$. This is a continuous function and it is always positive because $A \cap F = \emptyset$;. Therefore it takes a positive minimum value, say ϵ. If $y \in A \subseteq U(A; \epsilon)$ then $d(a; y) < \epsilon \leq d(a; F)$ implies that $y \notin F$, and therefore $U(A; \epsilon)$ is contained in the complement of F, which is U. $\square$

8.5.48 Problem. Show that the preceding conclusion need not hold if A is not compact.

8.5.48.1 Solution. Take X to be all real numbers with positive first coordinate, let A be the points of X satisfying $y = 0$, and let U be the set of all points such that $y < 1/|x|$. Then for every $\epsilon > 0$ there is a point not in U whose distance from A is less than ϵ. For example, consider the points $(2n, 1/n)$. $\square$

8.5.49 Problem. For each of the following statements, determine whether it is true or false and justify your answer.

1. Every bounded set is closed.

2. Every closed set is bounded.

3. Every closed set is compact.

4. Every bounded set is compact.
5. A subset of a compact set is also compact.

8.5.49.1 Solution.

1. (2,4)
2. $[2, \infty)$.
3. $[2, \infty)$.
4. $[2, 3)$.
5. $(3, 4) \subseteq [2, 5]$.

8.5.50 Problem (Lindelöff). Let $\{A_\alpha; \alpha \in \Lambda\}$ be a collection of open subsets of $\mathbb{R}$. Then there is a countable subset $\{\alpha_1, \alpha_2, ..., \alpha_n, ...\} \subseteq \Lambda$ such that

$$\bigcup_{\alpha \in \Lambda} A_\alpha = \bigcup_{i=1}^{\infty} A_{\alpha_i}.$$

8.5.50.1 Solution. Let $A = \bigcup_{\alpha \in \Lambda} A_\alpha$. Then, $\forall\, x \in A$ we have $x \in A_{\alpha_x}$ for some $\alpha_x \in \Lambda$, and since A_{α_x} is open, we can find $\epsilon_x > 0$ with $x \in B(x; \epsilon_x) \subseteq A_{\alpha_x}$. Using the fact that the set $\mathbb{Q}$ of rational numbers is dense in $\mathbb{R}$; we can find a rational number $r_x > 0$ such that $x \in B(x; r_x) \subseteq B(x; \epsilon_x)$. Now the set $\{r_x; x \in A\}$ is countable and hence can be written as $\{r_x; x \in A\} = \{r_1, r_2, .., r_n, ...\}$ where $r_k = r_{x_k}$ for some $x_k \in A$. If for each $k \in \mathbb{N}$ we pick $\alpha_k \in \Lambda$ such that $B(x_k; r_{x_k}) \subseteq A_{\alpha_k}$; then we have a countable subcollection $\{A_{\alpha_k}; k \in \mathbb{N}\} \subseteq \{A_\alpha; \alpha \in \Lambda\}$ which satisfies $A = \bigcup_{i=1}^{\infty} A_{\alpha_i}$. This is also known as Finally Compact or Lindelöf compact.

8.5.51 Problem. (**Bolzano-Weierstrass**). If a subset S of $\mathbb{R}$ is bounded, then every sequence in S has a convergent subsequence.

8.5.51.1 Solution. Let (x_n) be a sequence in S. If the set $\{x_k; k \in \mathbb{N}\}$ is finite, the existence of a convergent subsequence is immediate by the above theorem. So assume that $\{x_n; n \in \mathbb{N}\}$ has an infinite number of elements. Since S is bounded, there is a closed interval $I_0 = [a, b],\ a < b$, containing S. Bisect I_0 to obtain two closed intervals and I_0, I_0'. At least one of I_0, I_0' contains infinitely many terms of the sequence (x_n). Call it (I_1). Repeat the process on (I_1). Continue inductively to obtain a sequence (I_n) of nested closed intervals with the properties
(i) $\ell(I_n) = (b - a)/2^n$, and
(ii) I_n contains infinitely many terms of (x_n) for $n = 1, 2, ...$ By the nested interval theorem, there is exactly one element x common to all I_n. Now, choose any x_{k_1} such that $x_{k_1} \in I_1$. Assume $x_{k_1}, x_{k_1}, ..., x_{k_m}$ have been chosen such that $x_{k_i} \in I_{k_i}$, and $k_i < k_{i+1}$ for $i = 1, 2, ..., m - 1$. Then choose $k_{m+1} > k_m$ so that $k_{m+1} \in I_{k_{m+1}}$. Then (x_{k_m}) converges to x since

$$|x_{k_m} - x| \leq \frac{(b - a)}{2^{k_m}}$$

and $k_m \to \infty$. □

8.5.52 Problem (Heine-Borel Theorem). A subset S of $\mathbb{R}$ is compact iff it closed and bounded.

8.5.52.1 Solution. Suppose S is a closed and bounded subset of $\mathbb{R}$. Then Bolzano-Weierstrass theorem implies that every sequence in S has a convergent subsequence and, since S is closed, this subsequence converges to a point in S. It follows that S is compact. Suppose next that S is compact. Then every convergent sequence in S converges to a point in S, and it follows that S is closed. If S is not bounded, then for each $k \in \mathbb{N}$, there is $x_k \in S$ such that $x_k > k$. Consequently, no subsequence of (x_n) can converge and so S is not compact.

8.5.52.2 Solution. Suppose first that S compact. We show that S is then closed and bounded. To show that S is bounded, let $G_x = B(x;1) = (x-1, x+1)$. Then $\{G_x; x \in S\}$ is an open cover of S and so has a finite subcover, say, $G_{x_1}, G_{x_2}, .., G_{x_k}$. Let $M = \max\{|x_i|; i = 1, 2, .., k\}$. If $x \in S$, then $x \in G_{x_i}$ for some i and $|x| \leq |x_i| + 1 \leq M + 1$. Thus, S is bounded.
To show that S is closed, consider an $x_0 \in \mathbb{R} \setminus S$. For $x \in S$, there are disjoint neighborhoods U_x and $V_{x_0}(x)$ of x and x_0, respectively, which generally will depend on x. The family $\{U_x : x \in S\}$ is an open cover of S and so has a finite subcover, say, $U_{x_1}, U_{x_2}, .., U_{x_k}$. Then $S \subseteq \bigcup_{i=1}^{k} U_{x_i}$ which is disjoint from the open set $\bigcap_{i=1}^{k} V_{x_i}(x)$. The latter is a neighborhood of x_0 contained entirely in $\mathbb{R} \setminus S$. It follows that $\mathbb{R} \setminus S$ is open and so S is closed.
Suppose that S is a compact subset of $\mathbb{R}$ and $\mathcal{F}$ is an open cover of S such that no finite subfamily of $\mathcal{F}$ is a cover for S. Since S is bounded, there is a closed interval $I_0 = [a, b]$ such that $S \subseteq I_0$. Bisect to obtain two closed subintervals I_0', I_0'', Then either $S \cap I_0$ or $S \cap I_0''$ is not covered by a finite subfamily of $\mathcal{F}$. Denote such a subinterval by I_1. Continue this process to obtain a sequence (I_n) of nested closed intervals whose lengths approach zero and each with the property that no finite subfamily of $\mathcal{F}$ will cover $S \cap I_k, k = 0, 1, ...$ Let $\{x\} = \bigcap_{k=0}^{n} I_k$ and for each k, choose $x_k \in S \cap I_k$ Then $x_k \to x$. Since S is closed, it follows that $x \in S$. Therefore $x \in G_x$ for some $G_x \in \mathcal{F}$. Since G is open, there is k_0 such that $x \in I_{k_0} \subseteq G_x$. Thus the family $\{G_x\}$ is a finite subcover of $I_{k_0} \cap S$. This is a contradiction.

8.5.4 Remark. An advantage of this characterization of compactness is that it remains meaningful when the discussion is extended to topological spaces much more general than $\mathbb{R}$ and thus serves as the basis for a definition of compactness in these more general spaces.

8.5.52.3 Solution. Let S be a bounded and closed subset of $\mathbb{R}$, and $\mathcal{F}$ be an open cover of S. First, suppose that $S \subseteq [a, b]$. Now, consider the set

$$T = \{x \leq b; [a, x] \text{ has a finite subcover of } \mathcal{F}.\}$$

$T \neq \emptyset$, for $[a, a] = \{a\} \in F$, for some $F \in \mathcal{F}$. Since T is bounded above by b, so $c = \sup T \in [a, b]$ exists. If $c = b$, then we are done. Let $c < b$, so there is an $F \in \mathcal{F}$ such that $c \in F$. Since F is open, there exists $\epsilon > 0$ such that $(c - \epsilon, c + \epsilon) \subseteq F$. So, $\exists\, y \in T$ such that $y > c - \epsilon$. Thus $[a, y]$ has a fnite subcover $\{F_1, F_2, .., F_k\}$. Consequently, the collection $\{F_1, F_2, .., F_k, F\}$ covers $[a, c + \epsilon)$, so each point of $[c, c + \epsilon)$ is a member of T and this contradicts that $c = \sup T$. Hence $c = b$. Thus $[a, b]$ can be covered by a finite number of sets from $\mathcal{F}$.
Now, let $\mathcal{A}$ be the collection obtained by addition of S^C i.e. $\mathcal{A} = \mathcal{F} \cup \{S^C\}$, as S^C open. Hence $\mathcal{A}$ is a cover of $\mathbb{R}$ and therefore of $[a, b]$. By our previous case, there exists a finite subcover $\{F_1, F_2, .., F_n, S^C\}$ of $[a, b]$. Thus

$$\begin{aligned} &[a, b] \subseteq F_1 \cup F_2 \cup .. \cup F_n \cup S^C \\ \Rightarrow &S \subseteq [a, b] \subseteq F_1 \cup F_2 \cup .. \cup F_n. \end{aligned}$$

Converse part follows from the previous proof. □

8.5.53 Problem. (**Bolzano-Weierstrass**). Every infinite subset X of a closed bounded interval $[a, b]$ has an accumulation point in $[a, b]$.
Equivalent Statement. Every subset X of $[a, b]$ which has no accumulation point in $[a, b]$ is finite.

8.5.53.1 Solution. If no point x of $[a, b]$ is an accumulation point of X, every x has an open neighborhood $N(x; \epsilon_x)$ containing at most one point of X, namely, x itself. These $N(x; \epsilon_x)$ form an open covering of $[a, b]$; since $[a, b]$ is compact, there exists a finite number of these $N(x; \epsilon)$, say $N(x_i; \epsilon_{x_i})(i = 1, 2, ..., n)$, which cover $[a, b]$. Thus X contains at most the points $x_i(i = 1, 2, ..., n)$.

8.5.5 Remark. Here again let us note that the assertion of above theorem does not extend either to bounded intervals which are not closed or to unbounded intervals. For example, the infinite sequence of points $1/n$ of the semi-open interval $(0, 1]$ has no accumulation point in $(0, 1]$; in fact its only accumulation point in $\mathbb{R}$ is the point 0, which does not belong to $(0, 1]$. □

In order to characterize compact sets, we need to know some of their elementary properties.

8.5.54 Problem. Let K be a compact subset of $\mathbb{R}$. Then

1. K is closed.
2. K is bounded.
3. K is complete.
4. There exists a sequence D that is dense in K.
5. If p is the unique cluster point of a sequence (p_n) of elements p_n of K, then p is the limit of (p_n).

8.5.54.1 Solution.

1. Use Heine-Borel theorem.
2. Use Heine-Borel theorem.
3. Left to the reader.
4. Let K be a compact subset of $\mathbb{R}$. For each rational number $r > 0$, the collection $\{B(p; r); p \in K\}$ of open balls of radius r as p varies in K is an open cover of K, let

   $$\{B(p_{r_1}; r), ..., B(p_{r_{k(r)}}; r)\}$$

   be a finite subcover. (Note that the number of such balls, $k(r)$, may depend on r.) We claim that the set $D = \{p_{r_{k(r)}}; r \in \mathbb{Q}\}$ of centers of all of these balls as $r > 0$ varies is then the countable, dense subset of K we are looking for. Indeed, we can express D as

   $$D = \bigcup_{0<r\in\mathbb{Q}} \{p_{r_1}, ..., p_{r_{k(r)}}\},$$

   so it is the countable union of finite sets, and is thus countable. To show that D is dense in K, it suffices to show that any open ball of K contains an element of D. To this end, let $p \in K$ and let $\epsilon > 0$; we want to show that $B(p; r)$ contains an element of D. Let t be a rational

number between 0 and ϵ. Then p is in at least one of the balls of radius t forming our open cover above: without loss of generality, say that $p \in B(p_{t_1}; t)$. Then

$$|p_{t_1} - p| < t,$$

so p_{t_1} is in the ball $B(p; t)$, which is itself contained in the ball $B(p; \epsilon)$, showing that this ball contains an element (namely p_{t_1}) of D as required. We conclude that D is a countable dense subset of K.

Another proof: We shall construct the sequence (a_n) by recursion. Take $a_0 = x_0 \in \mathbb{R}$. We construct a_1 such that

$$|a_1 - a_0| \geq \frac{1}{2} d_0 \text{ where } d_0 = \sup_{x \in K} |a_0 - x|$$

and knowing the a_i's for $i \leq n-1$, we construct a_n such that

$$\min_{0 \leq i \leq n-1} |a_n - a_i| \geq \frac{1}{2} d_{n-1} \text{ where } d_{n-1} = \sup_{x \in K} \min_{0 \leq i \leq n-1} |x - a_i|.$$

It is clear that $d_0 \geq d_1 ... \geq d_n \geq ...$ The sequence d_n converges to 0. Indeed, there exists a subsequence (a_{n_k}) that is convergent and, therefore, is a Cauchy sequence: For every $\epsilon > 0$, there is k_0 such that

$$d_{n_{k-1}} \leq 2|a_{n_{k_0}} - a_{n_k}| \leq \epsilon \text{ for } k \geq k_0.$$

Thus the subsequence converges to 0, and since (d_n) is decreasing, the sequence d_n converges to 0. Thus for every $x \in K$ and for every $\epsilon > 0$, there exists $n(\epsilon)$ such that for $n > n(\epsilon)$ we have

$$\min_{0 \leq i \leq n} |x - a_i| \leq \sup_{x \in K} \min_{0 \leq i \leq n} |x - a_i| = d_n \leq \epsilon.$$

Hence there exists $a_i(x, \epsilon)$ such that $|x - a_i(x, \epsilon)| \leq \epsilon$, which implies that $D = \{a_0, a_1, .., a_n, ..\}$ is dense in K.

5. Suppose that (p_n) does not converge to p. Then there exists $\epsilon > 0$ such that for every $N \in \mathbb{N}$, there exists $n >\in \mathbb{N}$ such that
$$|p_n - p| \geq \epsilon.$$
First pick such an $n_1 > 1$, so that $|p_{n_1} - p| \geq \epsilon$. Then there exists $n_2 > n_1$ such that $|p_{n_2} - p| \geq \epsilon$, and then $n_3 > n_2$ so that $|p_{n_3} - p| \geq \epsilon$. Continuing in this manner it produces a subsequence (p_{n_k}) of (p_n) so that $|p_{n_3} - p| \geq \epsilon \ \forall \ k \in \mathbb{N}$. Since K is compact, this sequence has a convergent subsequence (p_{n_k}), which converges to p since this is also a convergent subsequence of the original sequence (p_n). This is a contradiction since each term in this subsequence is at a distance at least ϵ away from p, and so cannot converge to p. We conclude that (p_n) converges to p. □

8.5.55 Problem. Construct a sequence (x_n) of real numbers that has exactly one cluster point but (x_n) is not convergent. Can (x_n) be bounded?

8.5.55.1 Solution. Put $x_{2n} = \frac{1}{n}, x_{2n-1} = n$. No. □

8.5.56 Problem. Prove that every compact subset of $\mathbb{R}$ has a countable dene set.

8.5.56.1 Solution. Hint: Consider the centers of finite coverings by open balls of radii 2^{-n}, for $n \in \mathbb{N}$. □

8.5.57 Problem. Let the function $f : \mathbb{R} \to \mathbb{R}$ be bounded on bounded sets and have the property that $f^{-1}(K)$ is closed whenever K is compact. Prove f is continuous.

8.5.57.1 Solution. We need only show that $f^{-1}(K)$ is closed whenever F is closed. Let F be a closed subset of $\mathbb{R}$. Let (x_n) be a convergent sequence in $f^{-1}(F)$ with limit x_0. We need only show that x_0 is in $f^{-1}(F)$. For $n > 0$, let $y_n = f(x_n)$. The sequence (y_n) is in F and it is bounded (since the convergent sequence (x_n) is). Passing to a subsequence, we can assume the sequence (y_n) converges, say to y_0, which lies in F because F is closed. The set $K = \{y_1, y_2, ...\}$ is then compact, so $f^{-1}(K)$ is closed. Hence $f^{-1}(K)$ contains its limit points. But the sequence (x_n) lies in $f^{-1}(K)$ and converges to x_0. Therefore x_0 is in $f^{-1}(K)$, and so also in $f^{-1}(F)$, as desired. □

8.5.58 Problem. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and $A \subseteq \mathbb{R}$ compact. Must it be true that $f^{-1}(A)$ is compact? Prove it or provide a counterexample.

8.5.58.1 Solution. No. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \dfrac{1}{1+x^2}$ and observe that $0 < f(x) \leq 1$ for all $x \in \mathbb{R}$. Now $[0, 1] \subseteq \mathbb{R}$ is compact but $f^{-1}([0, 1]) = \mathbb{R}$ is not compact. □

8.6 Additional Exercises on Compactness.

8.6.1 Exercise. Show that any compact set $K \subseteq \mathbb{R}$ is complete; i.e., every Cauchy sequence $(x_n) \in K^{\mathbb{N}}$ converges to a limit in K.

8.6.2 Exercise. Give an example of a continuous function with domain $\mathbb{R}$ such that the inverse image of a compact set is not compact.

8.6.3 Exercise. For any sets $A, B \subseteq \mathbb{R}$, define $A + B = \{a + b; a \in A; b \in B\}$:

1. Show that, if A and B are compact, then so is $A + B$:

2. Give an example to show that if A and B are closed, then $A + B$ need not be closed.

3. Show, however, that if A is compact and B is closed, then $A + B$ is closed.

8.6.4 Exercise. Show that a set $K \subseteq \mathbb{R}$ is compact if and only if every countable open cover $\{A_n; n \in \mathbb{N}\}$ of K has a finite subcover.

8.6.5 Exercise. Let $\mathcal{A} = \{A_\alpha; \alpha \in \Lambda\}$ be an open cover of a compact set $K \subseteq \mathbb{R}$. Show that there is an $\epsilon > 0$ such that, $\forall\, x \in K$, we have $B(x; \epsilon) \subseteq A_\alpha$ for some $\alpha \in \Lambda$.

8.6.6 Exercise. Let S be a subset of $\mathbb{R}$ such that every infinite subset of S has at least one limit point in S. Prove that S is a closed set.

8.6.7 Exercise. If p be a limit point of a set $S \subseteq \mathbb{R}$ prove that there exists a countably infinite subset of S having p as its only limit point.

8.6.8 Exercise. Regard $\mathbb{Q}$, the set of all rational numbers, as a metric space with $d(x, y) = |x - y|$. Let E be the set of all $x \in \mathbb{Q}$ such that $2 < x^2 < 3$. Show that E is closed and bounded in $\mathbb{Q}$, but that E is not compact in $\mathbb{Q}$. Is E open in $\mathbb{Q}$?

8.6.9 Exercise. Show that a set $S \subseteq \mathbb{R}$ is *totally bounded* if and only if it is *bounded.*

8.6.10 Exercise. For each of the following statements, determine whether it is true or false and justify your answer.

1. The set of irrational numbers is closed.
2. The set of rational numbers in the interval [0,1] is compact.
3. The set of negative numbers is closed.

8.6.11 Exercise. Let S be compact and K_n a decreasing sequence of closed subsets of S. If $f : S \to S$ is continuous, show that

$$f\left(\bigcap_{n=1}^{\infty} K_n\right) = \bigcap_{n=1}^{\infty} f(K_n).$$

8.6.12 Exercise. Let $f : [a, b] \to \mathbb{R}$ be continuous. Show that the graph of f, $G(f) = \{(x, f(x)); x \in [a, b]\}$, is a compact subset . Can the compactness be dropped? Can you generalize?

8.6.13 Exercise. Let A be subset of $\mathbb{R}$ that is not compact. Show that $\exists$ a continuous function $f : A \to \mathbb{R}$ that does not attain a largest value.

8.6.14 Exercise. If K is a compact subset of $\mathbb{R}$, E is a closed subset of K, and if $f : K \to \mathbb{R}$, is continuous, show that $f(E)$ is closed. Can the compactness of K be dropped?

8.6.15 Exercise. If K is a compact subset of $\mathbb{R}$, and $f : K \to \mathbb{R}$ is continuous, and $f(x) > 0 \;\forall\, x \in K$, show that $\exists\, \delta > 0$ such that $f(x) > \delta \;\forall\, x \in K$.

8.6.16 Exercise. Let $Y \subseteq \mathbb{R}$ be compact and $f : Y \to \mathbb{R}$ be continuous. If $Z = \{z \in Y; f(z) = \sup\{f(y) : y \in Y\}\}$, show that $Z \neq \emptyset$ and Z is compact.

8.6.17 Exercise. For each $x \in (0, 1)$, let I_x denote the open interval $\left(\frac{1}{2}x, \frac{1}{2}(x + 1)\right)$, show that $\mathcal{F} = \{I_x; x \in (0, 1)\}$ is an open cover of $(0, 1)$ which admits of no finite subcover of $(0, 1)$.

8.6.18 Exercise. Let $f : E \to \mathbb{R}$ have this property. For every $a \in E$ there is an $\epsilon > 0$ so that

$$f(x) > \epsilon \text{ if } x \in E \cap (a - \epsilon, a + \epsilon).$$

1. Show that if the set E is compact then there is some positive number c so that
$$f(a) > c \;\forall\, a \in E.$$
2. Show that if E is not compact this conclusion may not be valid.

8.6.19 Exercise. Prove that every open (bounded or unbounded) interval can be expressed as a countable union of compact sets.

8.6.20 Exercise. Use the Heine-Borel Theorem to prove the following version of the Bolzano-Weierstrass Theorem: Every bounded infinite subset of $\mathbb{R}$ has a cluster point in $\mathbb{R}$.

8.6.21 Exercise. Prove that the intersection of an arbitrary collection of compact sets in $\mathbb{R}$ is compact.

8.6.22 Exercise. Let A be a subset of $\mathbb{R}$ and assume that A contains a sequence $a_1, a_2, ...$ such that, for some positive number δ, $|a_m - a_n| \geq \delta$ for each $m \neq n$. Show that A is not compact.

8.6.23 Exercise. Let A be a compact subset of $\mathbb{R}$, let L be any fixed real number, and let $f : A \to \mathbb{R}$ be a function with the property that $|f(x) - f(y)| \leq L|x - y|$ for all $x, y \in A$. Show that $f(A)$ is compact.

8.6.24 Exercise. Let $f : E \to \mathbb{R}$ have this property: For every $e \in E$ there is an $\epsilon > 0$ so that $f(x) > \epsilon$ if $x \in E \cap (e - \epsilon, e + \epsilon)$. Show that if the set E is compact then there is some positive number c so that $f(e) > c$ for all $e \in E$. Show that if E is not closed or is not bounded, then this conclusion may not be valid.

8.6.25 Exercise. (Zan Armstrong). Prove or give a counter-example that: A set $S \subseteq \mathbb{R}$ is compact if every continuous function on S is uniformly continuous.

8.6.26 Exercise. A subset $E \subseteq \mathbb{R}$ is called **discrete** if all of its points are isolated points. Give a characterization of compact discrete sets. Give an example of a noncompact discrete set.

8.6.27 Exercise. Show that a finite union of compact sets is compact.

8.6.28 Exercise. Show that an infinite union of compact sets need not be compact.

8.6.29 Exercise. Give an example of a set E (other than $\emptyset$; and $\mathbb{R}$) that has the following property or else show that such a set cannot exist:

1. E has infinitely many points but no interior points.
2. E has infinitely many points but no points of accumulation.
3. E is open and unbounded.
4. E is closed and unbounded.
5. E has infinitely many points of accumulation but no interior points.
6. E is open but has no points of accumulation.
7. E is closed but has no points of accumulation.
8. E is compact and has no interior points.
9. E, E' and E'' are different.
10. E is countable and $E' = \{0, 1\}$.
11. E is countable and $E' = [0, 1]$.
12. E is countable and $E' = (0, 1)$.

8.6.30 Exercise. Prove that the following are equivalent:

1. **LUB Property**: Each non-empty set of real numbers that is bounded above has a supremum.
2. **GLB Property**: Each non-empty set of real numbers that is bounded below has an infimum.
3. **Completeness Axiom**: Every Cauchy sequence of real numbers converges.
4. **Monotone Convergence Theorem**: Every bounded monotone sequence of real numbers converges.
5. **Bolzano-Weirstrass Theorem**: Every bounded infinite subset of $\mathbb{R}$ has at least one limit point.
6. **Nested intervals Theorem**: If $([a_n, b_n])$ is a nested sequence of closed and bounded intervals, then there a point ξ that belongs to all of the intervals $[a_n, b_n]$.
7. **Sequentially compact**: Every bounded sequence has a cluster point.
8. **Open-cover compact**: Every open cover of a closed and bounded subset of $\mathbb{R}$ has a finite subcover.
9. **Compact**: Every sequence in a closed and bounded subset $A \subset \mathbb{R}$ has a cluster point in A.

8.7 Connectedness

Introduction

The intuitive meaning of a connected set in $\mathbb{R}$ that it is one piece; that is, it is not possible to represent it as the union of two seperated sets A and B in $\mathbb{R}$. By seperated sets A and B we mean $\overline{A} \cap B = \emptyset$ and $A \cap \overline{B} = \emptyset$. This is a stronger on A and B than disjointness but not as strong as requiring that the distance between them should be positive. For example, in $\mathbb{R}$ the sets $[0, 1]$ and $(1, 2]$ are disjoint but not not seperated. On the other hand the sets $[0, 1)$ and $(1, 2]$ are seperated but distance between them is 0. Thus, a set X is **not connected** if there are two non-empty sets A and B such that $X = A \cup B$ and $\overline{A} \cap B = \emptyset$ and $A \cap \overline{B} = \emptyset$. If such is the case then $\overline{A} \subseteq B^C = A$ and hence, A is closed. Similarly B must be closed set. If A and B are closed sets then the statements $\overline{A} \cap B = \emptyset$ and $A \cap \overline{B} = \emptyset$ are both equivalent to the single statement $A \cap B = \emptyset$. Thus, we can say that a set X is **connected** if there exists no pair of closed sets A and B in $\mathbb{R}$ such that $X = A \cup B$ and $A \cap B = \emptyset$. Since the sets A and B in such a pair are complemenary we can eqally well say that they are both open and the definition is usually given in these terms.

8.7.1 Definition. Two nonempty subsets A and B of $\mathbb{R}$ which satisfy

$$A \cap \overline{B} = \overline{A} \cap B = \emptyset.$$

are said to be **mutually separated** in $\mathbb{R}$.

To study connectedness we review certain examples in $\mathbb{R}$.

8.7.2 Example. Let $A = (-1, 0) \cup (0, 1)$ and $f : A \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} -1, & \text{if} -1 < x < 0 \\ 1, & \text{if } 0 < x < 1. \end{cases}$$

Now, let any point $a \in A$, suppose $a \in (0, 1)$, then

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = 0.$$

Hence for all $x \in A,\ f'(x) = 0$, but f is not constant on A.

8.7.3 Example. Again, f is continuous on A (in the previous example) and for $a \in (-1, 0)$ and $b \in (0, 1)$ $f(a) < 0,\ f(b) > 0$, but there is no $c \in A$ such that $f(c) = 0$.

8.7.4 Example. Let $f : \mathbb{Q} \to \mathbb{R}$ and $f(x) = x^2 - 2$, then f is continuous on $\mathbb{Q}$ and $f(2) > 0$ and $f(1) < 0$, but there is no $x \in \mathbb{Q}$ such that $f(x) = 0$.

All the pathological behavior in the above examples can be remedied, if the domain of the functions were connected.

8.7.5 Definition. Let $A \subseteq \mathbb{R}$, then A is an **interval** iff for each $a, b \in A$ with $a < b$, the set $(a, b) = \{x; a < x < b\} \subseteq A$.

8.7.6 Remark. We shall see that, for any disjoint open sets U, V with $A \subseteq U \cup V$ implies either $A \subseteq U$ or $A \subseteq V$. The analogue of the interval in an abstract metric space (X, d) are **connected** sets.

8.8 Disconnected, Connected:

8.8.1 Definition. A set $E \subseteq \mathbb{R}$ is said to be **disconnected**, if there exist two nonempty disjoint subsets U and V of E, each of them is open relatively to E, so that $E = U \cup V$. If a set E is not disconnected then it is said to be **connected.** In this case, we say that $(U|V)$ is a seperation of E. Finally, a set $S \subseteq \mathbb{R}$ is called **totally disconnected** if for any $x, y \in S$ such that $x < y$, there exists $z \in (x, y)$ such that $z \notin S$.

8.8.2 Remark. Assume that $U \subseteq E$ is open relatively to E. Then $E \setminus U$ is closed relatively to E. So, an equivalent definition is that E is disconnected whenever there exist a nonempty subset U of E that $U \neq E$ and U is simultaneously open and closed relatively to E.
Observe that the set $\mathbb{Q} \cap [0, 1]$ is disconnected. Indeed, the two sets

$$U = \mathbb{Q} \cap \left(-\infty, \frac{\sqrt{2}}{2}\right) \cap [0, 1], \text{ and } V = \mathbb{Q} \cap \left(\frac{\sqrt{2}}{2}, \infty\right) \cap [0, 1]$$

are both open relatively to $\mathbb{Q} \cap [0, 1]$, disjoint and nonempty, and $U \cup V = \mathbb{Q} \cap [0, 1]$. Another example of a disconnected subset of $\mathbb{R}$ is the set $(0, 2) \setminus \{1\}$, since it can be written as $(0, 1) \cup (1, 2)$.

8.8.3 Definition. A real-valued function f on a set $S \subseteq \mathbb{R}$ is said to be two-valued on S; if $f(S) \subseteq \{a, b\}, a \neq b$.

8.8.4 Theorem. The following are equivalent:

1. $\mathbb{R}$ is connected.
2. The only subsets of $\mathbb{R}$ that are both open and closed are the empty set and $\mathbb{R}$ itself.
3. Every continuous function $f : \mathbb{R} \to \{0, 1\}$ is constant.

8.8.5 Theorem (Intermediate value theorem for real-valued continuous functions)**.** Let f be real-valued and continuous on a connected subxet S of $\mathbb{R}$. If f takes on two different values in S, say a and b, then for each real c between a and b there exists a point $x \in S$ such that $f(x) = c$.

8.8.6 Theorem. If $\mathcal{S} = \{S_\alpha; \alpha \in \Lambda\}$ is a family of connected subsets of $\mathbb{R}$ and if $T = \bigcap_{\alpha\in\Lambda} S_\alpha \neq \emptyset$, then $U = \bigcup_{\alpha\in\Lambda} S_\alpha$ is also connected.

Every point x in $\mathbb{R}$ belongs to at least one connected subset of $\mathbb{R}$, namely $\{x\}$. By the above theorem, the union of all the connected subsets which contain x is also connected. We call this union a **component** of $\mathbb{R}$, and we denote it by $C(x)$. Thus, $C(x)$ is the **maximal connected** subset of $\mathbb{R}$ which contains x.

8.8.7 Theorem. Every point of $\mathbb{R}$ belongs to a uniquely determined component of $\mathbb{R}$. In other words, the components of $\mathbb{R}$ form a collection of disjoint sets whose union is $\mathbb{R}$.

8.9 Problems and Solutions on Connectedness.

Warning! *Never Never Ever read this solution unless you tried problems quite long time and gave up. Doing so may impair your ability to think and solve problems.*

8.9.1 Problem. A subset E of $\mathbb{R}$ is connected if and only if eveiy continuous function $f : E \to \{p, q\}; p \neq q$ is constant.

8.9.1.1 Solution. Suppose that f is not constant on E. Let $0 < \epsilon = \frac{|p-q|}{2}$, since f is continuous then both $f^{-1}(B(p;\epsilon))$ and $f^{-1}(B(q;\epsilon))$ are open, disjoint with

$$f^{-1}(B(p;\epsilon)) \cup f^{-1}(B(q;\epsilon)) = E,$$

which is impossible, as E is connected.
If f is constant, then $f(x) = p$ or $f(x) = q$ $\forall$ $x \in E$. Hence $\forall$ $\epsilon > 0$ $f^{-1}(B(p;\epsilon)) = E$ or $f^{-1}(B(q;\epsilon)) = E$. Thus f is continuous on E. □

8.9.2 Problem. The function $f : \mathbb{R} \to \mathbb{R}$ continuous if and only if its graph $gr(f)$ is a closed and connected subset of $\mathbb{R}^2$.

8.9.2.1 Solution. First, let $gr(f)$ be closed and connected. Since $gr(f)$ is connected, it is easy to see that f will have IVP. So, $f : \mathbb{R} \to \mathbb{R}$ has the IVP and $gr(f)$ is closed (and connected) and we are to show that f is continuous. Suppose f is not continuous at $x = c$. Then for some $\epsilon > 0$, there exists a sequence (b_n) of real numbers tending to c, such that $|f(b_n) - f(c)| > \epsilon$ for each $n \in \mathbb{N}$, i.e., $-\epsilon > f(b_n) - f(c) > \epsilon$. Let us consider the right hand inequality $f(b_n) > f(c) + \epsilon$. By the IVP, there exists c_n between b_n and c such that $f(c_n) = f(c) + \epsilon$. Thus, the sequence of points $\{(c_n, f(c_n))\} \to (c, f(c) + \epsilon) \notin gr(f)$, a contradiction, since $gr(f)$ is closed.
Conversely, suppose f is continuous and we have to show that (i) $gr(f)$ is closed and (ii) $gr(f)$ is

connected.
Proof of (i). If not, then there exists a sequence of points $(a_n, f(a_n)) \in gr(f), a_n \to a$, with limit not in $gr(f)$. However, this limit is $(a, f(a))$ as $a_n \to a$ and $f(a_n) \to f(a)$ (since f is continuous), so $(a, f(a)) \notin gr(f)$, which is absurd since for every $x \in \mathbb{R}, (x, f(x)) \in gr(f)$, by the definition of $gr(f)$.
Proof of (ii). If not, let $gr(f) \subseteq H \cup K$, where $gr(f) = \{(a, f(a)); a \in \mathbb{R}\},\ H, K \neq \emptyset,\ H \cap K = \emptyset$ and H, K are open. Now put $S_1 = \{a;\ (a, f(a)) \in H\},\ S_2 = \{a;\ (a, f(a)) \in K\}$. Then S_1, S_2 are non-empty, open and disjoint subsets of $\mathbb{R}$ such that $S_1 \cup S_2 = \mathbb{R}$, giving a contradiction, since $\mathbb{R}$ is connected. □

8.9.3 Problem. In each case, give an example of a real-valued function f, continuous on S and such that $f(S) = T$, or else explain why there can be no such f:

1. $S = (0, 1),\ T = (0, 1]$.
2. $S = (0, 1),\ T = (0, 1) \cup (1, 2)$.
3. $S = \mathbb{R},\ T =$ the set of rational numbers.
4. $S = [0, 1] \cup [2, 3],\ T = \{0, 1\}$.
5. $S = [0, 1] \times [0, 1],\ T = \mathbb{R}^2$.
6. $S = [0, 1] \times [0, 1],\ T = (0, 1) \times (0, 1)$.
7. $S = (0, 1) \times (0, 1),\ T = \mathbb{R}^2$.

8.9.3.1 Solution.

1. Define $f : (0, 1) \to (0, 1]$ by $f(x) = \begin{cases} 2x & \text{if } x \in \left(0, \frac{1}{2}\right) \\ 1 & \text{if } x \in \left[\frac{1}{2}, 1\right] . \end{cases}$
2. S is connected, but T is disconnected.
3. S is connected, but T is disconnected.
4. Define $f : [0, 1] \cup [2, 3] \to \{0, 1\}$ by $f(x) = \begin{cases} 0 & \text{if } x \in [0, 1] \\ 1 & \text{if } x \in [2, 3]. \end{cases}$
5. S is compact, but T is not compact.
6. S is compact, but T is not compact.
7. Define $f : (0, 1) \times (0, 1) \to \mathbb{R}^2$ by
$$f(x, y) = \left(\frac{1 - 2x}{x(1 - x)}, \frac{1 - 2y}{y(1 - y)} \right) \text{ if } x \in (0, 1), y \in (0, 1). \quad \square$$

8.9.4 Problem. Let $f : [a, b] \to \mathbb{R}$. The following conditions are equivalent.

1. f is Darboux continuous.
2. For any connected set $C \subseteq [a, b], f(C)$ is connected.

3. For any closed subinterval $[p, q]$ of $[a, b]$, $f([p, q])$ is connected.

8.9.4.1 Solution.

1. (1) implies (2): Let C be a connected subset of $[a, b]$ and let $f(\alpha)$ and $f(\beta)$ be elements of $f(C)(\alpha, \beta \in C, \alpha \leq \beta)$. Then $[\alpha, \beta] \subseteq C$. For every c between $f(\alpha)$ and $f(\beta)$ there is an s between $f(\alpha)$ and $f(\beta)$ such that $f(s) = c$. Hence, $f(C)$ contains the closed interval with end points $f(\alpha)$ and $f(\beta)$. Thus,$f(C)$ is connected.

2. (2) implies (3) is trivial.

3. (3) implies (1): Let $p, q \in [a, b], p < q$ and let c be between $f(p)$ and $f(q)$. Since $f([p, q])$ is connected, we have $c \in f([p, q])$. □

8.9.5 Problem. A subset A of $\mathbb{R}$ is disconnected if and only if it is the union of two nonempty sets mutually separated in $\mathbb{R}$.

8.9.5.1 Solution. If A is disconnected, it can be written as the union of two disjoint nonempty sets U and V which are open in A. (These sets need not, of course, be open in $\mathbb{R}$.) We show that U and V are mutually separated. It suffices to prove that $U \cap V$ is empty, that is, $U \subseteq \mathbb{R} \setminus V$. To this end suppose that $u \in U$. Since U is open in A, there exists $r > 0$ such that

$$A \cap B(u; r) = \{x \in A; |x - u| < r\} \subseteq U \subseteq \mathbb{R} \setminus V.$$

The interval $B(u; r) = (u - r, u + r)$ is the union of two sets: $A \cap B(u; r)$ and $A^C \cap B(u; r)$. We have just shown that the first of these belongs to $\mathbb{R} \setminus V$. Certainly the second piece contains no points of A and therefore no points of V. Thus $B(u; r) \subseteq \mathbb{R} \setminus V$. This shows that u does not belong to the closure (in $\mathbb{R}$) of the set V; so $u \in \mathbb{R} \setminus V$. Since u was an arbitrary point of U, we conclude that $U \subseteq \mathbb{R} \setminus V$. Conversely, suppose that $A = U \cup V$ where U and V are nonempty sets mutually separated in $\mathbb{R}$. To show that the sets U and V disconnect A, we need only show that they are open in A, since they are obviously disjoint. Let us prove that U is open in A. Let $u \in U$ and notice that since $U \setminus V$ is empty, u cannot belong to V. Thus there exists $r > 0$ such that $B(u; r)$ is disjoint from V. Then certainly $A \cap B(u; r)$ is disjoint from V. Thus $A \cap B(u; r)$ is contained in U. Thus U is open in A. □

8.9.6 Problem.

1. Give an example of a function $f \in C(\mathbb{R})$ and a compact set $K \subseteq \mathbb{R}$ such that $f^{-1}(K)$ is not compact.

2. Give an example of a function $f \in C(\mathbb{R})$ and a connected set $C \subseteq \mathbb{R}$ such that $f^{-1}(C)$ is not connected.

8.9.6.1 Solution.

1. Define $f : \mathbb{R} \to \mathbb{R}$ by
$$f(x) = \frac{1}{1 + x^2}$$
Observe that $f \in C(\mathbb{R})$. Now $[0, 1] \subseteq \mathbb{R}$ is compact but $f^{-1}([0, 1]) = \mathbb{R}$ is not compact.

2. Let $f : (1, 2) \cup (3, 4) \to \mathbb{R}$ be a function defined by $f(x) = 1; \forall\, x \in (1, 2) \cup (3, 4)$. Any connected set C containing 1, $f^{-1}(C) = (1, 2) \cup (3, 4)$ which is not connected. □

8.9.7 Problem. No subset of $\mathbb{R}$ but $\mathbb{R}$ itself or may be simultaneously open and closed in $\mathbb{R}$. In other words, $\mathbb{R}$ is connected.

8.9.7.1 Solution. Let $\mathbb{R} = U \cup V$ be a separation, and let $a \in U, b \in V$. We may assume $a < b$. Let

$$Y = \{x; x \in [a,b], x \in U\}$$

and let $c = \sup Y$. We have $c < b$ since $(b-\epsilon, b] \subseteq V$ for some $\epsilon > 0$. Either $c \in U$ or $c \in V$. If $c \in U$, then $[c, c+\epsilon) \subseteq U$ for some $\epsilon > 0$, contradicting that $c = \sup Y$. If $c \in V$, then $(c-\epsilon, c] \subseteq V$ for some $\epsilon > 0$, so $c - \epsilon$ would be an upper bound for Y, another contradiction. So no separation exists. $\square$

8.9.8 Problem. Prove that The Following Are Equivelent:

1. $\mathbb{R}$ is a connected set.
2. $\mathbb{R}$ cannot be written as the union of two disjoint non-empty open sets.
3. the only subsets of $\mathbb{R}$ which both open and closed are $\emptyset$ and $\mathbb{R}$.

8.9.8.1 Solution. We prove the equivalence by negations of (i),(ii) and (iii).

1. not (i)$\Rightarrow$ not (ii): Suppose $\mathbb{R} = A \cup B$ where A and B are non-empty and $A \cap \overline{B} = B \cap \overline{A} = \emptyset$. (Such sets A, B are called **seperated** sets.) This implies that $B \subseteq (\overline{A})^C$ and $A \subseteq (\overline{B})^C$ and hence that $(\overline{A})^C$ and $(\overline{B})^C$ are non-empty. Note also that $(\overline{A})^C \cap (\overline{B})^C = (\overline{A} \cup \overline{B})^C = \mathbb{R}^C = \emptyset$. Therefore $\mathbb{R} = (\overline{A})^C \cup (\overline{B})^C$ shows that $\mathbb{R}$ is the union of two non-empty disjoint open sets.
2. not (ii)$\Rightarrow$ not (iii): Assume that $\mathbb{R} = U \cup V$, where U and V are non-empty open sets, and $U \cap V = \emptyset$. Then $U = V^C$ is closed as V is open. Since $U^C = V$ is non-empty we see that U is distinct from $\emptyset$ and $\mathbb{R}$, and is both open and closed.
3. not (iii)$\Rightarrow$ not (i): Let A be a set which is both open and closed, and distinct from $\emptyset$ and $\mathbb{R}$. Therefore $\mathbb{R} = A \cup A^C$ shows that $\mathbb{R}$ is disconnected metric space. $\square$

8.9.9 Problem. Every general interval in $\mathbb{R}$ is a connected set, and, conversely, every connected subset of $\mathbb{R}$ is a general interval.

8.9.9.1 Solution. Assume that $(a,b) = A \cup B$, where A and B are disjoint nonempty subsets of (a,b), both open relatively to (a,b). Since (a,b) is already open, the two sets A and B are open in $\mathbb{R}$. Take $a_0 \in A$ and $b_0 \in B$. Without loss of generality, we may assume that $a_0 < b_0$. Let $C = \{x \in (a_0, b) : (a_0, x] \subseteq A\}$. This set is nonempty, due to the fact that A is open. Let $a_1 = \sup C (> a_0)$. It exists since C is bounded. Note that $[a_0, a_1) \subseteq A$. If $a_1 \in A$, then we may find $a_2 \in (a_1, b)$ such that $[a_1, a_2) \subseteq A$, hence $(a_0, a_2] \subseteq A$, contradicting the definition of a_1. Thus, $a_1 \in B$. Since B is open, we can find $b_1 \in B, b_1 < a_1$, such that $(b_1, a_1] \subseteq B$, and this contradict again that for some $x \in (b_1, a_1]$ we have $(a_0, x] \subseteq A$. That the converse holds, i.e., that every connected subset of $\mathbb{R}$ is a general interval, is easy: assume that S is a connected subset of $\mathbb{R}$. If S fails to be an interval, there exists $x_0 \in \mathbb{R} \setminus S$ such that $S_1 = (-\infty, x_0) \cap S = \emptyset$, and $S_2 = (x_0, \infty) \cap S = \emptyset$. Then $S = S_1 \cup S_2$, and $S_1 \cap S_2 = \emptyset$. Since S_1 and S_2 are two open relatively to S subsets of S, we reach a contradiction. $\square$

8.9.10 Problem. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. If E is connected, then $f(E)$ is also connected.

8.9.10.1 Solution. Suppose that $f(E)$ is not connected, then there exists open sets U, V such that $(U|V)$ a seperation of $f(E)$, i.e. $f(E) = U \cup V$ with $U \cap V = \emptyset$. Hence $E = f^{-1}(U) \cup f^{-1}(V)$. Since f is continuous, so $f^{-1}(U)$ and $f^{-1}(V)$ are both open and $f^{-1}(U) \cap f^{-1}(V) = \emptyset$. Thus $\big(f^{-1}(U)|f^{-1}(V)\big)$ is a seperation of E, hence E is not connected, a contradiction. $\square$

8.9.11 Problem. Show that if $\mathcal{S} = \{S_\alpha; \alpha \in \Lambda\}$ is a family of connected subsets of $\mathbb{R}$ and if $T = \bigcap_{\alpha\in\Lambda} S_\alpha \neq \emptyset$, then $U = \bigcup_{\alpha\in\Lambda} S_\alpha$ is also connected.

8.9.11.1 Solution. Since $T \neq \emptyset$, there is some $t \in T$. Let f be a two-valued function on U. We will show that f is constant on U by showing that $f(x) = f(t) \forall x \in U$. If $x \in U$, then $x \in S_\alpha$ for some $S_\alpha \in \mathcal{S}$. Since S_α is connected, f is constant on S_α and, since $t \in S_\alpha, f(x) = f(t)$. $\square$

8.9.12 Problem. Let F_n be a nest of connected compact sets ($F_{n+1} \subseteq F_n$) in $\mathbb{R}$. Show that $K = \cap_{n=1}^{\infty} F_n$ is connected.

8.9.1 Remark. The Nested property is crucial since the intersection of two compact connected sets may not be connected. The example may be easily found in $\mathbb{R}^2$, not in $\mathbb{R}$ where the result is true. The above conclusion is false when we only assume that the sequence $\{F_n\}$ are closed. Indeed, take

$$F_n = \mathbb{R}^2 \setminus \left\{(x, y); |x| < \frac{1}{n} \text{ and } |y| < n\right\}.$$

It is easy to see that F_n are closed and

$$\bigcap_{n=1}^{\infty} F_n = \mathbb{R}^2 \setminus \{(0, y); y \in \mathbb{R}\}.$$

Moreover each F_n is connected while their intersection is not.

8.9.13 Problem. The set $\{1, 4, 8\}$ is disconnected.

8.9.13.1 Solution. Let $A = \{1, 4, 8\} \subseteq \mathbb{R}$. Notice that the sets $\{1\}$ and $\{4, 8\}$ are open subsets of A. For $(-1, 2) \cap A = \{1\}$ and $(2, 1) \cap A = \{4, 8\}$; so $\{1\}$ is the intersection of an open subset of $\mathbb{R}$ with A, and so is $\{4, 8\}$. Thus $\{1\}$ and $\{4, 8\}$ are disjoint nonempty open subsets of A whose union is A. That is, the sets $\{1\}$ and $\{4, 8\}$ disconnect A. $\square$

8.9.14 Problem. The set $\mathbb{Q}$ of rational numbers is disconnected.

8.9.14.1 Solution. The pair of sets $\{x \in \mathbb{Q}; x < \pi\}$ and $\{x \in \mathbb{Q}; x > \pi\}$ is a disconnection of $\mathbb{Q}$. $\square$

8.9.15 Problem. The set $\mathbb{Q}^2$ of points in $\mathbb{R}^2$ both of whose coordinates are rational is a disconnected subset of $\mathbb{R}^2$.

8.9.15.1 Solution. The subset $\mathbb{Q}^2$ is disconnected by the sets $\{(x, y) \in \mathbb{Q}^2; x < \pi\}$ and $\{(x, y) \in \mathbb{Q}^2; x > \pi\}$.

8.9.16 Problem. Let S be a nonempty subset of $\mathbb{R}$ such that its boundary is empty. Show that $S = \emptyset$ or $S = \mathbb{R}$.

8.9.16.1 Solution. $\partial S = \emptyset \Rightarrow \overline{S} \setminus S^\circ = \emptyset \Rightarrow \overline{S} \subseteq S^\circ \subseteq S$ and $S^\circ \subseteq S \subseteq \overline{S}$ implies $S^\circ = S = \overline{S}$. That is, S is both open and closed. By connectedness of $\mathbb{R}$, we can infer that either $S = \emptyset$ or $S = \mathbb{R}$. $\square$

8.9.17 Problem. The following result says that "at any time, on the surface of the earth, there exists two diametrically opposite points at which the temperature is the same." To prove this statement let $T : S \to \mathbb{R}$ be a continuous function, $S = \{z = (x, y) \in \mathbb{R}; x^2 + y^2 = 1\}$. Show that there is $p \in S$ such that $T(p) = T(-p)$.

8.9.17.1 Solution. If possible, let $\exists$ no $z \in S$ such that $T(z) = T(-z)$. Then consider the function $f : S \to \mathbb{R}$ defined by

$$f(t) = \frac{|T(t) - T(-t)|}{T(t) - T(-t)}.$$

We see that $f(S) = \{0, 1\}$. Now S is connected but $f(S)$ is not connected, a contradiction. $\square$

8.9.18 Problem. The function $f : \mathbb{R} \to \mathbb{R}$ continuous if and only if its graph $G(f) = \{(x, f(x)); x \in \mathbb{R}\}$ is a closed and connected subset of $\mathbb{R}^2$.

8.9.18.1 Solution. First, let $G(f)$ be closed and connected. Since $G(f)$ is connected, it is easy to see that f will have IVP. So, $f : \mathbb{R} \to \mathbb{R}$ has the IVP and $G(f)$ is closed (and connected) and we are to show that f is continuous. Suppose f is not continuous at $x = c$. Then for some $\epsilon > 0$, there exists a sequence (b_n) of real numbers tending to c, such that $|f(b_n) - f(c)| > \epsilon$ for each n, i.e., $-\epsilon > f(b_n) - f(c) > \epsilon$. Let us consider the right hand inequality $f(b_n) > f(c) + \epsilon$. By the IVP, there exists c_n between b_n and c such that $f(c_n) = f(c) + \epsilon$. Thus, the sequence of points $((c_n, f(c_n))) \to (c, f(c) + \epsilon) \notin G(f)$, a contradiction, since $G(f)$ is closed.
Conversely, suppose f is continuous and we have to show that (i) $G(f)$ is closed and (ii) $G(f)$ is connected. Proof of (i). If not, then there exists a sequence of points $(a_n, f(a_n)) \in G(f)$, as $a_n \to a$, with limit not in $G(f)$. However, this limit is $(a, f(a))$ as $a_n \to a$, and $f(a_n) \to f(a)$ (since f is continuous), so $(a, f(a)) \in G(f)$, which is absurd since for every $x \in \mathbb{R}, (x, f(x)) \in G(f)$, by the definition of $G(f)$.
Proof of (ii). If not, let $G(f) \subseteq H \cup K$, where $G(f) = \{(a, f(a)); a \in \mathbb{R}\}, H \cap K \neq \emptyset$, and H, K are open. Now put $S_1 = \{a; (a, f(a)) \in H\}, S_2 = \{a; (a, f(a)) \in K\}$. Then S_1, S_2 are non-empty, open and disjoint subsets of $\mathbb{R}$ such that $S_1 \cup S_2 = \mathbb{R}$, giving a contradiction, since $\mathbb{R}$ is connected. $\square$

8.9.19 Problem. Suppose that $f : \mathbb{R} \to \mathbb{R}$ has a closed graph $G(f)$ (closed as a subset of $\mathbb{R}^2$). Then each of the following conditions implies that f is continuous:

1. f is locally bounded,
2. f has IVP,
3. $G(f)$ is connected.

8.9.19.1 Solution.

(i) $\Rightarrow$ f is continuous: Suppose f is not continuous at the point $x = \xi$. Then there exists a sequence (a_n) such that $f(a_n) \to \eta$ with $\eta \neq f(\xi) \in \mathbb{R}$, as f is locally bounded. Thus, the sequence of points $((a_n, f(a_n))) \in \mathbb{R}^2$, contradicting the fact that $G(f)$ is closed, since the point $(\xi, \eta) \notin G(f)$ and is a limit point of $G(f)$.
(ii) $\Rightarrow$ f is continuous: already done in the proof of the above problem.
(iii) $\Rightarrow$ f is continuous. $\square$

8.9.20 Problem. Show that f has IVP does not imply f is locally bounded, and $gr(f)$ is connected does not imply f is locally bounded.

8.9.20.1 Solution. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \frac{1}{x}\sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Verify that this function has the IVP but it is not locally bounded at the point $x = 0$. Further, for this very function f, its graph is connected, showing $gr(f)$ is connected does not imply f is locally bounded. $\square$

8.9.21 Problem. Show that f is locally bounded does not imply f has the IVP and also that f is locally bounded does not imply its graph $gr(f)$ is connected.

8.9.21.1 Solution. The following is the example of such a function. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0. \end{cases}$$

8.9.22 Problem. Here is an application of connectedness. Is there a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f(\mathbb{Q}) \subseteq \mathbb{Q}^C$ and $f\left(\mathbb{Q}^C\right) \subseteq \mathbb{Q}$?

8.9.22.1 Solution. No. If such f exists, then both $f(\mathbb{Q})$ and $f(\mathbb{Q}^C)$ are countable. Hence, $f(\mathbb{R})$ is countable. In addition, $f(\mathbb{R})$ is connected. Since $f(\mathbb{R})$ contains rationals and irrationals, so $f(\mathbb{R})$ is an interval which implies that $f(\mathbb{R})$ is uncountable, a cotradiction. Hence, such f does not exist. $\square$

8.9.23 Problem. Suppose that $f : [a, b] \to \mathbb{R}$ satisfies $f^{-1}(y)$ is closed for all $y \in \mathbb{R}$ and $f([c, d])$ is connected for all $[c, d] \subseteq [a, b]$. Prove that f is continuous.

8.9.23.1 Solution. Let $x \in [a, b]$ and take a sequence (x_n) in $[a, b]$ such that $x_n \to x$. Then $I = \bigcap_{n=1}^{\infty} f([x_n, x])$ is an interval containing $f(x)$. We claim that $I = \{f(x)\}$ and therefore $f(x_n) \to f(x)$. Indeed, take $f(y) \in I$. Then there exist $t_n \in [x_n, x]$ such that $f(t_n) = f(y)$. Hence $t_n \to x$ and $t_n \in f^{-1}(\{f(y)\})$. Since $f^{-1}(\{f(y)\})$ is closed, it follows that $x \in f^{-1}(\{f(y)\})$, and so $f(x) = f(y)$. $\square$

8.9.24 Problem. If S, T are subsets of $\mathbb{R}$ such that S is connected and $S \subseteq T \subseteq \overline{S}$, then T is connected. In particular, $\overline{S}$ is connected.

8.9.24.1 Solution. Suppose that A, B are nonempty open sets in the subspace T such that $T = A \cup B$ and $A \cap B = \emptyset$. As S is dense in T , both $S \cap A$ and $S \cap B$ are nonempty. They are clearly disjoint, and, they are open in S. Since $S = (S \cap A) \cup (S \cap B)$, we have contradicted the fact that S is connected. $\square$

8.9.25 Problem. Let A and B be connected subsets in $\mathbb{R}$ with $A \setminus B$ not connected and suppose $A \setminus B = C_1 \cup C_2$ where $\overline{C_1} \cap C_2 = C_1 \cap \overline{C_2} = \emptyset$. Show that $B \cup C_1$ is connected.

8.9.25.1 Solution. Assume that $B \cup C_1$ is not connected. and so we will prove that C_1 is disconnected. Using the relations $\overline{C_1} \cap C_2 = C_1 \cap \overline{C_2} = \emptyset$, we get

$$\begin{aligned} & C_1 \cap \overline{C_2 \cup (A \cap B)} \\ &= C_1 \cap \left[\overline{C_2} \cup \overline{(A \cap B)}\right] \\ &= \left[C_1 \cap \overline{C_2}\right] \cup \left[C_1 \cap \overline{(A \cap B)}\right] \\ &= C_1 \cap \overline{(A \cap B)} \subseteq C_1 \cap \overline{B} \end{aligned} \tag{8.1}$$

and

$$\begin{aligned}
&\overline{C_1} \cap [C_2 \cup (A \cap B)] \\
&= \left[\overline{C_1} \cap C_2\right] \cup \left[\overline{C_1} \cap (A \cap B)\right] \\
&= \overline{C_1} \cap (A \cap B) \subseteq \overline{C_1} \cap B.
\end{aligned} \tag{8.2}$$

as at least one of (8.1) and (8.2) is nonempty by the hypothesis that A is connected. In addition, by (8.1) and (8.2), we know that at leaset one of $C_1 \cap \overline{B}$ and $\overline{C_1} \cap B$ is nonempty. So, C_1 is disconnected by the hypothesis that B is connected.
From above result and hypothesis, we now have

1. B is connected.

2. C_1 is disconnected.

3. $B \cup C_1$ is disconnected.

Let D be a component of $B \cup C_1$ so that $B \subseteq D$; and we have, $(B \cup C_1) \setminus D = E \subseteq C_1$, and $D \cap \overline{E} = \overline{D} \cap E = \emptyset$, which implies that $\overline{E} \cap (A \setminus E) = \emptyset$ and $\overline{A \setminus E} \cap E = \emptyset$. So, we have proved that A is disconnected wich is absurd. Hence, $B \cup C_1$ is connected.

Now we prove that $\overline{E} \cap (A \setminus E) = \overline{A \setminus E} \cap E = \emptyset$ as follows: Since $D \cap \overline{E} = \emptyset$, so we have

$$\begin{aligned}
&\overline{E} \cap (A \setminus E) \\
=&\overline{E} \cap [(D \cup C_2) \cup (A \cap B)] \\
\subseteq&\overline{E} \cap [(D \cup C_2) \cup B] \\
=&\overline{E} \cap [(D \cup C_2)] \text{ since } B \subseteq D \\
=&\overline{E} \cap C_2 \text{ since } D \cap \overline{E} = \emptyset \\
\subseteq&\overline{C_1} \cap C_2 \text{ since } E \subseteq C_1 \\
=&\emptyset.
\end{aligned}$$

Again, since $\overline{D} \cap E = \emptyset$, so we have

$$\begin{aligned}
&\overline{A \setminus E} \cap E \\
&= \left[\overline{(D \cup C_2) \cup (A \cap B)}\right] \cap E \\
&\subseteq \left[\overline{(D \cup C_2) \cup B}\right] \cap E \\
&= \left[\overline{(D \cup C_2)}\right] \cap E \text{ since } B \subseteq D \\
&=\overline{C_2} \cap E \text{ since } \overline{D} \cap E = \emptyset \\
&\subseteq\overline{C_2} \cap C_1 \text{ since } E \subseteq C_1 \\
&=\emptyset.
\end{aligned}$$

□

8.10 Additional Exercises on Connectedness.

8.10.1 Exercise. Let A, B be two nonempty subsets of $\mathbb{R}$ such that $A \cap \overline{B} = \overline{A} \cap B = \emptyset$. Show that there exists an open set $U \supseteq A$ and an open set $V \supseteq B$ such that $U \cap V = \emptyset$. Hint: Consider the function $f(x; A) = \inf\{|x - a|; a \in A\}$, then define a function ϕ by $\phi(x) \to f(x; A) - f(x; B)$.

8.10.2 Exercise. Let S, T be nonempty closed subsets of $\mathbb{R}$ such that $S \cup T$ and $S \cap B$ are connected. Prove that S and T are connected. Give an example to show that the conclusion no longer holds if we remove the hypothesis that S and T are closed.

8.10.3 Exercise. If we have $B \subseteq A \subseteq \mathbb{R}$, then the set B is a connected subset of A, if and only if B is a connected subset of $\mathbb{R}$.

8.10.4 Exercise. If A is a subset of $\mathbb{R}$ and $a \in A$, then there exists a largest (with respect to inclusion) connected subset $C(a)$ of A that contains a. (The connected set $C(a)$ is called the **component** of a with respect to A.)

8.10.5 Exercise. If a, b belong to a subset A of $\mathbb{R}$ and $C(a)$ and $C(b)$, are the components of a and b in A, then either $C(a) = C(b)$ or else $C(a) \cap C(b) = \emptyset$. Hence, the identity $A = \cup_{a \in A} C(a)$ shows that A can be written as a disjoint union of connected sets.

8.10.6 Exercise. A nonempty subset of $\mathbb{R}$ with at least two elements is a connected set if and only if it is an interval. Use this to infer that every open subset of $\mathbb{R}$ can be written as an at most countable union of disjoint open intervals.

8.10.7 Exercise. Let A, B be two closed subsets of $\mathbb{R}$ such that both the sets $A \cap B$ and $A \cup B$ are connected. Show that A and B are connected.

8.10.8 Exercise. According to the Intermediate Value theorem, any connected set has a connected image under continuous mapping. Verify if the same is true in the “inverse” direction: if f is continuous on $\mathbb{R}$, then any connected set in $f(X)$ has a connected inverse image. If not, construct a counterexample.

8.10.9 Exercise. Let $\mathcal{C}$ be a set of non empty open sets in $\mathbb{R}$ such that for $U, V \in \mathcal{C}$, either $U = V$ or $U \cap V = \emptyset$ (the sets in $\mathcal{C}$ are then said to be ‘pairwise disjoint’). Prove: Either $\mathcal{C}$ is finite, or the sets in $\mathcal{C}$ can be listed in a sequence (U_n) Hint: Enumerate the rational numbers in a sequence. (r_n); in each $U \in \mathcal{C}$ choose a rational number.

Chapter 9

Infinite Series

God made the integers, and all the rest is the work of man.
—Leopold Kronecker.

9.0.1 Definition. Let I be any set. and let $\{a_i; i \in I\}$ be a set of real numbers indexed by I. Then we call the expression

$$\sum_{i \in I} a_i$$

an **unordered** series.[1] We say that the series converges to a real number S if and only if for each $\epsilon > 0$ there exists a finite set F with the property that, for every finite set $J \subseteq I$ satisfying $F \subseteq J \subseteq I$. we have

$$\left| \sum_{i \in J} a_i - S \right| < \epsilon.$$

In this case we say S is the sum of the series. If the series does not converge to any number, it is said to diverge. If for every $N > 0$, there exists a finite set with the property that, for every finite set J satisfying $F \subseteq J \subseteq I$, we have

$$\sum_{i \in J} a_i > N,$$

we say that the series diverges properly or diverges to ∞.

9.0.1 Problem. Let I be an uncountable set, and let $\{a_i; i \in I\}$ be a set of real numbers indexed by I, and suppose that $\sum_{i \in I} a_i$ converges. Then $a_i = 0$ for all but at most countably many i.

9.0.1.1 Solution. Suppose that $\{a_i; i \in I\}$ converges and choose some $\epsilon > 0$. Then, by definition we can find a F such that for every finite set J satisfying $F \subseteq J \subseteq I$. we have

$$\left| \sum_{i \in J} a_i - S \right| < \epsilon/2.$$

[1] This notion of unordered series convergence is important in more complicated topological spaces where the ordinary notion of sequence is inadequate. The notion of **net** is closely related. (see special topics)

If in particular. $J = F \cup \{i\}$ for some $i \notin F$ we have

$$\epsilon > \left|\sum_{i\in J} a_i - S\right| + \left|\sum_{i\in F} a_i - S\right| \geq \left|\sum_{i\in J} a_i - \sum_{i\in F} a_i\right| = |a_i|.$$

That is $|a_i| < \epsilon$ for all $i \notin F$, in other words, there can be only finitely many i such that $|a_i| > \epsilon$ (because all such i must be in F. In other words

$$I_\epsilon = \{i; |a_i| > \epsilon\}$$

is a finite set. But now we notice

$$\{i; a_i \neq 0\} = \bigcup_{i=1}^{\infty} I_{\frac{1}{i}}$$

and so $a_i = 0$ for all but a countable union of finite sets, which is hence coumable. □

9.0.2 Definition. Let (a_n) be a given sequence of real numbers, and form a new sequence (s_n) as follows:

$$s_n = a_1 + ... + a_n = \sum_{k=1}^{n} a_k \ (n = 1, 2, ...),$$

and is called the n-th partial sum. We define the infinite series $\sum_{k=1}^{\infty} a_k$ to be convergent if and only if $\lim_{n\to\infty} s_n$ exists and is finite). We define the series to be divergent (not convergent) if and only if $\lim_{n\to\infty} s_n = \infty$ or $\lim_{n\to\infty} s_n = -\infty$ or $\lim_{n\to\infty} s_n$ fails to exist. In the case that $\lim_{n\to\infty} s_n$ converges to a limit S (i.e. $\lim_{n\to\infty} s_n = S$), we call this limit as the sum of the series and write

$$\sum_{n=1}^{\infty} a_n = S.$$

9.0.3 Definition. (**Absolute Convergence**) A series $\sum_{n=1}^{\infty} a_n$ is called **absolutely convergent** if $\sum_{n=1}^{\infty} |a_n|$ converges.

9.0.4 Definition. (**Conditional Convergence**) A series $\sum_{n=1}^{\infty} a_n$ converges **conditionally** if it converges, but $\sum_{n=1}^{\infty} |a_n|$ diverges.

9.0.5 Definition. Let p be a function from $\mathbb{N} \to \mathbb{N}$ such that

1. $p(n) < p(m)$, if $n < m$.
   Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series related as follows:

2. $b_1 = a_1 + a_2 + ... + a_{p(1)}$,
   $b_{n+1} = a_{p(n)+1} + a_{p(n)+2} + ... + a_{p(n+1)}$ if $n = 1, 2, ...$

Then we say that $\sum_{n=1}^{\infty} b_n$ is obtained from $\sum_{n=1}^{\infty} a_n$ by **inserting parentheses**, and that $\sum_{n=1}^{\infty} a_n$ is obtained from $\sum_{n=1}^{\infty} b_n$ by **removing parentheses.**

9.0.6 Definition. Let f be a function from $\mathbb{N} \to \mathbb{N}$ and assume that f is one-to-one. Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series such that $b_n = a_{f(n)}$ for $n = 1, 2, ...$ Then $\sum_{n=1}^{\infty} b_n$ is said to be a **rearrangement** of $\sum_{n=1}^{\infty} a_n$.

9.0.7 Definition. A series $\sum_{n=1}^{\infty} a_n$ of real numbers is said to be **unconditionally convergent** if every rearrangement of it converges.

9.0.8 Definition. Let (a_n) be a sequence of real numbers.

1. A **subseries** of the series $\sum_{n=1}^{\infty} a_n$ is a series $\sum_{n=1}^{\infty} a_{n_k}$, where (a_{n_k}) is a subsequence of the sequence (a_n). The series $\sum_{n=1}^{\infty} a_n$ is said to be **subseries convergent** if every subseries $\sum_{n=1}^{\infty} a_n$ of it converges.

2. The series $\sum_{n=1}^{\infty} a_n$ is said to be **bounded-multiplier convergent** if $\sum_{n=1}^{\infty} b_n a_n$ converges for every bounded sequence (b_n) of real numbers.

3. The series $\sum_{n=1}^{\infty} a_n$ is said to be **sign-multiplier-convergent** if $\sum_{n=1}^{\infty} \epsilon_n a_n$ converges for every sequence (ϵ_n), where $\epsilon_n \in \{-1, 1\}$ for all $n \in \mathbb{N}$.

9.0.9 Proposition. Let (a_n) be a sequence of real numbers. Then, the following are equivalent:

1. The series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

2. The series $\sum_{n=1}^{\infty} a_n$ is unconditionally convergent.

3. The series $\sum_{n=1}^{\infty} a_n$ is subseries convergent.

4. The series $\sum_{n\in\mathbb{N}} a_n$ is unordered convergent.

5. The series $\sum_{n=1}^{\infty} a_n$ is bounded-multiplier convergent

6. The series $\sum_{n=1}^{\infty} a_n$ is sign-multiplier-convergent

9.1 Basic Results

1. Let $\sum_{n=1}^{\infty} a_n$ be a convergent series, then $\lim_{n\to\infty} a_n = 0$.

2. (**Cauchy Criterion:**) Let $\sum_{n=1}^{\infty} a_n$ be an infinite series, then $\sum_{n=1}^{\infty} a_n$ converges if and only if for every $\epsilon > 0$ $\exists$ $N \in \mathbb{N}$ such that $\forall$ $m, n \in \mathbb{N}$ satisfies

$$m, n \geq N \Rightarrow |s_m - s_n| < \epsilon.$$

3. Assume that $a_n \geq 0$ for each $n = 1, 2, \ldots.$ Then $\sum_{n=1}^{\infty} a_n$ converges if and only if, the sequence of partial sums is bounded above.

4. (**Divergence Test:**) Suppose that $\sum_{n=1}^{\infty} a_n$ is a series, and $\lim_{n\to\infty} a_n \neq 0$. Then $\sum_{n=1}^{\infty} a_n$ diverges.

5. Let $\sum_{n=1}^{\infty} a_n$ be a series of positive terms and $\sum_{n=1}^{\infty} b_n$ be a rearrangement of the first series. Then the two series either bolh converge to the same limit or both diverge properly.

6. (**Comparison Test:**)
Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series of positive terms, and suppose that

$$a_n \leq b_n \ \forall \ n \geq N \text{ for some } N \in \mathbb{N}.$$

(a) If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

(b) If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

7. (**Limit Comparison Test:**)
Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series, $a_n \geq 0,\ b_n > 0$, assume that $\lim_{n\to\infty}(a_n/b_n) = L$ allowing the case $L = \infty$.

   (a) If $0 < L < \infty$, then either both converge or both diverge.

   (b) If $L = \infty$ and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} b_n$ converges also.

   (c) If $L = 0$ and $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges also.

8. (**Limit Comparison Test Strengthened:**) Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series, $a_n, b_n > 0$, and suppose that

$$\liminf_{n\to\infty}(a_n/b_n) = L_1 \text{ and } \limsup_{n\to\infty}(a_n/b_n) = L_2$$

   (a) If $L_2 < \infty$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

   (b) If $L_1 > 0$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

9. (**Ratio Comparison Test:**) Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series, $a_n, b_n > 0$, and suppose that

$$\frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n}.$$

   (a) If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

   (b) If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

10. (**Integral Test:**) Consider a series $\sum_{n=1}^{\infty} a_n$, where $a_n = a(n)$ for some function $a(x)$ defined on $[1, \infty)$ that is continuous, positive, and decreasing. Then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the improper integral

$$\int_1^{\infty} a(x)dx$$

converges. Furthermore, the limit

$$D = \lim_{n\to\infty}\left[\sum_{k=1}^{\infty} a_k - \int_1^n a(x)dx\right]$$

exists, and $0 \leq D \leq a_1$.

9.1.1 Note. Applying the second part to the function $f(x) = \frac{1}{x}$, we find that

$$\lim_{n\to\infty}\left[\sum_{k=1}^{\infty} \frac{1}{k} - \log n\right] = \gamma$$

where $0 \leq \gamma < 1$ is the Euler constant.

11. (**p-Series:**) Consider the series
$$\sum_{n=1}^{\infty} \frac{1}{n^p},$$
called a p-series. where p is a real constant.
    (a) If $p > 1$, the series converges.
    (b) If $p \leq 1$, the series diverges.
12. (**Ratio Test:**) Consider a strictly positive series $\sum_{n=1}^{\infty} a_n$, and define
$$r = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$$
assuming that it exists.
    (a) If $r < 1$, the series $\sum_{n=1}^{\infty} a_n$ converges.
    (b) If $r > 1$, the series $\sum_{n=1}^{\infty} a_n$ diverges.
    (c) If $r = 1$, the the test is inconclusive.
13. (**Ratio Test Strengthened:**) Consider a strictly positive series $\sum_{n=1}^{\infty} a_n$, and define
$$r_1 = \liminf_{n\to\infty} \frac{a_{n+1}}{a_n} \text{ and } r_2 = \limsup_{n\to\infty} \frac{a_{n+1}}{a_n}$$
assuming that it exists.
    (a) If $r_2 < 1$, the series $\sum_{n=1}^{\infty} a_n$ converges.
    (b) If $r_1 > 1$, the series $\sum_{n=1}^{\infty} a_n$ diverges.
14. (**Root Test:**) Consider a strictly positive series $\sum_{n=1}^{\infty} a_n$, and define
$$\rho = \lim_{n\to\infty} a_n^{\frac{1}{n}}$$
assuming that it exists.
    (a) If $\rho < 1$, the series $\sum_{n=1}^{\infty} a_n$ converges.
    (b) If $\rho > 1$, the series $\sum_{n=1}^{\infty} a_n$ diverges.
    (c) If $\rho = 1$, the the test is inconclusive.
15. (**Root Test Strengthened:**) Consider a strictly positive series $\sum_{n=1}^{\infty} a_n$, and define
$$\rho_1 = \liminf_{n\to\infty} a_n^{\frac{1}{n}} \text{ and } \rho_2 = \limsup_{n\to\infty} a_n^{\frac{1}{n}}$$
assuming that it exists.
    (a) If $\rho_2 < 1$, the series $\sum_{n=1}^{\infty} a_n$ converges.
    (b) If $\rho_1 > 1$, the series $\sum_{n=1}^{\infty} a_n$ diverges.

9.1.2 Definition. (**Absolute Convergence**) A series $\sum_{n=1}^{\infty} a_n$ converges absolutely if and only if $\sum_{n=1}^{\infty} |a_n|$ converges. A series that converges absolutely is often called absolutely convergent.

9.1.3 Definition. (**Alternating Series:**) If $a_n > 0$ for each $n \in \mathbb{N}$, then the series $\sum_{n=1}^{\infty}(-1)^n a_n$ is called an alternating series.

1. (**Alternating Series Test:**) Consider an alternaing series $\sum_{n=1}^{\infty}(-1)^n a_n$ and suppose that the terms a_n satisfy:

   (a) a_n is non-increasing eventually, i.e. for some some $N \in \mathbb{N}$ $n \geq N \Rightarrow a_{n+1} \leq a_n$.

   (b) $\lim_{n\to\infty} a_n = 0$.

   Then $\sum_{n=1}^{\infty}(-1)^n a_n$ converges to a sum S. Furthermore, assuming $k > N$ for some integer $N > 0$, we have

   $$S_k \leq S \leq S_{k+1} \text{ or } S_{k+1} \leq S \leq S_k$$

   according as k is odd or even. (Consequently, S and S_k can never differ by more than a_{k+1}.)

2. (**Cauchy Condensation Test:**) Let $\sum_{n=1}^{\infty} a_n$ be a positive infinite series and suppose that the terms a_n are eventually decreasing and b be a positive integer greater than 1. Then the series

   $$\sum_{n=1}^{\infty} a_n \text{ and } \sum_{n=1}^{\infty} b^n a_{b^n}$$

   either both converge or both diverge.

9.2 Kummer's Results:

1. Let $\sum_{n=1}^{\infty} a_n$ be a positive infinite series, and let (λ_n) be any sequence of positive numbers. If, eventually,

   $$\lambda_n - \frac{a_{n+1}}{a_n}\lambda_{n+1} \geq k > 0$$

   for some constant k, then $\sum_{n=1}^{\infty} a_n$ converges.

2. (**Kummer's Test:**) Let $\sum_{n=1}^{\infty} a_n$ be a positive infinite series, let $\sum_{n=1}^{\infty} \frac{1}{d_n}$ be a divergent positive series. and define

   $$\kappa_n = d_n - \frac{a_{n+1}}{a_n} d_{n+1}$$

   (a) If, eventually, $\kappa_n > k > 0$ for some constant k. then $\sum_{n=1}^{\infty} a_n$ converges.

   (b) If, eventually, $\kappa_n \leq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

3. (**Kummer's Test Simplified:**) Let $\sum_{n=1}^{\infty} a_n$ be a positive infinite series, let $\sum_{n=1}^{\infty} \frac{1}{d_n}$ be a divergent positive series. and define

   $$\kappa = \lim_{n\to\infty}\left(d_n - \frac{a_{n+1}}{a_n} d_{n+1}\right)$$

   assuming that it exists. If $\kappa > 0$ (resp. $\kappa < 0$), then $\sum_{n=1}^{\infty} a_n$ converges (resp. diverges).

9.3 The Tests of Raabe and Gauss

1. (**Raabe's Test**) Let $\sum_{n=1}^{\infty} a_n$ be a positive infinite series, and suppose we can write

$$\frac{a_{n+1}}{a_n} = 1 - \frac{\beta_n}{n}, \text{ where } \lim_{n\to\infty} \beta_n = \beta.$$

   (a) If $\beta > 1$, then $\sum_{n=1}^{\infty} a_n$ converges.

   (b) If $\beta < 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

2. (**Raabe's Test Strengthened:**) Let $\sum_{n=1}^{\infty} a_n$ be a positive infinite series, and suppose we can write

$$\frac{a_{n+1}}{a_n} = 1 - \frac{\beta_n}{n},$$
$$\text{where } \limsup_{n\to\infty} \beta_n = \beta_1 \text{ and } \liminf_{n\to\infty} \beta_n = \beta_2.$$

   (a) If $\beta_2 > 1$, then $\sum_{n=1}^{\infty} a_n$ converges.

   (b) If $\beta_1 < 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

   A stronger test that copes with the $\beta = 1$ case of Raabe's Test follows. This test is due to the prolific mathematician Karl Friedrich Gauss (1777-1855).

3. (**Raabe-Duhamel's rule.**) Consider the series $\sum_{n=1}^{\infty} a_n$ with $a_n > 0 \ \forall\, n \in \mathbb{N}$ such that there exist $p > 0$ and $q > 1$ such that the sequence

$$\left(n^q \left(1 - \frac{a_{n+1}}{a_n} - \frac{p}{n}\right)\right)$$

   is bounded. Show that

   (a) if $p \leq 1$, then $\sum_{n=1}^{\infty} a_n$ is divergent;

   (b) and if $p > 1$, then $\sum_{n=1}^{\infty} a_n$ is convergent.

4. (**Raabe-Duhamel's test**) Let $\sum_{n=1}^{\infty} a_n$ be a series of strictly positive real numbers, and let us note $r_n = n\left(\frac{x_n}{x_{n+1}} - 1\right)$ We claim that:

   (a) If there exist $n_0 \in \mathbb{N}$ and $r > 1$, such that $r_n > r$ holds for all $n \geq n_0$, then the series is convergent;

   (b) If there is some $n_0 \in \mathbb{N}$ such that $r_n \leq 1$ holds for all $n \geq n_0$, then the series is divergent.

5. (**Raabe-Duhamel's criterion** in limit form). Let $\sum_{n=1}^{\infty} a_n$ be a series of strictly positive real numbers, for which there exists

$$\ell = \lim_{n\to\infty} n\left(\frac{x_n}{x_{n+1}} - 1\right) > 0$$

   (a) If $\ell > 1$, then the series is convergent, and

   (b) If $\ell < 1$, then the series is divergent.

(c) The case $\ell = 1$ is undecided.

6. (**Gauss's Test**) Let $\sum_{n=1}^{\infty} a_n$ be a positive infinite series, and suppose we can write

$$\frac{a_{n+1}}{a_n} = 1 - \frac{\beta}{n} + \frac{\theta_n}{n^{1+k}},$$

where θ_n is a bounded sequence and $k > 0$

(a) If $\beta > 1$, then $\sum_{n=1}^{\infty} a_n$ converges.

(b) If $\beta \leq 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

7. (**Abel, Summation by Parts:**) Let a_n and b_n be two arbitrary sequences, and write

$$A_n = a_1 + a_2 + ... + a_n$$

for the partial sums of $\sum_{n=1}^{\infty} a_n$, and $A_0 = 0$. Then for each $m, 1 \leq m \leq n$, we have

$$\sum_{k=m}^{n} a_k b_k = \left[\sum_{k=m}^{n} A_k (b_k - b_{k+1}) \right] - A_{m-1} b_m + A_n b_{n+1}.$$

8. Let (a_n) and (b_n) be two arbitrary sequences, and write

$$A_n = a_1 + a_2 + ... + a_n$$

for the partial sum of $\sum_{n=1}^{\infty} a_n$, and $A_0 = 0$. Then if $\sum_{n=0}^{\infty} A_n(b_n - b_{n+1})$ converges, and $\lim_{n\to\infty} A_n b_{n+1}$ exists the series $\sum_{n=1}^{\infty} a_n b_n$ converges.

9. (**Abel**) Let $\sum_{n=1}^{\infty} a_n$ be a convergent series and let (b_n) be a bounded monotone sequence. Then $\sum_{n=1}^{\infty} a_n b_n$ converges.

10. (**Abel's test** or **Abel's theorem**.) Let (x_n) and (ϵ_n) be two sequences of real numbers such that

(a) the sequence of partial sums (s_n) of $\sum x_n$ is bounded, i.e., there exists $M > 0$ such that $|s_n| = |x_1 + ... + x_n| \leq M, n = 1, ...;$

(b) $\lim_{n\to\infty} \epsilon_n = 0$.

(c) the series $\sum_{n=1}^{\infty} |\epsilon_{n+1} - \epsilon_n|$ is convergent.

Then the series $\sum_{n=1}^{\infty} \epsilon_n x_n$ is convergent.

11. (**Dirichlet**) Let $\sum_{n=1}^{\infty} a_n$ be a series with bounded partial sums. and let (b_n) be a monotone sequence with $\lim_{n\to\infty} b_n = 0$. Then $\sum_{n=1}^{\infty} a_n b_n$ converges.

12. (**Dirichlet's Rearrangement Theorem**) Let $\sum_{n=1}^{\infty} x_n$ be an absolutely convergent series, with sum s, and let $\sum_{n=1}^{\infty} y_n$ be any rearrangement of $\sum_{n=1}^{\infty} x_n$. Show that $\sum_{n=1}^{\infty} y_n$ converges, and $\sum_{n=1}^{\infty} x_n = s$.
In other words, Every absolutely convergent series of real numbers is unconditionally convergent, and any of its rearrangements sums to the same number.

13. (**Dedekind, Bois-Reymond**) Let $\sum_{n=1}^{\infty} a_n$ be a convergent series. and let a sequence (b_n) be such that $\sum_{n=1}^{\infty}(b_n - b_{n+1})$ converges absolutely. Then $\sum_{n=1}^{\infty} a_n b_n$ converges.

14. (**Riemann's theorem on conditionally convergent series**) Let $\sum_{n=1}^{\infty} a_n$ be a conditionally convergent series. Let x and y be given numbers in the closed interval $[-\infty, +\infty]$, with $x \leq y$. Then there exists a rearrangement $\sum_{n=1}^{\infty} b_n$ of $\sum_{n=1}^{\infty} a_n$, such that
$$\liminf_{n\to\infty} t_n = x \text{ and } \limsup_{n\to\infty} t_n = y$$
where $t_n = b_1 + b_2 + ... + b_n$.

15. (**Mertens**) If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be convergent series with sums s and t respectively and one of the series, say $\sum_{n=1}^{\infty} a_n$ be absolutely convergent, then the series $\sum_{n=1}^{\infty} c_n$, where $c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + .. + a_n b_0$, is convergent and its sum is st.

16. (**Abel's theorem** or **Pringsheim's theorem**) If $\sum_{n=1}^{\infty} a_n$ be a convergent series positive real numbers and (a_n) is a monotone decreasing sequence then $\sum_{n\to\infty} na_n = 0$.

Let $\sum_{n=1}^{\infty} a_n$ be a series of real numbers. Let us define two associated sequences and as follows: For $n \in \mathbb{N}$, let
$$a_n^+ = \begin{cases} a_n & \text{if } a_n \geq 0 \\ 0 & \text{if } a_n < 0, \end{cases} \quad a_n^- = \begin{cases} 0 & \text{if } a_n \geq 0 \\ -a_n & \text{if } a_n < 0, \end{cases}$$
Note that $a_n^+ \geq 0$ and $a_n^- \geq 0 \ \forall\ n \in \mathbb{N}$. It is easy to show that $a_n^+ - a_n^- = a_n$ and $a_n^+ + a_n^- = |a_n| \ \forall\ n \in \mathbb{N}$.

9.3.1 Proposition. Let $\sum_{n=1}^{\infty} a_n$ be a series of real numbers.

1. $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if, and only if, the series $\sum_{n=1}^{\infty} a_n^+$ and $\sum_{n=1}^{\infty} a_n^-$ are both convergent, where a_n^+ and a_n^- are defined, for all $n \in \mathbb{N}$,. If this is the case, then $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_n^+ - \sum_{n=1}^{\infty} a_n^-$.

2. If $\sum_{n=1}^{\infty} a_n$ is convergent but not absolutely convergent, then $\sum_{n=1}^{\infty} a_n^+$ the series and $\sum_{n=1}^{\infty} a_n^-$ are both divergent.

9.3.2 Proposition. A series of real numbers is unconditionally convergent iff it is absolutely convergent and all rearrangements of an unconditionally convergent series sum to the same number.

9.3.3 Remark. The theory of infinite sequences and the theory of infinite series are logically equivalent in the sense that the behavior of any sequence is controlled by the behavior of an associated series, and vice versa. For, in defining the convergence of infinite series we established a correspondence from infinite series $\sum_{n=1}^{\infty} a_n$ to infinite sequences of partial sums $|S_n|$. We observe that, since $S_n = a_1 + a_2 + ... + a_{n-1} + a_n = S_{n-1} + a_n$ for $n > 1$. we have $a_n = S_n - S_{n-1}$ for $n > 1$ and also $a_1 = S_1$. That is, the terms a_n are uniquely determined by the partial sums S_n. In other words, the correspondence from infinite series to infinite sequences is in fact a bijection.
In the development of infinite series, the standard route of defining infinite sequences and defining the behavior of infinite series in terms of the associated sequence of partial sums. Instead, one could have defined the convergence of infinite series as our primitive concept. Then the convergence and divergence of infinite sequences could be defined in terms of the associated series. If (a_n) be a sequence of real numbers, and define (b_n) by $b_n = a_{n+1} - a_n$. Then $\sum_{n=1}^{\infty} b_n$ converges iff (a_n) converges.

9.4 Problems and Solutions on Chapter 9.

Warning! *Never Never Ever read this solution unless you tried problems quite long time and gave up. Doing so may impair your ability to think and solve problems.*

9.4.1 Problem. Let (x_n) be a sequence of real numbers. Show that $x = \lim_{n\to\infty} x_n$ if and only if

$$x = x_1 + \sum_{k=1}^{\infty} (x_{k+1} - x_k).$$

9.4.1.1 Solution. We have

$$\begin{aligned} x_1 + \sum_{k=1}^{\infty} (x_{k+1} - x_k) &= x_1 + \lim_{n\to\infty} \sum_{k=1}^{n} (x_{k+1} - x_k) \\ &= \lim_{n\to\infty} \left[x_1 + \sum_{k=1}^{n} (x_{k+1} - x_k) \right] = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} x_n \end{aligned}$$

Hence the result follows. □

9.4.2 Problem. Let E be a set of positive real numbers. We define $\sum_{x\in E} x$ to be $\sup_{F\in\mathcal{F}} S_F$, where $\mathcal{F}$ is the collection of finite subsets of E and S_F is the (finite) sum of the elements of F.

1. Show that $\sum_{x\in E} x < \infty$ only if E is countable.
2. Show that if E is countable and (x_n) is a one-to-one mapping of $\mathbb{N}$ onto E, then $\sum_{x\in E} x = \sum_{n=1}^{\infty} x_n$.

9.4.2.1 Solution.

1. Suppose $\sum_{x\in E} x < \infty$. For each n, let $E_n = \{x \in E; ; x \geq 1/n\}$. Then each E_n is a finite subset of E. Otherwise, if E_{n_0} is an infinite set for some n_0, then letting F_k be a subset of E_{n_0} with kn_0 elements for each $k \in \mathbb{N}, S_{F_k} \geq k$. Then $\sum_{x\in E} x \geq S_{F_k} \geq k$ for each k, a contradiction. Now $E = \cup_{n=1} E_n$, so E is countable.
2. Clearly, $\{x_1, ..., x_n\} \in \mathcal{F}$ for all n. Thus $\sup S_n \leq \sup_{F\in\mathcal{F}} S_F$. On the other hand, given $F \in \mathcal{F}$, there exists $n \in \mathbb{N}$ such that $F \subseteq \{x_1, ..., x_n\}$ so $S_F \leq S_n$ and $\sup S_F \leq \sup S_n$. Hence

$$\sum_{x\in E} x = \sup_{F\in\mathcal{F}} S_F = \sup S_n = \sum_{n=1}^{\infty} x_n. \quad \square$$

9.4.3 Problem. For a sequence (x_n) of real numbers show that the following conditions are equivalent:

1. The series $\sum_{n=1}^{\infty} x_n$ rearrangement invariant in $\mathbb{R}$.
2. For every permutation σ of $\mathbb{N}$ the series $\sum_{n=1}^{\infty} x_{\sigma_n}$ converges in $\mathbb{R}$.
3. The series $\sum_{n=1}^{\infty} |x_n|$ converges in $\mathbb{R}$.

4. For every sequence (s_n) of $\{-1,1\}$, the series $\sum_{n=1}^{\infty} s_n x_n$ converges in $\mathbb{R}$.

5. For every subsequence (x_{n_k}) of (x_n), the series $\sum_{n=1}^{\infty} x_{n_k}$ converges in $\mathbb{R}$.

6. For every $\epsilon > 0$, there exists an integer k (depending on ϵ) such that for every finite subset S of $\mathbb{N}$ with $\min S \geq k$, we have $\left|\sum_{n\in S} x_n\right| < \epsilon$.

(Any series $\sum_{n=1}^{\infty} x_{n_k}$ satisfying any one of the above conditions is also referred to as an **unconditionally convergent** series.)

9.4.3.1 Solution.

1. (1) implies (2) is obvious for every rearrangement is a permutation of $\mathbb{N}$.

2. (2) implies (3): Assume $\sum_{n=1}^{\infty} |x_n| = \infty$. From our hypothesis it follows that $x_n > 0$ and $x_n < 0$ both hold for infinitely many n. Split (x_n) into two subsequences (y_n) and (z_n) such that $y_n \geq 0$ and $z_n < 0$ hold for all n. We can assume that $\sum_{n=1}^{\infty} y_n = \infty$.
Now, use induction to construct a strictly increasing sequence of natural numbers (k_n) such that

   (a) $k_1 = 1$ and $z_1 + \sum_{i=1}^{k_n} y_i > 1$; and

   (b) $z_n + \sum_{i=k_n+1}^{k_{n+1}} y_i > 1$; for $n = 1, 2, ...$

   Then note that

$$y_1, ..., y_{k_1}, z_1, y_{k_1+1}, ..., y_{k_2}, z_2, y_{k_2+1} ....$$

   is a permutation of (x_n) whose series is not convergent, contrary to our hypothesis.

3. (3) implies (4) is obvious.

4. (4) implies (5): Let (x_{n_k}) be a subsequence of (x_n). Put $s_i = -1$ if $i \neq k_n$ for each n, and $s_{k_n} = 1$. Then

$$\sum_{n=1}^{\infty} x_{k_n} = \frac{1}{2}\left[\sum_{n=1}^{\infty} x_n + \sum_{n=1}^{\infty} s_n x_n\right]$$

   is a convergent series.

5. (5) implies (6): If (6) is false, then there exists some $\epsilon > 0$ and a sequence (S_n) of finite subsets of natural numbers such that $\max S_n < \min S_{n+1}$ and $\left|\sum_{i\in S_n} x_i\right| < \epsilon$. for all n. Let

$$\bigcup_{n=1}^{\infty} S_n = \{k_1, k_2, ...\},$$

   where k_n is increasing. Then, one can easily verify that the series $\sum_{n=1}^{\infty} x_{k_n}$ does not converge in $\mathbb{R}$, contradicting (5).

6. (6) implies (1): Let $\sigma : \mathbb{N} \to \mathbb{N}$ be a permutation. By our hypothesis, the partial sums of both series $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} x_{\sigma_n}$ form Cauchy sequences, and hence, both series converge in $\mathbb{R}$. Let $x = \sum_{n=1}^{\infty} x_n$ and $y = \sum_{n=1}^{\infty} x_{\sigma_n}$.
Now, if $\epsilon > 0$ is given, then choose k so large such that

$$\left| x - \sum_{n=1}^{r} x_n \right| < \frac{\epsilon}{3}, \ \left| y - \sum_{n=1}^{r} x_{\sigma_n} \right| < \frac{\epsilon}{3}, \text{ and } \left| \sum_{n\in S} x_n \right| < \frac{\epsilon}{3}$$

hold for all $r \geq> k$ and all finite subsets S of $\mathbb{N}$ with $\min S \geq k$. Fix some $r > k$ such that for each $1 \leq i \leq k$ there exists $1 \leq j \leq r$ with $x_i = x_{\sigma_j}$ and note that

$$\begin{aligned} |x-y| &\leq \left| x - \sum_{n=1}^{k} x_n \right| + \left| \sum_{n=1}^{k} x_n - \sum_{n=1}^{r} x_{\sigma_n} \right| + \left| \sum_{n=1}^{r} x_{\sigma_n} - y \right| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

holds for all $\epsilon > 0$, and so $x = y$. In other words, the series $\sum_{n=1}^{\infty} x_n$ rearrangement invariant. □

9.4.4 Problem. A product

$$\prod_{n=1}^{\infty} (1 + a_k)$$

where all the terms a_k are positive is convergent if and only if the series $\sum_{n=1}^{\infty} a_n$ converges.

9.4.4.1 Solution. A sequence that is monotonic is convergent if and only if it is bounded. Using this we see that

$$a_1 + a_2 + a_3 + ... + a_n \leq (1 + a_1)(1 + a_2)(1 + a_3)..(1 + a_n)$$

so that the convergence of the product gives an upper bound for the partial sums of the series. It follows that if the product converges so must the series. In the other direction we have

$$(1 + a_1)(1 + a_2)(1 + a_3)..(1 + a_n) \leq e^{a_1+a_2+a_3+...+a_n}$$

and so the convergence of the series gives an upper bound for the partial products of the infinite product. It follows that if the series converges, so must the product. □

9.4.5 Problem. If $\lim_{n\to\infty} a_n = 0$ and the sequence of the partial sums is bounded, then $\sum_{n=1}^{\infty} a_n$ is convergent. True or false?

9.4.5.1 Solution. Let us consider the following sequence (a_n) defined by

$$a_1 = 1, a_2 = a_3 = -\frac{1}{2}, a_4 = a_5 = a_6 = \frac{1}{3}, ...$$
$$a_{\frac{n(n-1)}{2}+1} = ... = a_{\frac{n(n+1)}{2}} = (-1)^{n-1}\frac{1}{n}, ...$$

We have $\lim_{n\to\infty} a_n = 0$ and the sequence of the partial sums S_k is bounded by 0 and 1, but the series $\sum_{n=1}^{\infty} a_n$ diverges, because the partial sums have no limit: for the partial sums with indices $k = 1, 6, ..., \frac{(2n-1)2n}{2}, ..\forall\ n \in \mathbb{N}$ one gets $S_k = 1 \to 1$, while for the partial sums with indices $k = 3, 10, ..., \frac{2n(2n+1)}{2}$ one obtains $S_k = 0 \to 0$. □

9.4.1 Remark. The converse is true.

9.4.2 Remark. For positive series, the condition of boundedness of partial sums is necessary and sufficient for series convergence.

9.4.6 Problem. The series $\sum_{n=1}^{\infty} a_n$ converges whenever $\lim_{n\to\infty} a_n = 0$. True or false?

9.4.6.1 Solution. False. Consider the series $\sum_{n=1}^{\infty} \frac{1}{n}$, here $\lim_{n\to\infty} \frac{1}{n} = 0$, but the series diverges. □

9.4.7 Problem. The series $\sum_{n=1}^{\infty} a_n$ converges whenever the sequence of partial sums converges to 0. True or false?

9.4.7.1 Solution. True. By definition, a series converges if and only if its sequence of partial sums converge to any number, and zero is acceptable. □

9.4.8 Problem. If $\sum_{n=1}^{\infty} a_n$ converges and $a_n > 0$, then $\sum_{n=1}^{\infty} a_n^2$ converges. True or false?

9.4.8.1 Solution. True. Since $\sum_{n=1}^{\infty} a_n$ converges and $a_n > 0$, we have $a_n \to 0$ and consequently $a_n < 1$ eventually. Then also $a_n^2 < a_n$ eventually, so that the comparison test applies. □

9.4.9 Problem. If $\sum_{n=1}^{\infty} a_n$ converges and $a_n > 0$, then $\sum_{n=1}^{\infty} \sqrt{a_n}$ converges. True or false?

9.4.9.1 Solution. False. Consider $a_n = 1/n^2$. The former converges and the latter diverges. □

9.4.10 Problem. If $\sum_{n=1}^{\infty} a_n$ converges and $a_n > 0$, then $\sum_{n=1}^{\infty} \frac{1+a_n}{2+a_n}$ converges. True or false?

9.4.10.1 Solution. False. For we have, $a_n \to 0$ and thus $\frac{1+a_n}{2+a_n} \to 1/2$. Thus the latter series' terms do not go to 0, hence it diverges. □

9.4.11 Problem. If $\sum_{n=1}^{\infty} a_n$ converges and $a_n > 0$, then $\sum_{n=1}^{\infty} \frac{2^n+a_n}{3^n+a_n}$ converges. True or false?

9.4.11.1 Solution. True. We have $a_n \to 0$. Now apply the ratio test. So

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1} + a_{n+1}}{2^n + a_n} . \frac{3^n + a_n}{3^{n+1} + a_{n+1}}.$$

For large n, the a_n are small and this ratio tends then to $2/3$. That is,

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{2}{3}.$$

Hence convergent. □

9.4.12 Problem. If $\sum_{n=1}^{\infty} a_n$ converges. then $\sum_{n=1}^{\infty} \frac{a_n}{n}$ converges. True or false?

9.4.12.1 Solution. True. Apply Abel' Test, with a_n as given, $b_n = \frac{1}{n}$. □

9.4.13 Problem. The series $\sum_{n=1}^{\infty} n^{\cos 3}$. converges. True or false?

9.4.13.1 Solution. False. This is a p-series with $p = -\cos 3 < 1$. □

9.4.14 Problem. If $a_n > 0$ and the sequence of partial sums for $\sum_{n=1}^{\infty} a_n$ is bounded above, then the series converges. True or false?

9.4.14.1 Solution. True. A monotone increasing sequence that is bounded above must converge. □

9.4.15 Problem. If the sequence of partial sums for $\sum_{n=1}^{\infty} a_n$ is bounded, then the series converges. True or false?

9.4.15.1 Solution. False. Consider the series $1 - 1 + 1 - 1 + 1 - 1 + ..$ Its partial sums oscillate between 1 and 0. Thus the partia1 sums are bounded but clearly do not converge. □

9.4.16 Problem. If $\sum_{n=1}^{\infty} a_n$ diverges and $a_n > 0$. then $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ diverges. True or false?

9.4.16.1 Solution. True. In the case that $a_n \to 0$, we eventually have

$$\frac{a_n}{1 + a_n} > \frac{a_n}{2}$$

and the comparison test applies. If on the other hand $a_n \nrightarrow 0$, then the terms $\frac{a_n}{1+a_n}$ do not go to 0 either and the divergence test applies. □

9.4.17 Problem. If $\sum_{n=1}^{\infty} a_n$ converges, where $a_n > 0$ and $a_n \neq 1 \; \forall \; n$, does

$$\sum_{n=1}^{\infty} \frac{a_n}{1 - a_n}$$

converge? (Prove or give a counterexample.)

9.4.17.1 Solution. There exists N such that $\forall \; n > N$, sufficiently large, $0 < a_n < 1/2$. Thus for $n > N, a_n/(1 - a_n) < 2a_n$. So

$$0 < \sum_{n=1}^{\infty} \frac{a_n}{1 - a_n} < \sum_{n=1}^{N} \frac{a_n}{1 - a_n} + 2 \sum_{n=N+1}^{\infty} a_n$$

and the series converges. □

9.4.18 Problem. If both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ diverge, then $\sum_{n=1}^{\infty}(a_n + b_n)$ diverges. True or false?

9.4.18.1 Solution. False. Take $a_n = 1, b_n = -1$. □

9.4.19 Problem. If both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ diverge, then $\sum_{n=1}^{\infty}(a_n + b_n)$ converges. True or false?

9.4.19.1 Solution. False. Take $a_n = 1, b_n = 1$. □

9.4.20 Problem. If $\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty}(a_n + b_n)$ diverges. True or false?

9.4.20.1 Solution. True. If possible, let $\sum_{n=1}^{\infty}(a_n + b_n)$ converge. Then

$$\sum_{k=1}^{n} b_n = \sum_{k=1}^{n}(a_n + b_n) - \sum_{k=1}^{n} a_n.$$

We see that the latter two sequences converge, hence $\sum_{k=1}^{n} b_n$ must converge, contrary to assumption. □

9.4.21 Problem. If $a_n > 0$ and $\sum_{k=1}^{\infty} a_n$ converges, then $\sum_{k=1}^{\infty} 1/a_n$ diverges. True or false?

9.4.21.1 Solution. True. $\sum_{k=1}^{\infty} a_n$ converges implies $a_n \to 0$ and then $1/a_n \to \infty$. Hence $\sum_{k=1}^{\infty} 1/a_n$ diverges. □

9.4.22 Problem. If $a_n > 0$ and $\sum_{k=1}^{\infty} a_n$ diverges, then $\sum_{k=1}^{\infty} 1/a_n$ converges. True or false?

9.4.22.1 Solution. False. Take $a_n = 1$. □

9.4.23 Problem. If $\lim_{n\to\infty} a_n/b_n = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges. True or false?

9.4.23.1 Solution. False. Without assuming the terms are positive this does not follow from the limit comparison test. Take $a_n = (-1)^n/\sqrt{n}, b_n = 1/(n \ln n) + (-1)^n/n$. Then $\sum_{n=2}^{\infty} b_n$ diverges because it is the sum of a divergent positive series and the convergent alternating harmonic series, but $\sum_{n=1}^{\infty} a_n$ is a convergent alternating series. Yet

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(-1)^n/\sqrt{n}}{1/(n\ln n) + (-1)^n/n} = \lim_{n\to\infty} \frac{\sqrt{n}}{1 + (-1)^n/\ln n} = \infty. \quad \square$$

9.4.24 Problem. If $\lim_{n\to\infty} a_n/b_n = 1$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. True or false?

9.4.24.1 Solution. False. Again, the limit comparison test does not apply because the terms need not be non-negative. Take $b_n = (-1)^n/\sqrt{n}$, the terms of a convergent alternating series, and take $a_n = b_n + 1/n$. Then $\sum_{n=1}^{\infty} a_n$ is divergent, because it is formed as the sum of a convergent series and a divergent (harmonic) series. We now check

$$\begin{aligned} \lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{(-1)^n/\sqrt{n} + 1/n}{(-1)^n/\sqrt{n}} \\ &= \lim_{n\to\infty} \left(1 + \frac{(-1)^n}{\sqrt{n}}\right) = 1. \quad \square \end{aligned}$$

9.4.25 Problem. If $\lim_{n\to\infty} a_n/b_n = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. True or false?

9.4.25.1 Solution. False. Interchanging the roles of a_n and b_n, this is the contrapositive of the previous item. □

9.4.26 Problem. If $\lim_{n\to\infty} a_n = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n b_n$ converges. True or false?

9.4.26.1 Solution. False. Take $a_n = b_n = (-1)^n/\sqrt{n}$. □

9.4.27 Problem. If $\lim_{n\to\infty} a_n = 0$ and $\sum_{n=1}^{\infty} b_n$ converges absolutely, then $\sum_{n=1}^{\infty} a_n b_n$ converges. True or false?

9.4.27.1 Solution. True. Apply the comparison test to $\sum_{n=1}^{\infty} |b_n|$, $\sum_{n=1}^{\infty} |a_n b_n|$, noticing that eventually we must have $|a_n| < 1$. □

9.4.28 Problem. If $a_n \geq 0$, $\sum_{n=1}^{\infty} a_n$ does not converge conditionally. True or false?

9.4.28.1 Solution. True. Since $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} |a_n|$ are in fact the same series, so they cannot have different behavior. $\square$

9.4.29 Problem. If $\sum_{n=1}^{\infty} a_n b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge. True or false?

9.4.29.1 Solution. False. Take $a_n = b_n = 1/n$. $\square$

9.4.30 Problem. If $a_n \geq 0$ and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} \sin a_n$ converges. True or false?

9.4.30.1 Solution. True. It is known in early calculus that $\lim_{x\to 0} \frac{\sin x}{x} = 1$. With that in mind, apply the limit comparison test, noting that $a_n \to 0$. $\square$

9.4.31 Problem. If $a_n \geq 0$ and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} \cos a_n$ converges. True or false?

9.4.31.1 Solution. False. We have $a_n \to 0$. Thus $\cos a_n \to \cos 0 = 1 \neq 0$ and the divergence test applies. $\square$

9.4.32 Problem. The root test cannot alone be used to determine conditional convergence. True or false?

9.4.32.1 Solution. True. In order to verify conditional convergence one must show 1) convergence for the series as given and 2) that the series does not converge absolutely. The root test may only give information as regards the second requirement. $\square$

9.4.33 Problem. The ratio test cnnnot alone be used to determine conditional convergence. True or false?

9.4.33.1 Solution. True. In order to verify conditional convergence one must show 1) convergence for the series as given and 2) that the series does not converge absolutely. The ratio test may only give information as regards the second requirement. $\square$

9.4.34 Problem.

$$\sum_{n=1}^{\infty} \frac{1}{n^n} \geq 2.$$

True or false?

9.4.34.1 Solution. False. We simply observe that

$$\sum_{n=1}^{\infty} \frac{1}{n^n} = 1 + \sum_{n=2}^{\infty} \frac{1}{n^n} < 1 + \sum_{n=2}^{\infty} \frac{1}{2^n} = \frac{3}{2}. \quad \square$$

9.4.35 Problem. If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} (-1)^n a_n$ converges. True or false?

9.4.35.1 Solution. False. Consider $a_n = (-1)^n/n$. $\square$

9.4.36 Problem. If $0 \leq a_n \leq b_n$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges also. True or false?

9.4.36.1 Solution. False. Take $a_n = 1/n^2, b_n = 1$. $\square$

9.4.37 Problem. If $a_n \leq b_n \leq 0$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges also. True or false?

9.4.37.1 Solution. True. Changing the sign of every term in a series does not affect its convergence. We apply the comparison test to the series $\sum_{n=1}^{\infty} -b_n$ and $\sum_{n=1}^{\infty} -a_n$. $\square$

9.4.38 Problem. If $a_n \leq b_n \leq 0$ and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} b_n$ converges also. True or false?

9.4.38.1 Solution. True. Changing the sign of every term in a series does not affect its convergence. We apply the comparison test to the series $\sum_{n=1}^{\infty} -b_n$ and $\sum_{n=1}^{\infty} -a_n$. $\square$

9.4.39 Problem. If $a_n > 0$ and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty}(-1)^n a_n$ converges also. True or false?

9.4.39.1 Solution. True. By assumption it converges absolutely, hence it is also convergent. $\square$

9.4.40 Problem. If the condition $0 \leq a_{n+1} \leq a_n$ fails when applying the alternating series test, then the series $\sum_{n=1}^{\infty}(-1)^n a_n$ diverges. True or false?

9.4.40.1 Solution. False. Consider $a_n = \frac{1}{k_n^n}$ where k_n is 2 or 3 according as n is odd or even. By the strengthened root test $\sum_{n=1}^{\infty}(-1)^n a_n$ converges absolutely. However, $a_{n+1} \leq a_n$ never holds for even n. $\square$

9.4.41 Problem. If a series converges, then no subsequence of the sequence of partial sums can be unbounded. True or false?

9.4.41.1 Solution. True. Since the series converges. its sequence of partial sums converges. Thus any subsequence would also converge to the same value. $\square$

9.4.42 Problem. If $a_n > 0$ for all n and if $\sum_{n=1}^{\infty} a_n$ converges, then

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L < 1.$$

True or false?

9.4.42.1 Solution. False. Consider any convergent p-series. (It is also possible for the limit not to exist.) $\square$

9.4.43 Problem. If $\left|\dfrac{a_{n+1}}{a_n}\right| < 1$ for all n, then $\sum_{n=1}^{\infty} |a_n|$ converges. True or false?

9.4.43.1 Solution. False. Consider the harmonic series. $\square$

9.4.44 Problem. If $\sum_{n=1}^{\infty} a_n$ converges and (b_n) is a bounded sequence, then $\sum_{n=1}^{\infty} a_n b_n$ conwrges. True or false?

9.4.44.1 Solution. False. Let $a_n = b_n = \frac{(-1)^n}{\sqrt{n}}$. $\square$

9.4.45 Problem. If $\sum_{n=1}^{\infty} a_n$ diverges. then $\sum_{n=1}^{\infty} |a_n|$ must diverge.

9.4.45.1 Solution. True. An absolutely convergcnt series must converge. This states the contrapositive. $\square$

9.4.46 Problem. A convergent series is either absolutely convergent or conditionally convergent.

9.4.46.1 Solution. True. Either $\sum_{n=1}^{\infty} a_n$ converges, in which case we have absolute convergence, or else it doesn't in which case we have conditional convergence. $\square$

9.4.47 Problem. If $a_n \geq 0$ and $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} \frac{a_n}{1+na_n}$ diverges. True or false?

9.4.47.1 Solution. False. Consider $a_n = \sqrt{n}$. $\square$

9.4.48 Problem. If $a_n \geq 0$ and $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} \frac{a_n}{1+na_n}$ converges. True or false?

9.4.48.1 Solution. False. Consider $a_n = 1$. $\square$

9.4.49 Problem. If $a_n \geq 0$ and $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n^2}$ converges. True or false?

9.4.49.1 Solution. False. Consider $a_n = n$. $\square$

9.4.50 Problem. If $a_n \geq 0$ and $\sum_{n=1}^{\infty} a_n$ diverges. then $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n^2}$ diverges. True or false?

9.4.50.1 Solution. False. Consider $a_n = 2^n$. $\square$

9.4.51 Problem. If a series converges, then any summation of a subset of the terms will form a convergent series. True or false?

9.4.51.1 Solution. False. As an explicit counterexample, the alternating harmonic series converges but the subseries formed by the positive terms diverges. $\square$

9.4.52 Problem. The alternating harmonic series cannot be rearranged to diverge to $-\infty$.

9.4.52.1 Solution. False. $\square$

9.4.53 Problem. If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} na_n = 0$. True or false?

9.4.53.1 Solution. The alternating series $\sum_{n=1}^{\infty}(-1)^n \frac{1}{\sqrt{n}}$ is convergent (both conditions of the Leibniz test are satisfied), but $\lim_{n\to\infty} n(-1)^n \frac{1}{\sqrt{n}} = \infty$ (the limit does not exist if one considers the specific infinity – positive or negative). $\square$

9.4.54 Problem. If $\sum_{n=1}^{\infty} na_n$ converges, then so does $\sum_{n=1}^{\infty} a_n$. True or false?

9.4.54.1 Solution. True. $\square$

9.4.55 Problem. If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent, then $\sum_{n=1}^{\infty} a_n b_n$ is also convergent. True or false?

9.4.55.1 Solution. If $a_n = (-1)^n \frac{1}{\sqrt{n}}$ and $b_n = (-1)^n \frac{1}{\sqrt[3]{n}}$. Then $\sum_{n=1}^{\infty}(-1)^n \frac{1}{\sqrt{n}}$ and $\sum_{n=1}^{\infty}(-1)^n \frac{1}{\sqrt[3]{n}}$ are convergent since both conditions of the Leibniz test are satisfied for these two series and $\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0, |a_{n+1}| = \frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}} = |a_n|\ \forall\ n \in \mathbb{N}$ and $\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{1}{\sqrt[3]{n}} = 0, b_{n+1} = \frac{1}{\sqrt[3]{n+1}} < \frac{1}{\sqrt[3]{n}} = |b_n|,\ \forall\ n \in \mathbb{N}$. However $\sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{\infty} \frac{1}{n^{5/6}}$ is divergent (it is a p-series with $p = 5/6$.). $\square$

9.4.3 Remark. As a particular case of this example, one can also formulate the following false statement: if a series $\sum_{n=1}^{\infty} a_n$ converges then $\sum_{n=1}^{\infty} a_n^2$ also converges. The corresponding counterexample is in line with the above: if $a_n = (-1)^n \frac{1}{\sqrt{n}}$, then the alternating series $\sum_{n=1}^{\infty}(-1)^n \frac{1}{\sqrt{n}}$ is convergent, but the series of the squares $\sum_{n=1}^{\infty} \frac{1}{n}$ is harmonic, that is divergent.

9.4.4 Remark. For positive series this statement is true.

9.4.56 Problem. If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are divergent, then $\sum_{n=1}^{\infty} a_n b_n$ is also divergent. True or false?

9.4.56.1 Solution. Consider $a_n = \frac{1}{n}$ and $b_n = \frac{1}{\sqrt{n}}$. □

9.4.57 Problem. Let $\sum_{n=1}^{\infty} a_n$ be a positive series. If the Cauchy, D'Alembert, Raabe and Integral tests are not conclusive, then neither is any other test. True or false?

9.4.57.1 Solution. Let us consider $a_n = e^{-\left(1+\frac{1}{2}+..+\frac{1}{n-1}\right)}$ and the corresponding series

$$\sum_{n=2}^{\infty} e^{-\left(1+\frac{1}{2}+..+\frac{1}{n-1}\right)}.$$

The limit of the D'Alembert test gives

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{e^{-\left(1+\frac{1}{2}+..+\frac{1}{n}\right)}}{e^{-\left(1+\frac{1}{2}+..+\frac{1}{n-1}\right)}} = \lim_{n\to\infty} e^{-\frac{1}{n}} = 1$$

that is the D'Alembert test is inconclusive. Since this limit exists, so does the limit of the Cauchy test and the former and latter coincide $\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$, which means that the Cauchy test is also inconclusive. Moreover, the Raabe test also fails:

$$\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(e^{\frac{1}{n}} - 1\right) = \lim_{x\to 0} \frac{e^x - 1}{x} = 1.$$

Finally, the Integral test is not applicable for this series (one cannot generate an integrable function f such that $f(n) = a_n$). Nevertheless, the Bertrand test shows that the series diverges:

$$\lim_{n\to\infty} \ln n\left(n\left(\frac{a_n}{a_{n+1}} - 1\right) - 1\right) = \lim_{n\to\infty} \ln n\left(ne^{\frac{1}{n}} - (n+1)\right)$$
$$= \lim_{x\to 0+} (-\ln x)\left(\frac{1}{x}e^x - \frac{1}{x} - 1\right) = -\lim_{x\to 0+}\left(\frac{e^x - 1 - x}{x(\ln x)^{-1}}\right) = 0 < 1. \quad \square$$

9.4.5 Remark. There is the well-known result in the theory of series stating that there is no definite test for verification of convergence or divergence of all series.

9.4.6 Remark.

1. It is crucial to note that the ratio test gives no information at all if $\lim_{n\to\infty}(a_{n+1}/a_n) = 1$. In both $\sum_{n=1}^{\infty} \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ we easily see that $\lim_{n\to\infty}(a_{n+1}/a_n) = 1$, but the former series diverges and the latter series converges.

2. By now it will be clear that the behaviour of an infinite series depends on the "ultimate" nature of its terms. When we apply the comparison test, for example, the first finite number of terms is of no consequence: it is the behaviour for sufficiently large n that matters. Though we have not emphasised this, the same applies to series of positive terms. All our results apply to ultimately positive series, by which of course we mean series $\sum_{n=1}^{\infty} a_n$.

9.4.58 Problem. Let (a_n) be a sequence such that $\sum_{n=1}^{\infty} a_n^2$ converges. Then $\sum_{n=1}^{\infty} \frac{a_n}{n}$ converges absolutely.

9.4.58.1 Solution. Hint: Apply $AM \geq GM$ to the numbers $|a_n|^2$ and $\frac{1}{n^2}$, to get $|a_n|^2 + \frac{1}{n^2} \geq 2\frac{|a_n|}{n}$. □

9.4.59 Problem. Let (a_n) and (b_n) be two sequences such that $\sum_{n=1}^{\infty} a_n^2$ and $\sum_{n=1}^{\infty} b_n^2$ converge. Then $\sum_{n=1}^{\infty} a_n b_n$ converges absolutely.

9.4.59.1 Solution. Similar to the above. □

9.4.60 Problem. Suppose that $\sum_{n=1}^{\infty} a_n$ converges while $\sum_{n=1}^{\infty} a_n^2$ diverges. Then $\sum_{n=1}^{\infty} a_n$ converges conditionally.

9.4.60.1 Solution. Assume for the sake of contradiction that $\sum_{n=1}^{\infty} a_n$ converges absolutely, i.e., $\sum_{n=1}^{\infty} |a_n|$ that converges, we must have $a_n \to 0$ so that eventually we have $|a_n| < 1$ and $a_n^2 < |a_n|$. By the Comparison Test, we then have $\sum_{n=1}^{\infty} a_n^2$ a convergent series, a contradiction. □

9.4.61 Problem.

1. Find a divergent sequence (a_n) such that $-1 < a_n < 1$ for all n.
2. Find two divergent series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ suct that $\sum_{n=1}^{\infty}(a_n + 2b_n)$ is convergent.
3. Find a series $\sum_{n=1}^{\infty} a_n$ such that $\sum_{n=1}^{\infty} a_n = 2$ and $\sum_{n=1}^{\infty} a_n^2 = 10$.

9.4.61.1 Solution.

1. $a_n = \frac{1}{2}(-1)^n$
2. $a_n = \frac{1}{n}$, $b_n = -\frac{1}{2n}$
3. $a_n = \frac{-20}{3}\left(\frac{-3}{7}\right)^n$ □

9.4.62 Problem. Test the convergence of the series. $\frac{1}{2} + \frac{1}{3} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^3} + ...$

9.4.62.1 Solution. We see that the following sequence (a_n) defined by

$$a_n = \begin{cases} \frac{1}{3^n} & \text{if } n \text{ is even} \\ \frac{1}{2^n} & \text{if } n \text{ is odd} \end{cases}$$

That is, $a_{2n} = \frac{1}{3^{2n}}$ and $a_{2n+1} = \frac{1}{2^{2n+1}}$, $a_{2n-1} = \frac{1}{2^{2n-1}}$ and $\frac{a_{2n}}{a_{2n-1}} = \left(\frac{2}{3}\right)^{2n}$ and $\frac{a_{2n+1}}{a_{2n}} = \frac{1}{3}\left(\frac{3}{2}\right)^{2n+1}$ So

$$\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} = \infty \text{ and } \limsup_{n\to\infty} \frac{a_{n+1}}{a_n} = 0.$$

Thus the ratio test is inconclusive. So by root test, we get

$$\limsup_{n\to\infty} \sqrt[n]{a_n} = \frac{1}{\sqrt{2}} \text{ and } \liminf_{n\to\infty} \sqrt[n]{a_n} = \frac{1}{\sqrt{3}}$$

Since $\limsup_{n\to\infty} \sqrt[n]{a_n} = \frac{1}{\sqrt{2}} < 1$. Therefore $\sum_{n=1}^{\infty} a_n$ is convergent by the root test. □

9.4.63 Problem. Test the series

$$a + b + a^2 + b^2 + a^3 + b^3 + ..., \ 0 < a < b < 1.$$

9.4.63.1 Solution. Similar to the above. □

9.4.64 Problem. A conditionally convergent series cannot be rearranged to form an absolutely convergent series.

9.4.64.1 Solution. If $\sum_{n=1}^{\infty} a_n$ could be rearranged to form an absolutely convergent series, then every rearrangement of that series converges to the same sum. This violates the fact that $\sum_{n=1}^{\infty} a_n$ a conditionally convergent series can be rearranged to sum to any preselected number. □

9.4.65 Problem. If $\sum_{n=1}^{\infty} a_n$ is conditionally convergent, then the terms can be grouped so as to form an absolutely convergenr series.

9.4.65.1 Solution. Let (s_n) be the sequence of partia1 sums for our series, with $s_n \to s$. Choose an increasing sequence $n_1 < n_2 < ... < n_k < ...$ such that for $k > 0$,

$$s - \frac{1}{2^k} < s_{n_k} < s + \frac{1}{2^k}$$

and group the series as $b_1 + b_2 + b_3 + ...$ where the b_i are

$$b_i = \sum_{n=n_{i-1}+1}^{n_i} a_n$$

Notice that $b_i = s_{n_i} - s_{n_{i-1}}$ so that for $i > 1$,

$$|b_i| \leq |s_{n_i} - s| + |s_{n_{i-1}} - s| < \frac{1}{2^i} + \frac{1}{2^{i-1}} = \frac{3}{2^i}.$$

Then the terms $|b_i|$ are dominated by the terms of a convergent geometric series, and so $\sum_{n=1}^{\infty} b_n$ is an absolutely convergent grouping of our series. □

9.4.66 Problem. Show that if $\sum_{k=1}^{\infty} a_k = 1$ and $0 < a_n < \sum_{k=n+1}^{\infty} a_k$ for $n = 1, 2, ...$, then for every $x \in (0, 1)$ there is a subseries $\sum_{k=1}^{\infty} a_{n_k}$ whose sum is x.

9.4.66.1 Solution. Note that, since the sum of the series is 1 and $x \in (0, 1)$, there exists $n_1 \in \mathbb{N}$ such that

$$\sum_{k=n_1}^{\infty} a_k > x \text{ and } \sum_{k=n_1+1}^{\infty} a_k \leq x$$

implying

$$\sum_{k=n_1+1}^{\infty} a_k > x - a_{n_1} \text{ and } a_{n_1} < \sum_{k=n_1+1}^{\infty} a_k \leq x.$$

Therefore there exists $n_2 > n_1$ such that

$$\sum_{k=n_2}^{\infty} a_k > x - a_{n_1} \text{ and } a_{n_2} < \sum_{k=n_2+1}^{\infty} a_k \leq x - a_{n_1}.$$

Continuing this way, we can find a sequence of integers $n_1 < n_2 < ....$ such that

$$0 \leq x - \sum_{k=1}^{m} a_{n_k} < \sum_{k=n_m+1}^{\infty} a_k.$$

Taking $m \to \infty$, we conclude that $\sum_{k=1}^{\infty} a_{n_k} = x$. □

9.4.67 Problem. If $\sum_{n=1}^{\infty} c_n$ is a convergent series of positive terms. Then $\sum_{n=1}^{\infty} \frac{\sqrt{c_n}}{n}$ converges.

9.4.67.1 Solution. By Cauchy's inequality

$$\left(\sum_{n=1}^{\infty} \frac{\sqrt{c_n}}{n}\right)^2 \leq \left(\sum_{n=1}^{\infty} c_n\right)\left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right).$$

Since both series on the right converge, so also does the series on the left. □

9.4.68 Problem. If $\sum_{n=1}^{\infty} c_n$ is a convergent series of positive terms. Then $\sum_{n=1}^{\infty} \sqrt{\frac{c_n}{n}}$ may converge or diverge.

9.4.68.1 Solution. Convergent: We take $c_n = \frac{1}{n^3}$, then

$$\sum_{n=1}^{\infty} \sqrt{\frac{c_n}{n}} = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Divergent: Take $c_n = \frac{1}{n(\ln n)^2}$ then $\sum_{n=2}^{\infty} \sqrt{\frac{c_n}{n}} = \sum_{n=2}^{\infty} \frac{1}{n \ln n}$. □

9.4.69 Problem. For each real number x, $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n}$ converges.

9.4.69.1 Solution. If x is an integral multiple of π, the series is identically 0 and there is nothing to prove. Assume that $x \neq 0$. Let $a_n = \sin(nx)$ and $b_n = 1/n$ and appeal to the result by Dirichlet. Consider the trigonometric identity

$$2\sin(x/2)\sin(nx) = \cos(n-1/2)x - \cos(n+1/2)x$$

Let $s_n = \sum_{k=1}^{n} a_n$, assume x is not an integer multiple of π, and using telescoping to see that

$$s_n = \frac{1}{2}\frac{\cos(x/2) - \cos(n+1/2)x}{\sin(x/2)} = \frac{1}{2}\left[\frac{\cos(x/2)}{\sin(x/2)} - \frac{\cos(n+1/2)x}{\sin(x/2)}\right].$$

Then the partial sums s_n are bounded. Since (b_n) is a monotone sequence tending to 0, the Dirichlet result guarantees

$$\sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{\infty} \frac{\sin(nx)}{n}$$

converges. □

9.4.70 Problem. For any positive integer n, let $\langle n \rangle$ denote the closest integer to $\sqrt{n}$. Prove that

$$\sum_{n=1}^{\infty} \frac{2^{\langle n \rangle} + 2^{-\langle n \rangle}}{2^n} = 3.$$

9.4.70.1 Solution. We regroup this sum based on the value of $\langle n \rangle$. First, we notice that

$$\begin{aligned}\langle n \rangle = k &\Leftrightarrow k - 1/2 \leq \sqrt{n} < k + 1/2\\ &\Leftrightarrow k^2 - k + 1/4 \leq n < k^2 + k + 1/4\\ &\Leftrightarrow k^2 - k + 1/4 \leq n < k^2 + k\end{aligned}$$

Now we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{2^{\langle n\rangle}+2^{-\langle n\rangle}}{2^n} &= \sum_{k=1}^{\infty} \sum_{n=k^2-k+1}^{n=k^2+k} \frac{2^k+2^{-k}}{2^n} \\
&= \sum_{k=1}^{\infty} \left(2^k+2^{-k}\right) \sum_{n=k^2-k+1}^{n=k^2+k} 2^{-n} \\
&= \sum_{k=1}^{\infty} \left(2^k+2^{-k}\right)\left(2^{-k^2+k}-2^{-k^2-k}\right) \\
&= \sum_{k=1}^{\infty} \left(2^{-k^2+2k}+2^{-k^2}-2^{-k^2}-2^{-k^2-2k}\right) \\
&= \sum_{k=1}^{\infty} \left(2^{-k^2+2k}-2^{-k^2-2k}\right) \\
&= \sum_{k=1}^{\infty} 2^{-k(k-2)} - \sum_{k=1}^{\infty} 2^{-k(k+2)} \\
&= \sum_{k=1}^{\infty} 2^{-k(k-2)} - \sum_{k=3}^{\infty} 2^{-k(k-2)} \\
&= \sum_{k=1}^{2} 2^{-k(k-2)} = 2+1 = 3. \quad \square
\end{aligned}$$

9.4.71 Problem. Prove that the series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$ converges to $\ln 2$, but the re-arranged series

1. $\sum_{n=1}^{\infty}(\frac{1}{2n-1}-\frac{1}{4n-2}-\frac{1}{4n})=\frac{1}{2}\ln 2.$

2. $\sum_{n=1}^{\infty}(\frac{1}{2n-1}-\frac{1}{8n-6}-\frac{1}{8n-4}-\frac{1}{8n-2}-\frac{1}{8n})=0.$

3. $\sum_{n=1}^{\infty}(\frac{1}{4n-3}+\frac{1}{4n-1}-\frac{1}{2n})=\frac{3}{2}\ln 2.$

4. $\sum_{n=1}^{\infty}(\frac{1}{4n-3}+\frac{1}{4n-1}-\frac{1}{4n-2}-\frac{1}{4n})=\ln 2.$

5. $\sum_{n=1}^{\infty}(\frac{1}{6n-5}+\frac{1}{6n-3}-\frac{1}{6n-1}-\frac{1}{2n})=\frac{1}{2}\ln 12.$

6. $\sum_{n=1}^{\infty}(\frac{1}{3n-2}+\frac{1}{3n-1}-\frac{1}{3n})$ is divergent.

9.4.71.1 Solution. In these problems, we use the important sequence (γ_n) defined by $\gamma_n = 1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}-\ln n$, with $\lim_n \gamma_n = \gamma$, the Euler number. Let $s_n = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+....+(-1)^{n-1}\frac{1}{n}$.

Then

$$\begin{aligned}
s_{2n} &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ... + \frac{1}{2n-1} - \frac{1}{2n} \\
&= \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ... + \frac{1}{2n}\right) - 2\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + ... + \frac{1}{2n}\right) \\
&= \ln 2n + \gamma_{2n} - \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ... + \frac{1}{n}\right) \\
&= \ln 2n + \gamma_{2n} - (\ln n + \gamma_n) \\
&= \ln 2 + \gamma_{2n} - \gamma_n.
\end{aligned}$$

Therefore $\lim_n s_{2n} = \ln 2$. Again, $s_{2n+1} = s_{2n} + \frac{1}{2n+1} \Rightarrow \lim_n s_{2n+1} = \lim_n s_{2n} = \ln 2$. Hence $\lim_n s_n = \ln 2$.

1. Let $t_n = 1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + ... + \frac{1}{2n-1} - \frac{1}{4n-2} - \frac{1}{4n}$ to n terms.

$$\begin{aligned}
s_{3n} &= \left(1 - \frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{6} - \frac{1}{8}\right) + ... + \left(\frac{1}{2n-1} - \frac{1}{4n-2} - \frac{1}{4n}\right) \\
&= \left(1 + \frac{1}{3} + \frac{1}{5} + ... + \frac{1}{2n-1}\right) - \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ... + \frac{1}{2n}\right) \\
&= \left(1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{2n}\right) - \left(\frac{1}{2} + \frac{1}{4} + ... + \frac{1}{2n}\right) - \frac{1}{2}\left(1 + \frac{1}{2} + .. + \frac{1}{2n}\right) \\
&= \ln 2n + \gamma_{2n} - \frac{1}{2}(\ln n + \gamma_n) - \frac{1}{2}(\ln 2n + \gamma_{2n}) \\
&= \frac{1}{2}(\ln 2n + \gamma_{2n}) - \frac{1}{2}(\ln n + \gamma_n) \\
&= \frac{1}{2}\ln 2 + \frac{1}{2}\gamma_{2n} - \frac{1}{2}\gamma_n.
\end{aligned}$$

Therefore $\lim_n s_{3n} = \frac{1}{2}\ln 2$. Again, $s_{3n+1} = s_{3n} + \frac{1}{2n+1}$ and $s_{3n+2} = s_{3n+1} + \frac{1}{4n+2}$. Therefore $\lim s_{3n+1} = \lim s_{3n} = \frac{1}{2}\ln 2$, and $\lim s_{3n+2} = \lim s_{3n+1} = \frac{1}{2}\ln 2$. This proves that $\lim_n t_n = \frac{1}{2}\ln 2$ and hence the series converges to $\frac{1}{2}\ln 2$.

2. Let

$$s_{8n} = 1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \frac{1}{8} + \frac{1}{3} - \frac{1}{10} - \frac{1}{12} - \frac{1}{14} - \frac{1}{16} + ... + \frac{1}{2n-1} - \frac{1}{8n-6} - \frac{1}{8n-4} - \frac{1}{8n-2} - \frac{1}{8n}.$$

Now,

$$\begin{aligned}
t_{5n} &= \left(1-\frac{1}{2}-\frac{1}{4}-\frac{1}{6}-\frac{1}{8}\right)+\left(\frac{1}{3}-\frac{1}{10}-\frac{1}{12}-\frac{1}{14}-\frac{1}{16}\right)+..\\
&....+\left(\frac{1}{2n-1}-\frac{1}{8n-6}-\frac{1}{8n-4}-\frac{1}{8n-2}-\frac{1}{8n}\right)\\
&=\left(1+\frac{1}{3}+\frac{1}{5}+...+\frac{1}{2n-1}\right)-\frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{4n}\right)\\
&=\left(1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{2n}\right)-\frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}\right)\\
&-\frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{4n}\right)\\
&=\ln 2n+\gamma_{2n}-\frac{1}{2}(\ln n+\gamma_n)-\frac{1}{2}(\ln 4n+\gamma_{4n})\\
&=\ln 2+\ln n+\gamma_{2n}-\frac{1}{2}\ln n-\frac{1}{2}\gamma_n-\ln 2-\frac{1}{2}\ln n-\frac{1}{2}\gamma_{4n}\\
&=\gamma_{2n}-\frac{1}{2}\gamma_n-\frac{1}{2}\gamma_{4n}.
\end{aligned}$$

Now, $\lim_n t_{5n}=\lim_n \gamma_{2n}-\frac{1}{2}\lim_n \gamma_n-\frac{1}{2}\lim_n \gamma_{4n}=\gamma-\frac{1}{2}\gamma-\frac{1}{2}\gamma=0.$

3. Let $s_{9n}=1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+...+\frac{1}{4n-3}+\frac{1}{4n-1}-\frac{1}{2n}$. So,

$$\begin{aligned}
t_{3n} &= \left(1+\frac{1}{3}-\frac{1}{2}\right)+\left(\frac{1}{5}+\frac{1}{7}-\frac{1}{4}\right)+...+\\
&\left(\frac{1}{4n-3}+\frac{1}{4n-1}-\frac{1}{2n}\right)\\
&=\left(1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+...+\frac{1}{4n-1}\right)-\frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{2n}\right)\\
&-\frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}\right)\\
&=\ln 4n+\gamma_{4n}-\frac{1}{2}(\ln 2n+\gamma_{2n})-\frac{1}{2}(\ln n+\gamma_n)\\
&=2\ln 2+\ln n+\gamma_{4n}-\frac{1}{2}\ln 2-\frac{1}{2}\ln n-\frac{1}{2}\gamma_{2n}-\frac{1}{2}\ln n-\frac{1}{2}\gamma_n\\
&=\frac{3}{2}\ln 2+\gamma_{4n}-\frac{1}{2}\gamma_{2n}-\frac{1}{2}\gamma_n
\end{aligned}$$

Thus $\lim_n t_{3n}=\frac{3}{2}\ln 2$.

4. Let

$$
\begin{aligned}
t_{4n} &= \left(1+\frac{1}{3}-\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{7}-\frac{1}{6}-\frac{1}{8}\right)+\left(\frac{1}{9}+\frac{1}{11}-\frac{1}{10}-\frac{1}{12}\right)+..\\
&\quad ...+\left(\frac{1}{4n-3}+\frac{1}{4n-1}-\frac{1}{4n-2}-\frac{1}{4n}\right)\\
&= \left(1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+...+\frac{1}{4n-1}\right)-\frac{1}{2}\left(1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+...+\frac{1}{2n-1}\right)\\
&\quad -\frac{1}{4}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+...+\frac{1}{n}\right)\\
&= \left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+...+\frac{1}{4n}\right)-\frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+...+\frac{1}{2n}\right)\\
&\quad -\frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+...+\frac{1}{2n}\right)+\frac{1}{4}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+...+\frac{1}{n}\right)\\
&\quad -\frac{1}{4}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+...+\frac{1}{n}\right)\\
&= \ln 4n+\gamma_{4n}-\ln 2n-\gamma_{2n}\\
&= 2\ln 2+\ln n+\gamma_{4n}-\ln 2-\ln n-\gamma_{2n}\\
&= \ln 2+\gamma_{4n}-\gamma_{2n}.
\end{aligned}
$$

Thus $\lim_n t_{4n} = \ln 2$.

5. Left to the reader.

6. Let s_n be the n-th partial sum. Then

$$
\begin{aligned}
t_{3n} &> \left(\frac{1}{3}+\frac{1}{3}-\frac{1}{3}\right)+\left(\frac{1}{6}+\frac{1}{6}-\frac{1}{6}\right)+...+\left(\frac{1}{3n}+\frac{1}{3n}-\frac{1}{3n}\right)\\
&> \frac{1}{3}+\frac{1}{6}+\frac{1}{9}+....+\frac{1}{3n}\\
&> \frac{1}{3}\left(1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}\right). \quad \square
\end{aligned}
$$

9.4.72 Problem. Let I be the set of positive integers that do not contain the digit 9 in their decimal expansions. Prove that

$$\sum_{n\in I}\frac{1}{n} \text{ is convergent.}$$

9.4.72.1 Solution. Let $I_k = I\cap[10^k, 10^{k+1})$ and $S_k = \sum_{n\in I_k} 1/n$. For a fixed $k\geq 1$, each $n\in I_k$ has the decimal representation

$$\overline{n_k n_{k-1}...n_1 n_0}, n_i \neq 9 \text{ for } 0\leq i\leq k \text{ and } n_k\neq 0.$$

Hence $n = 10^k n_k + j$, where $1 \le n_k \le 8$ and $j \in I_s$ for some $0 \le s \le k-1$ or else $j = 0$. It follows in particular that $|I_k| = 8\sum_{s=0}^{k-1} |I_s| + 8$ which implies by an induction that $|I_k| = 8.9^k$. Thus

$$\begin{aligned} S_k &= \sum_{i-1}^{8}\sum_{s=0}^{k-1}\sum_{j\in I_s} \frac{1}{10^k i + j} + \sum_{i=1}^{8} \frac{1}{10^k i} \\ &\le 8\sum_{s=0}^{k-1}\sum_{j\in I_s} \frac{1}{10^k} + \sum_{i=1}^{8} \frac{8}{10^k i} \\ &= 10^{-k}\left(8\sum_{s=0}^{k-1} |I_s| + 8\right) \\ &= 10^{-k}|I_k| \le 8(0.9)^k. \end{aligned}$$

This implies $\sum_{n\in I} \dfrac{1}{n} = S_0 + \sum_{k=1}^{\infty} S_k \le S_0 + 8\sum_{k=1}^{\infty}(0.9)^k = S_0 + 72 < 75.$ □

9.4.73 Problem. Show that every positive real number is a sum (possibly infinite) of a subset of the numbers $\left\{1, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{n}, ...\right\}$.

9.4.73.1 Solution. Suppose that $t \in \mathbb{R}$ is not a finite sum of elements from $S = \left\{1, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{n}, ...\right\}$. Define $f(x) = \max\left\{\frac{1}{n} \in S; \frac{1}{n} \le x\right\}$. Since 0 is a limit point of S, f is well defined for $x > 0$. Finally, define $t_0 = f(t)$, $t_{i+1} = t_i + f(t - t_i)$. Since t is not a finite sum of elements from S, we must have $t - t_i \ne 0$ for all i. Now, since f only attains positive values, the sequence (t_i) is monotone increasing. Furthermore, since $f(t-t_i) < t - t_i$, we have $t_i < t \ \forall\, i$ so that (t_i) is bounded above. Every bounded monotone sequence converges, so $t_i \to s$ for some $s \in \mathbb{R}$. If $s < t$, then let $\frac{1}{m} = f(t - s)$. As $t_i \to s$, so there exists a t_i for which $s - t_i < \frac{1}{m}$. It follows immediately that $t_{i+1} > t_i + \frac{1}{m} > s$ contradiction. Therefore $t = s$. □

9.4.73.2 Solution.

Case 1: Assume that our positive number p, satisfies $0 < p < 1$. Then p has a binary representation of the form

$$p = 0.b_1b_2b_3.... = b_1\frac{1}{2} + b_2\frac{1}{4} + b_3\frac{1}{8} + ...$$

with each b_k either 0 or 1. This gives a representation of p as a sum (possibly infinite) of a subset of the numbers $\{1, 1/2, 1/4, 1/8, , ..\}$.
Case 2: Assume that our positive number M satisfies $M > 1$. Since the series $\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + ...$ diverges, there is a finite sum $S = \sum_{k=1}^{n} \frac{1}{2k+1}$ such that $M = S + q$ with $0 < q < 1$. Applying case 1 to this q gives the result. □

9.4.73.3 Solution. Since the series $1 - \frac{1}{2} + \frac{1}{3} - ... + (-1)^n \frac{1}{n} + ...$is conditionally convergent, so by Riemann's theorem, the conclusion follows.

9.4.74 Problem. Let $a_1, a_2, a_3, ...$ be positive numbers.

1. Prove that $\sum_{n=1}^{\infty} a_n < \infty$ implies $\sum_{n=1}^{\infty} \sqrt{a_n a_{n+1}} < \infty$.

2. Prove that the converse of the above statement is false.

9.4.74.1 Solution.

1. Assume that $\sum_{n=1}^{\infty} a_n < \infty$. Applying $AM > GM$ between a_{n+1} and a_n, we get
$$\sum_{n=1}^{\infty} \sqrt{a_{n+1}a_n} \leq \frac{1}{2}\sum_{n=1}^{\infty}(a_{n+1}+a_n) = \frac{1}{2}a_1 + \sum_{n=2}^{\infty} a_n < \infty. \quad \square$$

2. Since $\sum(a_{n+1}+a_n) = 2\sum\sqrt{a_{n+1}a_n} + \sum\left(\sqrt{a_{n+1}}-\sqrt{a_n}\right)^2$, we require a sequence $a_n = b_n^2,\ b_n > 0$, such that $\sum\sqrt{b_{n+1}b_n} < \infty$ but $\sum\left(\sqrt{b_{n+1}}-\sqrt{b_n}\right)^2 \to \infty$. One such example is
$$b_n = \begin{cases} \frac{1}{\sqrt{n}} & \text{if } n \text{ is odd} \\ \frac{1}{n} & \text{if } n \text{ is even.} \end{cases}$$

$\square$

9.4.75 Problem. Let $a_1, a_2, a_3, ...$ be positive numbers.

1. Prove that $\sum_{n=1}^{\infty} a_n < \infty$ implies $\sum_{n=1}^{\infty}\left(a_n^{-1}+a_{n+1}^{-1}\right)^{-1} < \infty$.
2. Prove that the converse of the above statement is false.

9.4.75.1 Solution.

1. Assume that $\sum_{n=1}^{\infty} a_n < \infty$. Applying $HM < GM < AM$ between a_{n+1} and a_n, we get
$$2\left(a_{n+1}^{-1}+a_n^{-1}\right)^{-1} \leq \sqrt{a_{n+1}a_n} \leq \frac{1}{2}(a_{n+1}+a_n)$$
Then proceed as above.

2. Let
$$a_n = \begin{cases} \frac{1}{n} & \text{if } n \text{ is even} \\ \frac{1}{n^3} & \text{if } n \text{ is odd.} \end{cases}$$

$\square$

9.4.76 Problem. If $\sum_{n=1}^{\infty} a_n$ be a convergent series of positive real numbers prove that $\sum_{n=1}^{\infty} a_n^2$ is convergent.

9.4.76.1 Solution. Hint. Since $a_n \to 0$, so there exists an $m \in \mathbb{N}$ such that $a_n < 1$ for all $n \geq m$, so $a_n^2 \leq a_n$ for all $n \geq m$. $\square$

9.4.77 Problem. If $\sum_{n=1}^{\infty} a_n$ be a convergent series of positive real numbers prove that $\sum_{n=1}^{\infty} \frac{a_n}{n}$ is convergent.

9.4.77.1 Solution. Hint: Apply AM-GM inequality between a_n^2 and $\frac{1}{n^2}$ to get
$$a_n.\frac{1}{n} < \frac{a_n^2 + \frac{1}{n^2}}{2}$$

$\square$

9.4.78 Problem. If $\sum_{n=1}^{\infty} a_n$ be a convergent series of positive real numbers prove that $\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}$ is convergent.

9.4.78.1 Solution. Hint: Apply AM-GM inequality between $\sqrt{a_n}$ and $\frac{1}{n^2}$ to get

$$\sqrt{a_n}.\frac{1}{n} < \frac{a_n + \frac{1}{n^2}}{2} = \frac{a_n}{2} + \frac{1}{2n^2}. \quad \square$$

9.4.79 Problem. If $\sum_{n=1}^{\infty} a_n$ be a convergent series of positive real numbers, prove that $\sum_{n=1}^{\infty} a_{2n}$ is convergent.

9.4.79.1 Solution. Hint. Let $s_n = a_1 + a_2 + ... + a_n,\ t_n = a_2 + a_4 + ... + a_{2n}$. Then $t_n < s_{2n}$ for all $n \in \mathbb{N}$. The sequence (t_n) is a monotone increasing sequence bounded above. $\square$

9.4.80 Problem. If $\sum_{n=1}^{\infty} a_n$ be a series of positive real numbers, prove that $\sum_{n=1}^{\infty} b_n$ is divergent, where $b_n = \frac{\sum_{k=1}^{n} a_k}{n}$.

9.4.80.1 Solution. Hint: $b_1 + b_2 + b_3 + ... + b_n > a_1 \left(1 + \frac{1}{2} + \frac{1}{3} + .. + \frac{1}{n}\right)$ $\square$

9.4.81 Problem. If $\sum_{n=1}^{\infty} a_n$ be a divergent series of positive real numbers, prove that $\sum_{n=1}^{\infty} \frac{a_n}{s_n}$ is divergent, where $s_n = \sum_{k=1}^{n} a_k$.

9.4.81.1 Solution. Hint: Since (s_n) is a monotone increasing sequence diverging to ∞, so for every natural number n, we can choose a natural number p such that $s_{n+p} > 2s_n$. Then

$$\frac{a_{n+1}}{s_{n+1}} + \frac{a_{n+2}}{s_{n+2}} + ... + \frac{a_{n+p}}{s_{n+p}} > \frac{1}{2}. \quad \square$$

9.4.82 Problem. If $\{a_1, a_2, a_3, ...\}$ be the collection of those natural numbers that end with 1, prove that the series $\sum_{n=1}^{\infty} \frac{1}{a_n}$ is divergent.

9.4.82.1 Solution. Hint. Considering the collection as an increasing sequence of natural numbers, $a_n = 10n - 9$ for all $n \in \mathbb{N}$ and $\frac{1}{a_n} > \frac{1}{10n}$ for all $n \in \mathbb{N}$. $\square$

9.4.83 Problem. Test the convergence of the following series:

1. $\frac{2}{1^p} + \frac{3}{2^p} + \frac{4}{3^p} + ...$
2. $\frac{1}{1+2} + \frac{1}{1+2^2} + \frac{1}{1+2^3} + ...$
3. $\frac{1}{1+2^{-1}} + \frac{1}{1+2^{-2}} + \frac{1}{1+2^{-3}} + ...$
4. $\sin\frac{\pi}{2} + \sin\frac{\pi}{4} + \sin\frac{\pi}{6} + ...$(Use the inequality $\frac{2x}{\pi} < \sin x$ for $0 < x < \frac{\pi}{2}$.)
5. $\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + ...$

9.4.83.1 Solution.

1. $a_n = \frac{n+1}{n^p} = \frac{1}{n^{p-1}} + \frac{1}{n^p}$.
2. $a_n = \frac{1}{1+2^n} < \frac{1}{2^n}$.
3. $a_n = \frac{1}{1+2^{-n}} \nrightarrow 0$.

4. Using the inequality $\frac{2x}{\pi} < \sin x$, we get $\sin\frac{\pi}{2n} > \frac{1}{n}$.

5. $\frac{1}{1.2.3} < \frac{1}{1^3}, \frac{1}{2.3.4} < \frac{1}{2^3}, \frac{1}{3.4.5} < \frac{1}{3^3}, \ldots\ldots$ □

9.4.84 Problem. Test the series $\sum_{n=1}^{\infty} a_n$ for convergence where a_n is given by

$$(1)\ \frac{2^n+1}{3^n+2}, \qquad (2)\ \sqrt{n^4+1}-n^2, \ \ (3)\ \frac{\sqrt{n+1}-\sqrt{n-1}}{n}$$
$$(4)\ \frac{1}{\sqrt{n}}\tan\frac{1}{n}, \ \ (5)\ \frac{1}{n}\sin\frac{1}{n}, \ \ (6)\ \frac{3^n}{2^n+3^n}, \ \ (7)\ \frac{1}{n\ln n}, n \geq 2.$$

9.4.84.1 Solution.

1. $a_n = \frac{2^n+1}{3^n+2} < \frac{2^n+1}{3^n} = \left(\frac{2}{3}\right)^n + \frac{1}{3^n}$.

2. $a_n = \sqrt{n^4+1} - n^2 = \frac{1}{\sqrt{n^4+1}+n^2} < \frac{1}{n^2}$.

3. $a_n = \frac{\sqrt{n+1}-\sqrt{n-1}}{n} = \frac{2}{n\sqrt{n+1}+n\sqrt{n-1}} \leq \frac{2}{n\sqrt{n+1}} < \frac{2}{n\sqrt{n}}$.

4. Let $a_n = \frac{1}{\sqrt{n}}\tan\frac{1}{n}$ and $b_n = \frac{1}{n\sqrt{n}}$, then $\frac{a_n}{b_n} = n\tan\frac{1}{n} \to 1$ as $n \to \infty$.

5. $a_n = \frac{1}{n}\sin\frac{1}{n} < \frac{1}{n^2}$.

6. $a_n = \frac{3^n}{2^n+3^n} = \frac{1}{(\frac{2}{3})^n+1}$ does not tend to 0.

7. Use Cauchy's condensation test with appropriate choice of a_n and b.

9.4.85 Problem. Test the convergence of the following series:

1. $1 + \frac{2^2}{2!} + \frac{3^2}{3!} + \ldots$

2. $\frac{1^2.2^2}{1!} + \frac{2^2.3^2}{2!} + \frac{3^2.4^2}{3!} + \ldots$

3. $\frac{1^2}{2^2} + \frac{1^2.3^2}{2^2.4^2} + \frac{1^2.3^2.5^2}{2^2.4^2.6^2} + ..$

4. $\left(\frac{2^2}{1^2} - \frac{2}{1}\right)^{-1} + \left(\frac{3^3}{2^3} - \frac{3}{2}\right)^{-2} + \left(\frac{4^4}{3^4} - \frac{4}{3}\right)^{-3} + ..$

5. $1 + \frac{1}{2} + \frac{1}{4^2} + \frac{1}{2^3} + \frac{1}{4^4} + \frac{1}{2^5} + \frac{1}{4^6} + \ldots$

6. $1 + \frac{1}{2} + \frac{1}{2.3} + \frac{1}{2^2.3} + \frac{1}{2^2.3^2} + \frac{1}{2^3.3^2} + \ldots$

7. $1 + \frac{1}{2^3} + \frac{1}{2^2} + \frac{1}{2^5} + \frac{1}{2^4} + \ldots$ Here $a_n = \left\{2^{n+(-1)^n}\right\}^{-1}$ and $\frac{u_{n+1}}{u_n} = 1/8$ if n be odd and $= 2$ if n is even.

8. $\frac{1}{\log 2} + \frac{1}{\log 3} + \frac{1}{\log 4} + \ldots$ Use $\log(1+x) < x$ for $x > 0$.

9. $\tan\frac{\pi}{4} + \tan\frac{\pi}{8} + \tan\frac{\pi}{12} + \ldots$.Use $\tan x < x$ for $0 < x < \frac{\pi}{2}$.

10. The series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\log n}$ is convergent by Abel's test, since $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is a convergent series and the sequence $\left(\frac{1}{\log n}\right)$ is a monotone decreasing sequence bounded below.

9.4.86 Problem. Prove that

$$1+\frac{n}{2}<\sum_{k=1}^{2^n}\frac{1}{k}<1+n,\ \forall\, n\in\mathbb{N}.$$

9.4.86.1 Solution. Observe that

$$\begin{aligned}1+\left(\frac{1}{2}\right)+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+...+\left(\frac{1}{2^{n-1}+1}+...+\frac{1}{2^n}\right)\\ >1+\frac{1}{2}+\frac{2}{4}+\frac{4}{8}+...+\frac{2^{n-1}}{2^n}=1+\frac{n}{2}.\end{aligned}$$

On the other hand

$$\begin{aligned}1+\left(\frac{1}{2}\right)+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+...+\left(\frac{1}{2^{n-1}+1}+...+\frac{1}{2^n}\right)\\ <1+1+\frac{2}{2}+\frac{4}{4}+...+\frac{2^{n-1}}{2^{n-1}}=1+n.\quad\square\end{aligned}$$

9.4.87 Problem. Let $\mathbb{N}$ denote the positive integers, let $a_n=(-1)^n\frac{1}{n}$, and let α be any real number. Prove there is a one-to-one and onto mapping $\sigma:\mathbb{N}\to\mathbb{N}$ such that

$$\sum_{n=1}^{\infty}a_{\sigma(n)}=\alpha.$$

9.4.87.1 Solution. We proceed for a general series which is conditionally convergent but not absolutely convergent. This series is conditionally convergent by the alternating series test, but not absolutely convergent. This is clear from the comparsion

$$\sum_{n=1}^{\infty}|a_n|=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+....\geq\frac{1}{1}+\frac{1}{2}+\frac{1}{4}+\frac{1}{4}+...$$

since the sum on the right is $1+\frac{1}{2}+\frac{1}{2}+...$, which does not converge.
Define $p_j=\frac{1}{2}(|a_j|+a_j)$ and $q_j=\frac{1}{2}(|a_j|-a_j)$. Then $p_j=a_j$ if a_j is non-negative, and $q_j=-a_j$ if a_j is negative. If both $\sum_{j=1}^{\infty}p_j$ and $\sum_{j=1}^{\infty}q_j$ converge, then $\sum_{j=1}^{\infty}a_n=\sum_{n=1}^{\infty}|p_j-q_j|$ converges, a contradiction. Thus either $\sum_{j=1}^{\infty}p_j$ or $\sum_{j=1}^{\infty}q_j$ diverges. Suppose the former. If $\sum_{j=1}^{\infty}q_j$ converges, then $\sum_{j=1}^{\infty}p_j=\sum_{j=1}^{\infty}(a_j+q_j)$converges, a contradiction. Hence both $\sum_{j=1}^{\infty}p_j$ and $\sum_{j=1}^{\infty}q_j$ converge. Re-index p_j and q_j to eliminate the 0 terms which do not correspond with some a_j and preceed a term corresponding to some a_j. Clearly both $\sum_{j=1}^{\infty}p_j$ and $\sum_{j=1}^{\infty}q_j$ still diverge.
Suppose $\alpha>0$ (the other cases are similar). Let P_j and Q_j, $j\geq 1$ be the partial sums of $\sum_{j=1}^{\infty}p_j$ and $\sum_{j=1}^{\infty}q_j$ respectively. Select the smallest N_1 such that $P_{N_1}>\alpha$. (Such an N_1 exists since $\sum_{j=1}^{\infty}p_j$ is divergent and positive.) Then select the smallest N_2 such that $P_{N_1}-Q_{N_2}<\alpha$. Next, select $N_3>N_1$ such that $P_{N_3}-Q_{N_2}>\alpha$. By the zero test for sequences, p_j and q_j approach 0 as $j\to\infty$. Thus it follows that when continuing this procedure,

$$P_{N_k}-Q_{N_{k+1}}$$

approaches α. Define the bijection $\sigma : \mathbb{N} \to \mathbb{N}$ such that $a_{\sigma(1)} = p_1, ..., a_{\sigma(N_1)} = p_{N_1}, a_{\sigma(N_1+1)} = q_1, ..., a_{\sigma(N_2)} = q_{N_2}$. Then by construction, $\sum_{j=1}^{N} a_{\sigma(j)}$ converges to α as $N \to \infty$, so

$$\sum_{n=1}^{\infty} a_{\sigma(n)} = \alpha. \quad \square$$

9.4.88 Problem. Suppose that $\sum_{n=1}^{\infty} x_n$ is a series of positive terms which is convergent. Show that $\sum_{n=1}^{\infty} \frac{1}{x_n}$ is divergent. What about the converse?

9.4.88.1 Solution. Note that if a series $\sum_{n=1}^{\infty} x_n$ is convergent, then we must have $\lim_{n\to\infty} x_n = 0$. Obviously this will imply that $\left(\frac{1}{x_n}\right)$ is divergent. Hence $\sum_{n=1}^{\infty} \frac{1}{x_n}$ is divergent. For the converse, take $x_n = \sqrt{n}$, then $\sum_{n=1}^{\infty} x_n$ both $\sum_{n=1}^{\infty} \frac{1}{x_n}$ and are divergent. So the converse is false. $\square$

9.4.89 Problem. Show that the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n} + (-1)^n}$$

is divergent, while

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

is convergent. Deduce from this that the limit convergence test does not work for non-positive series.

9.4.89.1 Solution. It is well known from the alternating series test that the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ is convergent. Let us focus on the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}+(-1)^n}$. We have

$$\begin{aligned}
\frac{(-1)^n}{\sqrt{n} + (-1)^n} - \frac{(-1)^n}{\sqrt{n}} + \frac{1}{n} &= (-1)^n \left[\frac{-(-1)^n}{\sqrt{n}\,(\sqrt{n} + (-1)^n)}\right] + \frac{1}{n} \\
&= \frac{-1}{n + (-1)^n\sqrt{n}} + \frac{1}{n} \\
&= \frac{(-1)^n\sqrt{n}}{n(n + (-1)^n\sqrt{n})}.
\end{aligned}$$

Using the inequality $||a| - |b|| \leq |a-b|$ for any real numbers a and b, we get $n^2 - n \leq n^2 - (-1)^n n\sqrt{n}$, for any $n \geq 2$. This will imply

$$\left|\frac{(-1)^n}{\sqrt{n} + (-1)^n} - \frac{(-1)^n}{\sqrt{n}} + \frac{1}{n}\right| \leq \frac{\sqrt{n}}{n^2 - n\sqrt{n}}.$$

for any $n \geq 2$. Note that for any $n \geq 4$, we have $2\sqrt{n} < n$ or $\sqrt{n} < n - \sqrt{n}$. Hence

$$\left|\frac{(-1)^n}{\sqrt{n} + (-1)^n} - \frac{(-1)^n}{\sqrt{n}} + \frac{1}{n}\right| \leq \frac{\sqrt{n}}{n\sqrt{n}}$$

for any $n \geq 4$. Since the series $\sum_{n=1}^{\infty} \frac{1}{n^{1.5}}$ is convergent, from the basic comparison theorem we get that $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{\sqrt{n}+(-1)^n} - \frac{(-1)^n}{\sqrt{n}} + \frac{1}{n}\right|$ is convergent.

Hence the series $\sum_{n=1}^{\infty}\left(\frac{(-1)^n}{\sqrt{n}+(-1)^n}-\frac{(-1)^n}{\sqrt{n}}+\frac{1}{n}\right)$ is absolutely convergent which implies that it is also convergent. Since

$$\frac{(-1)^n}{\sqrt{n}+(-1)^n}=\frac{(-1)^n}{\sqrt{n}}-\frac{1}{n}+\left(\frac{(-1)^n}{\sqrt{n}+(-1)^n}-\frac{(-1)^n}{\sqrt{n}}+\frac{1}{n}\right)$$

and the series $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ is convergent while $\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent, we can deduce that $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}+(-1)^n}$ is divergent. Because

$$\lim_{n\to\infty}\frac{\frac{(-1)^n}{\sqrt{n}+(-1)^n}}{\frac{(-1)^n}{\sqrt{n}}}=\lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n}+(-1)^n}=\lim_{n\to\infty}\frac{1}{1+\frac{(-1)^n}{\sqrt{n}}}=1$$

Hence, we conclude that the limit test is not valid for non-positive series. □

9.4.90 Problem. Let (x_n) and (ϵ_n) be two sequences of real numbers such that

1. the sequence of partial sums (s_n) of $\sum x_n$ is bounded, i.e., there exists $M>0$ such that $|s_n|=|x_1+...+x_n|\leq M, n=1,...;$
2. $\lim_{n\to\infty}\epsilon_n=0$.
3. the series $\sum_{n=1}^{\infty}|\epsilon_{n+1}-\epsilon_n|$ is convergent.

Then the series $\sum_{n=1}^{\infty}\epsilon_n x_n$ is convergent. This conclusion is known as Abel's test or Abel's theorem.

9.4.90.1 Solution. Indeed, let $n\geq 2$ and $N>n$. Then

$$\sum_{k=n}^{N}\epsilon_k x_k=\sum_{k=n}^{N}\epsilon_k(s_k-s_{k-1})=\sum_{k=n}^{N}\epsilon_k s_k-\sum_{k=n}^{N}\epsilon_k s_{k-1}.$$

But

$$\sum_{k=n}^{N}\epsilon_k s_{k-1}=\sum_{k=n-1}^{N-1}\epsilon_{k+1}s_k.$$

Hence

$$\sum_{k=n}^{N}\epsilon_k x_k=\sum_{k=n}^{N-1}\left(\epsilon_k-\epsilon_{k+1}\right)s_k+\epsilon_N s_N-\epsilon_n s_{n-1}.$$

Let $\epsilon>0$. Then, since $\epsilon_n\to 0$ and (ϵ_n) is bounded, there exists $n_0\geq 2$ such that for any $n,\ N\geq n_0$ we have $|\epsilon_N s_N-\epsilon_n s_{n-1}|<\epsilon/2$. Also since $|\epsilon_{n+1}-\epsilon_n|$ is convergent and sn is bounded, the basic comparison test implies that $|(\epsilon_{n+1}-\epsilon_n)s_n|$ is also convergent. Therefore there exists $n_1\geq 2$ such that for any $n,\ N\geq n_1$ we have

$$\sum_{k=n}^{N-1}\left|\left(\epsilon_k-\epsilon_{k+1}\right)s_k\right|<\frac{\epsilon}{2}.$$

Let $n_2 > \max\{n_0, n_1\}$. Then for any $n,\ N \geq n_2$, we have

$$\left|\sum_{k=n}^{N} \epsilon_k x_k\right| \leq \sum_{k=n}^{N-1} |(\epsilon_k - \epsilon_{k+1})\, s_k| + |\epsilon_N s_N - \epsilon_n s_{n-1}| < \epsilon.$$

The Cauchy criterion for series will imply that the series $\sum_{n=1}^{\infty} \epsilon_n x_n$ is convergent as claimed. □

9.4.91 Problem. (**Dirichlet's Rearrangement Theorem**) Let $\sum_{n=1}^{\infty} x_n$ be an absolutely convergent series, with sum s, and let $\sum_{n=1}^{\infty} y_n$ be any rearrangement of $\sum_{n=1}^{\infty} x_n$. Show that $\sum_{n=1}^{\infty} y_n$ converges, and $\sum_{n=1}^{\infty} x_n = s$.

9.4.91.1 Solution. Since $\sum_{n=1}^{\infty} y_n$ is a rearrangement of $\sum_{n=1}^{\infty} x_n$, there is a bijection $f : \mathbb{N} \to \mathbb{N}$ such that $y_r = x_{f(r)}$ and $x_r = y_{f^{-1}(r)}$. Let $\epsilon > 0$. Since $\sum_{n=1}^{\infty} x_n$ is absolutely convergent, it follows from the General Principle of Convergence that there exists N such that $n \geq m \geq M$

$$\sum_{r=m}^{n} |x_r| < \frac{\epsilon}{2}.$$

In fact, if S is any finite subset of $\{r \in \mathbb{N}; r > N\}$, then

$$\sum_{r \in S} |x_r| < \frac{\epsilon}{2}. \tag{9.1}$$

Next, we show that

$$m \geq N \Rightarrow \left|s - \sum_{r=1}^{m} x_r\right| \leq \frac{\epsilon}{2}. \tag{9.2}$$

The reason for this is that

$$\begin{aligned}\left|s - \sum_{r=1}^{m} x_r\right| &= \lim_{n\to\infty} \left|\sum_{r=1}^{n} x_r - \sum_{r=1}^{m} x_r\right| \\ &= \lim_{n\to\infty} \left|\sum_{r=m+1}^{n} x_r\right| \leq \lim_{n\to\infty} \sum_{r=m+1}^{n} |x_r| \leq \frac{\epsilon}{2}\end{aligned}$$

by (10.1).
Now, let $M = \max\{f^{-1}(1), ..., f^{-1}(N)\}$, and note that $M \geq N$. Since $x_r = y_{f^{-1}(r)}$, it follows that $\{y_1, ..., y_M\} \supseteq \{x_1, .., x_N\}$. So, if $n \geq M$, then $\{y_1, .., y_n\} = \{x_1, ..., x_N\} \cup \{x_r; r \in S_n\}$, for some finite subset S_n of $\{r \in \mathbb{N}\}$. Thus

$$\begin{aligned} n \geq M \Rightarrow \left|s - \sum_{r=1}^{n} y_r\right| &\leq \left|s - \sum_{r=1}^{N} x_r\right| + \sum_{r \in S_n} |x_r| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

by (10.2) and (10.1). This shows that $\sum_{n=1}^{\infty} y_n$ converges to s. □

9.4.92 Problem. Consider the series $\sum_{n=1}^{\infty} x_n$ with $x_n > 0 \; \forall \; n \in \mathbb{N}$. Assume that there exists $N \geq 1$ and $p > 1$ is a real number such that

$$n\left(1 - \frac{x_{n+1}}{x_n}\right) \geq p$$

for any $n \geq N$. Show that $\sum_{n=1}^{\infty} x_n$ is convergent.

9.4.92.1 Solution. Let $y_n = \frac{1}{(n-1)^p}$. Then

$$\frac{y_{n+1}}{y_n} = \frac{n-1}{n} = 1 - \frac{1}{n}$$

Using the inequality $(1-x)^p \geq 1 - px$ for any $x \in [0,1]$, we get

$$\frac{y_{n+1}}{y_n} \geq 1 - p\frac{1}{n} \geq \frac{x_{n+1}}{x_n}$$

for any $n \geq N$, because of our assumption on (x_n). Since $p > 1$, the series $\sum_{n \geq N}^{\infty} y_n$ is convergent. From the previous problem we can deduce that $\sum_{n \geq N}^{\infty} x_n$ is convergent, which implies that $\sum_{n=1}^{\infty} x_n$ is convergent. $\square$

9.4.93 Problem. Consider the series $\sum_{n=1}^{\infty} x_n$ with $x_n > 0 \; \forall \; n \in \mathbb{N}$. Assume that there exists $N \geq 1$ such that

$$n\left(1 - \frac{x_{n+1}}{x_n}\right) \leq 1$$

for any $n \geq N$. Show that $\sum_{n=1}^{\infty} x_n$ is divergent.

9.4.93.1 Solution. Let $y_n = \frac{1}{n-1}$. Then

$$\frac{y_{n+1}}{y_n} = \frac{n-1}{n} = 1 - \frac{1}{n}$$

Hence

$$\frac{y_{n+1}}{y_n} \geq \frac{x_{n+1}}{x_n}$$

for any $n \geq N$, because of our assumption on (x_n). From the previous problem we can deduce that $\sum_{n \geq N}^{\infty} x_n$ is convergent, which implies that $\sum_{n=1}^{\infty} x_n$ is convergent. $\square$

9.4.94 Problem. Consider the series $\sum_{n=1}^{\infty} x_n$ with $x_n > 0 \; \forall \; n \in \mathbb{N}$.

1. if $\lim_{n\to\infty} n\left(1 - \frac{x_{n+1}}{x_n}\right) > 1$, then $\sum_{n=1}^{\infty} x_n$ is convergent and
2. if $\lim_{n\to\infty} n\left(1 - \frac{x_{n+1}}{x_n}\right) < 1$, then $\sum_{n=1}^{\infty} x_n$ is divergent.

Show that we do not have any conclusion when

$$\lim_{n\to\infty} n\left(1 - \frac{x_{n+1}}{x_n}\right) = 1.$$

9.4.94.1 Solution. Left to the reader.

9.4.95 Problem. Consider the series $\sum_{n=1}^{\infty} x_n$ with $x_n > 0 \ \forall\ n \in \mathbb{N}$ such that there exist $p > 0$ and $q > 1$ such that the sequence

$$\left(n^q\left(1 - \frac{x_{n+1}}{x_n} - \frac{p}{n}\right)\right)$$

is bounded. Show that

1. if $p \leq 1$, then $\sum_{n=1}^{\infty} x_n$ is divergent, and
2. if $p > 1$, then $\sum_{n=1}^{\infty} x_n$ is convergent.

This is known as **Raabe-Duhamel's rule.**

9.4.95.1 Solution. Our assumption implies the existence of $M > 0$ such that for any $n \geq 1$, we have

$$\left|1 - \frac{x_{n+1}}{x_n} - \frac{p}{n}\right| \leq \frac{M}{n^q}$$

So,

$$\left|n\left(1 - \frac{x_{n+1}}{x_n}\right) - p\right| \leq \frac{M}{n^{q-1}}.$$

Hence

$$\lim_{n\to\infty} n\left(1 - \frac{x_{n+1}}{x_n}\right) = p.$$

The previous problem implies that if $p < 1$, then $\sum_{n=1}^{\infty} x_n$ is divergent; and if $p > 1$, then $\sum_{n=1}^{\infty} x_n$ is convergent. So assume $p = 1$ and let us prove that $\sum_{n=1}^{\infty} x_n$ is divergent Let

$$\epsilon_n = n^q\left(\frac{x_{n+1}}{x_n} - 1 + \frac{1}{n}\right),$$

then

$$\frac{x_{n+1}}{x_n} = 1 - \frac{1}{n} + \frac{\epsilon_n}{n^q}.$$

Again, let $y_n = nx_n$ then

$$\frac{y_{n+1}}{y_n} = \frac{n+1}{n}.\frac{x_{n+1}}{x_n} = \left(1 + \frac{1}{n}\right)\left(1 - \frac{1}{n} + \frac{\epsilon_n}{n^q}\right)$$

which implies

$$\frac{y_{n+1}}{y_n} = 1 - \frac{1}{n^2} + \frac{\delta_n}{n^q}$$

where $\delta_n = \frac{(n+1)}{n}\epsilon_n$, since (δ_n) is bounded,

$$\lim_{n\to\infty} n^\alpha \ln\left(\frac{y_{n+1}}{y_n}\right) = 1$$

where $\alpha = \min(2, q)$. Thus the series

$$\sum_{n=1}^{\infty} \log\left(\frac{y_{n+1}}{y_n}\right)$$

is convergent by the limit test which implies that $\lim_{n\to\infty} \log y_n = l$ exists. Hence $\lim_{n\to\infty} y_n = e^l$. i.e.,

$$\lim_{n\to\infty} n x_n = e^l$$

The limit test again implies that $\sum_{n=1}^{\infty} x_n$ is divergent, since $\sum_{n=1}^{\infty} \frac{e^l}{n}$ is divergent. □

9.4.96 Problem. Discuss the convergence or divergence of $\sum_{n=1}^{\infty} x_n$ where

$$x_n = \frac{1.3.5...(2n-1)}{2.4.....(2n)}.$$

9.4.96.1 Solution. Let us first show that $\lim_{n\to\infty} x_n = 0$. Note that $0 < x_n < 1$ for any $n \geq 1$. Since

$$\frac{x_{n+1}}{x_n} = \frac{2n+1}{2n+2},$$

we have $x_{n+1} < x_n$ for any $n \geq 1$, which implies that $(x_n$ is decreasing. So $\lim_{l\to\infty} x_n$ exists. Define

$$y_n = \frac{2.4.6.....2n}{3.5...(2n+1)}$$

Since $4n^2 - 1 < 4n^2$, we get $\frac{2n-1}{2n} < \frac{2n}{2n+1}$ for any $n \geq 1$. This obviously implies $x_n < y_n$ for any $n \geq 1$. Since $x_n y_n = \frac{1}{2n+1}$, we deduce that $x_n^2 < x_n y_n = \frac{1}{2n+1}$ for any $n \geq 1$. Hence $\lim_{n\to\infty} x_n^2 = 0$ which implies $\lim_{n\to\infty} x_n = 0$. This conclusion may suggest that the series $\sum_{n=1}^{\infty} x_n$ is convergent. Since

$$\frac{x_{n+1}}{x_n} = \frac{2n+1}{2n+2},$$

we have

$$n\left(1 - \frac{x_{n+1}}{x_n}\right) = \frac{n}{2n+2}.$$

Hence $\lim_{n\to\infty} n\left(1 - \frac{x_{n+1}}{x_n}\right) = \frac{1}{2} < 1$, then $\sum_{n=1}^{\infty} x_n$ is divergent based on the previous problem. Note that the ratio test does not help since $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = 1$. In fact, the root test will also be not conclusive. □

9.4.97 Problem. If $\sum_{n=1}^{\infty} a_n$ converges, where $a_n > 0$ and $a_n \neq 1 \, \forall \, n$, does

$$\sum_{n=1}^{\infty} \frac{a_n}{1-a_n}$$

converge?

9.4.97.1 Solution. There exists N such that for all $n > N$, sufficiently large, $0 < a_n < 1/2$. Thus for $n > N, a_n/(1-a_n) < 2a_n$. So

$$0 < \sum_{n=1}^{\infty} \frac{a_n}{1-a_n} < \sum_{n=1}^{N} \frac{a_n}{1-a_n} + 2 \sum_{n=N+1}^{\infty} a_n,$$

and the series converges. □

9.4.98 Problem. Show elementary that the series

$$\sum_{n=1}^{\infty} \sin n$$

has bounded partial sums. This is in contrast with the series $\sum_{n=1}^{\infty} |\sin n|$, that cannot have bounded partial sums.

9.4.98.1 Solution. Hint.

$$\begin{aligned}
&(\sin 1 + \sin 2 + ... + \sin n)\left(\sin \frac{1}{2}\right) \\
&= \frac{1}{2}\left(\cos\frac{1}{2} - \cos\frac{3}{2} + \cos\frac{3}{2} - \cos\frac{5}{2} + ... + \cos\left(n - \frac{1}{2}\right) - \cos\left(n + \frac{1}{2}\right)\right) \\
&= \frac{1}{2}\left(\cos\frac{1}{2} - \cos\left(n + \frac{1}{2}\right)\right).
\end{aligned}$$

Thus

$$|\sin 1 + \sin 2 + ... + \sin n| \leq \frac{1}{\sin \frac{1}{2}}$$

for each $n \in \mathbb{N}$. The series $\sum_{n=1}^{\infty} |\sin n|$ cannot have bounded partial sums, as otherwise it would converge and thus $\sin n \to 0$ which is not true. □

9.4.99 Problem. Show that $\sum_{n=1}^{\infty} \frac{|\sin n|}{n}$, is divergent.

9.4.99.1 Solution. Hint. Assume it is convergent. Since $|\sin n| \geq \sin^2 n$ for each n, we would get that $\frac{1}{2} \sum_{n=1}^{\infty} \frac{1-\cos 2n}{n}$, is convergent. Since $\sum_{n=1}^{\infty} \frac{\cos 2n}{n}$ is convergent by the Dirichlet test, we would get that the sum of the last two series, i.e., $\sum \frac{1}{n}$ is convergent, a contradiction. □

9.4.100 Problem. There exist two divergent series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ of positive terms with $a_1 \geq a_2 \geq ...$ and $b_1 \geq b_2 \geq ...$ such that if $c_n = \min\{a_n, b_n\}$ then $\sum_{n=1}^{\infty} c_n$ converges.

9.4.100.1 Solution. Hint. Let

$$a_k = 1/2^k, b_k = 1/2^n, \text{ if } 2^n \leq k < 2^{n+1},\ n \text{ is even}$$
$$a_k = 1/2^n, b_k = 1/2^k, \text{ if } 2^n \leq k < 2^{n+1},\ n \text{ is odd.} \quad \square$$

9.4.101 Problem. (There is no convergent series with largest terms) Suppose that $a_n > 0$ and $\sum_{n=1}^{\infty} a_n$ converges. Define $r_n = \sum_{k=n}^{\infty} a_k$.

1. Prove that $\sum_{n=1}^{\infty} \frac{a_n}{r_n}$ diverges.
2. Prove that $\sum_{n=1}^{\infty} \frac{a_n}{\sqrt{r_n}}$ conerges.

9.4.101.1 Solution. Hint.

1. Show that
$$\frac{a_m}{r_m} + \frac{a_{m+1}}{r_{m+1}} + ... + \frac{a_n}{r_n} > 1 - \frac{r_n}{r_m}$$
for all $m < n$.
2. Show that
$$\frac{a_n}{\sqrt{r_n}} < 2\left(\sqrt{r_n} - \sqrt{r_{n+1}}\right),\ \forall\, n. \quad \square$$

9.4.102 Problem. Suppose (a_n) decreases monotonically to a limit of 0, and $\sum_{n=1}^{\infty} a_n$ diverges to infinity.

1. Does the series $a_1 + a_2 - a_3 + a_4 + a_5 - a_6 + a_7 + a_8 - a_9 + ...$ converge of diverge?
2. Does the series $a_1 + a_2 - a_3 - a_4 + a_5 + a_6 - a_7 - a_8 + a_9 + ...$ converge of diverge?
3. Generalize (1) and (2).

9.4.102.1 Solution.

1. We claim that the first series diverges. First, consider the grouping $a_1 + (a_2 - a_3) + a_4 + (a_5 - a_6) + a_7 + (a_8 - a_9) + ...$Since (a_n) is non-increasing, each of the terms in parentheses is either positive or zero, but never negative. Using this we can observe the following:
$$\begin{aligned} a_1 + (a_2 - a_3) + a_4 + &(a_5 - a_6) + a_7 + (a_8 - a_9) + ... \\ &\geq a_1 + a_4 + a_7 + a_{10} + ... \\ &= \frac{1}{3}(3a_1 + 3a_4 + 3a_7 + 3a_{10} + ..) \\ &\geq \frac{1}{3}(a_1 + a_2 + a_3 + a_4 + a_5 + ...) \end{aligned}$$
The last series is just $\frac{1}{3}\sum_{n=1}^{\infty} a_n$, which diverges, so the the original series must also diverge.
2. Consider the sequence $(x_n) = (1, 1, -1, -1, 1, 1, -1, -1, 1, 1, ...)$. The partial sums of this series are bounded by $M > 2$. Using the fact that (a_n) is non-increasing with limit equal to zero, we can apply Dirichlet's test to conclude that $\sum x_n a_n = a_1 + a_2 - a_3 + a_4 + a_5 - a_6 + a_7 + a_8 - a_9 + ...$ converges.

3. Any sequence of the form

$$(x_n) = \left(\overbrace{1,1,..,1}\{n_1 times\}\overbrace{-1,.,-1}\{n_1 times\}\overbrace{1,1,..,1}\{n_2 times\}\overbrace{-1,..,-1}\{n_2 times\}\right)$$

will be bounded by $M > \max\{n_k\}$, so that Dirichlet's test can be applied to $\sum x_n a_n$.. $\square$

9.4.103 Problem. Let $a_1 < a_2 < a_3 < ...$ be an increasing sequence of positive integers. Let the series

$$\sum_{n=1}^{\infty} \frac{1}{a_n}$$

be convergent. For any number x, let $k(x)$ be the number of the a_n's which do not exceed x. Show that $\lim_{x\to\infty} \frac{k(x)}{x} = 0$.

9.4.103.1 Solution. Suppose that, for some $\epsilon > 0$ there are $x_j \to \infty$ with $k(xj)/x_j \geq \epsilon$. Note that if $1 \leq n \leq k(x_j)$, then (because a_n increasing) $a_n \leq a_{k(x_j)} \leq x_j$ and $1/a_n \geq 1/x_j$. Now for any positive integer N,

$$\sum_{n=N}^{\infty} \frac{1}{a_n} \geq \sup_j \sum_{n=N}^{k(x_j)} \frac{1}{a_n} \geq \sup_j \frac{k(x_j) - N}{x_j} \geq \sup_j (\epsilon - N/x_j) = \epsilon.$$

But this contradicts the convergence of $\sum_{n=1}^{\infty} \frac{1}{a_n}$ which implies

$$\lim_{N\to\infty} \sum_{n=N}^{\infty} \frac{1}{a_n} = 0.$$

$\square$

9.4.104 Problem. If $\sum_{n=1}^{\infty} u_n^2$ and $\sum_{n=1}^{\infty} v_n^2$ are convergent series of real constants, prove that $\sum_{n=1}^{\infty} (u_n - v_n)^p$; $p \geq 2$, is convergent.

9.4.104.1 Solution. Let $A = \sum_{n=1}^{\infty} u_n^2$ and $B = \sum_{n=1}^{\infty} v_n^2$. Since

$$(u_i + v_i)^2 + (u_i - v_i)^2 = 2u_i^2 + 2v_i^2$$

we have, for any positive integer n,

$$\sum_{i=1}^{n} (u_i - v_i)^2 \leq 2\sum_{i=1}^{n} u_i^2 + 2\sum_{i=1}^{n} v_i^2 \leq 2A + 2B$$

Since the terms are all non-negative, it follows that $\sum_{i=1}^{\infty}(u_i - v_i)^2$ is convergent. Therefore, the term $(u_n - v_n)^2$ approaches zero, so there exists an integer k such that $(u_i - v_i)^2 < 1$ for all $i \geq k$. If p is an integer and $p \geq 2$, then

$$|u_i - v_i|^p \leq (u_i - v_i)^2, \ \forall\, i \geq k,$$

so the series

$$\sum_{n=1}^{\infty} (u_n - v_n)^p$$

is absolutely convergent, and therefore convergent. $\square$

9.4.105 Problem. Let $a_1, a_2, ...$ be positive numbers such that

$$\sum_{n=1}^{\infty} a_n < \infty.$$

Prove that there are positive numbers $c_1, c_2, ...$ such that

$$\sum_{n=1}^{\infty} c_n \to \infty \text{ and } \sum_{n=1}^{\infty} c_n a_n < \infty.$$

9.4.105.1 Solution. Let $\sum_{n=1}^{\infty} a_n$ converge to S. For $n \in \mathbb{N}$, let $r_n = \sum_{k=1}^{n} a_k$ and $r_0 = 0$, so we have $a_n = r_n - r_{n-1}$ $\forall\ n \in \mathbb{N}$. Consider the sequence (c_n) defined by $c_n = 1/(\sqrt{r_n} + \sqrt{r_{n-1}})$. As $r_n \to S$, we get $\sum_{n=1}^{\infty} c_n \to \infty$. We have

$$\begin{aligned}\sum_{n=1}^{\infty} c_n a_n &= \sum_{n=1}^{\infty} \frac{r_n - r_{n-1}}{\sqrt{r_n} + \sqrt{r_{n-1}}} \\ &= \sum_{n=1}^{\infty} \left(\sqrt{r_n} - \sqrt{r_{n-1}}\right) \\ &= \sqrt{S}. \quad \square\end{aligned}$$

9.4.106 Problem. Show that the alternating harmonic series

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - .. + \frac{1}{2n-1} - \frac{1}{2n} + \frac{1}{2n+1} - .....$$

converges according to following steps. □

9.4.106.1 Solution.

Step(1) Prove that (s_{2n}) is an increasing and (s_{2n-1}) is a decreasing sequence.
Step(2) $(s_{2n}) < (s_{2n-1})$.
Step(3) Conclude that (s_{2n}) converges to a real number S and (s_{2n-1}) converges to a real number T and $S \leq T$.
Step(4) Prove that $S = T$ to conclude that the series converges.

9.5 Additional Exercises on Chapter 9.

9.5.1 Exercise. Are the following statements true or false? If true, give a proof; if false, give a counterexample. The numbers a_n and b_n are positive.

1. If, for all $n > 1, a_{n+1}/a_n < 1$, then $\sum_{n=1}^{\infty} a_n$ is convergent.
2. If $\lim_{n\to\infty}(a_n - b_n) = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
3. If $\lim_{n\to\infty}(a_n/b_n) = 1$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
4. If $\sum_{n=1}^{\infty} a_n$ converges, then so does $\sum_{n=1}^{\infty} a_n^2$. converges.

5. If $\sum_{n=1}^{\infty} a_n^2$ converges, then so does $\sum_{n=1}^{\infty} a_n$. converges.

6. If $\lim_{n\to\infty}(a_{n+1}+a_{n+2}+...+a_{2n})=0$ converges, then $\sum_{n=1}^{\infty} a_n$ is convergent.

9.5.2 Exercise. If (a_n) be a monotone decreasing sequence of positive real numbers and $\lim a_n = 0$ prove that the following series are convergent.

1. $a_1 - \frac{1}{2}(a_1+a_2) + \frac{1}{3}(a_1+a_2+a_3) - .....$

2. $a_1 - \frac{1}{2}(a_1+a_3) + \frac{1}{3}(a_1+a_3+a_5) - ...$

3. $a_1 - \frac{1}{3}(a_1+a_3) + \frac{1}{5}(a_1+a_3+a_5) - ....$

9.5.3 Exercise.

1. (Abel) Show that, if (x_n) is a decreasing sequence of positive numbers such that $\sum_{n=1}^{\infty} x_n$ is convergent, then $\lim_n x_n = 0$.

2. Show by a counterexample that in the previous problem we cannot remove the condition that (x_n) be decreasing.

3. Show, however, that for any convergent series $\sum_{n=1}^{\infty} x_n$ of positive numbers, we have

$$\lim_{n\to\infty} n\,(x_1x_2..x_n)^{\frac{1}{n}} = 0.$$

9.5.4 Exercise. A sequence (x_n) of real numbers is said to be of **bounded variation** if the series

$$\sum_{n=2}^{\infty} |x_k - x_{k-1}|$$

converges.

1. Show that every sequence of bounded variation is convergent.

2. Show that not every convergent sequence is of bounded variation.

3. Show that all monotonic convergent sequences are of bounded variation.

4. Show that any linear combination of two sequences of bounded variation is of bounded variation.

5. Is the product of of two sequences of bounded variation also of bounded variation?

9.5.5 Exercise.

1. Let (a_n) be a sequence of positive numbers and write

$$L_n = \frac{\log\left(\frac{1}{a_n}\right)}{\log n}$$

Show that if $\liminf L_n > 1$, then $\sum_{n=1}^{\infty} a_n$ converges. Show that if $L_n < 1$ for all sufficiently large n, then $\sum_{n=1}^{\infty} a_n$ diverges.

2. Apply the test in (1) to obtain convergence or divergence of the following series (x is positive):

(a) $\sum_{n=2}^{\infty} x^{\log n}$,

(b) $\sum_{n=2}^{\infty} x^{\log\log n}$,

(c) $\sum_{n=2}^{\infty} (\log n)^{-\log n}$.

9.5.6 Exercise. Show that $\sum_{n=1}^{\infty}\left(\frac{a}{n}+\frac{b}{n+1}-\frac{c}{n+2}\right)$ converges if and only if $c=a+b$.

9.5.7 Exercise. Prove that the series

$$\left(\frac{1}{2}\right)^p+\left(\frac{1.3}{2.4}\right)^p+\left(\frac{1.3.5}{2.4.6}\right)^p+....$$

is convergent for $p>2$ and divergent for $p\leq 2$.

9.5.8 Exercise. Determine whether the series $\sum_{n=1}^{\infty} a_n$ is convergent or divergent, where a_n is

1. $\frac{\sqrt{n}}{n+1}$.
2. $\frac{(n+1)^n}{n^{n+1}}$.
3. $\frac{1}{n^\alpha(\log)^\beta}$ $\alpha<0, \beta\in\mathbb{R}$.
4. $a^{(\log n)^\alpha}, a>0, \alpha>0$.
5. $\frac{(-1)^n\log n}{n}$.
6. $\frac{(-1)^n}{n+1}\left(1+\frac{1}{2}+..+\frac{1}{n}\right)$.

9.5.9 Exercise. Why is the following arguement incorrect?

$$\begin{aligned}\sum_{n=1}^{\infty} n(a_n-a_{n+1}) &= 1(a_1-a_2)+2(a_2-a_3)+3(a_3-a_4)+...\\ &= a_1-a_2+2a_2-2a_3+3a_3-3a_4+....\\ &= a_1+a_2+a_3+....=\sum_{n=1}^{\infty} a_n.\end{aligned}$$

Prove that if either of the two series $\sum_{n=1}^{\infty} n(a_n-a_{n+1}), \sum_{n=1}^{\infty} a_n$ is convergent, and $na_n\to 0$ as $n\to\infty$, then both series are convergent, and their sums are equal.

9.5.10 Exercise. Decide whether the following statements are true or false, giving a general proof for a true statement and a counter-example for a false tatement.

1. If $\sum_{n=1}^{\infty} a_n$ is a convergent series of positive terms, then $na_n\to 0$ as $n\to\infty$.
2. If (a_n) is a decreasing sequence of positive numbers such that $\sum_{n=1}^{\infty} a_n$ is convergent, then $\sum_{n=1}^{\infty} na_n^2$ is convergent.
3. If $(a_n),(b_n)$ are sequences of positive numbers such that $\sum_{n=1}^{\infty}(-1)^n a_n$ is convergent and that $a_n\geq b_n$ for each n, then $\sum_{n=1}^{\infty}(-1)^n b_n$ is convergent.

4. If $\sum_{n=1}^{\infty} a_n$ is a convergent series of non-negative terms, then $\sum_{n=1}^{\infty} a_n^k$ is convergent for all $k \geq 1$.

5. If $\sum_{n=1}^{\infty} a_n$ is convergent, then there exists $k_0 > 0$ such that $\sum_{n=1}^{\infty} |a_n|^k$ is convergent for all $k \geq k_0$.

6. If (a_n) is a sequence of positive numbers such that a_n tends to a finite non-zero limit l as $n \to \infty$, then, for each positive integer p, $\frac{a_{n+p}}{a_n} \to l$ as $n \to \infty$.

7. If (a_n) is a sequence of positive numbers such that for each positive integer p, $\frac{a_{n+p}}{a_n} \to l$ as $n \to \infty$ then a_n tends to a finite non-zero limit l or to $\to \infty$ as $n \to \infty$.

9.5.11 Exercise. Prove by induction that for all integers $n \geq 1$

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ... + \frac{1}{2n-1} - \frac{1}{2n} = \frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{2n}$$

Use Euler's limit $\lim_n \gamma_n = \gamma$, where $\gamma_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} - \ln n$, to find

$$\lim_{n\to\infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{2n} \right),$$

and hence find the sum of the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ...$

9.5.12 Exercise. Find

1. $\lim_{n\to\infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} ... + \frac{1}{5n} \right)$,

2. $\lim_{n\to\infty} \left(\frac{1}{n+1} + \frac{1}{n+3} + \frac{1}{n+5} ... + \frac{1}{5n-1} \right)$,

3. $\lim_{n\to\infty} \left(\frac{1}{1(n-1)} + \frac{1}{2(n-2)} + \frac{1}{3(n-3)} + ... + \frac{1}{(n-1)1} \right)$.

9.5.13 Exercise. Prove that if $0 < \alpha < 1$, then

$$\sum_{r=1}^{n} \frac{1}{r^\alpha} = \frac{n^{1-\alpha}}{1-\alpha} - \beta + \mu_n$$

where β is a fixed number such that $\alpha/(1-\alpha) \leq \beta \leq 1/(1-\alpha)$, and $\mu_n \to 0$ as $n \to \infty$. Hence show that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^\alpha} = \beta \left(2^{1-\alpha} - 1\right).$$

9.5.14 Exercise. Prove that if p is a non-negative integer and $\alpha > 1$, then

$$\sum_{r=p+1}^{\infty} \frac{1}{r^\alpha} - \frac{1}{(\alpha-1)(p+1)^{\alpha-1}} \leq \frac{1}{(p+1)^\alpha}.$$

9.5.15 Exercise. If $a_n > -1, (n \geq 1)$ and both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} a_n^2$ are convergent, then $\sum_{n=1}^{\infty} \log(1 + a_n)$ is convergent.

9.5.16 Exercise. Prove that if $\sum_{n=1}^{\infty} a_n$ is a convergent series of non-negative terms, then $\sum_{n=1}^{\infty} n^{-\alpha} a_n^{1/2}$ is convergent for $\alpha > \frac{1}{2}$. Give an example of a convergent series non-negative terms for which $\sum_{n=1}^{\infty} n^{-\frac{1}{2}} a_n^{\frac{1}{2}}$ is divergent.

9.5.17 Exercise. Show that the series

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k+|x|}$$

converges $\forall\, x \in \mathbb{R}$ and the sum is a Lipschitz function.

9.5.18 Exercise. Using the Cauchy-Schwarz inequality, show that if (a_n) is a sequence of nonnegative numbers for which $\sum_{n=1}^{\infty} a_n$ converges, then the series

$$\sum_{n=0}^{\infty} \frac{\sqrt{a_n}}{n^p}$$

also converges for any $p > 1/2$. Without the Cauchy-Schwarz inequality what is the best you can prove for convergence?

9.5.19 Exercise. Suppose that $\sum_{n=1}^{\infty} a_n^2$ converges. Show that

$$\limsup_{n\to\infty} \frac{a_1 + \sqrt{2}a_2 + \sqrt{3}a_3 + \sqrt{4}a_4 + \ldots + \sqrt{n}a_n}{n} < \infty.$$

9.5.20 Exercise. Let $x_1, x_2, .., x_n..$ be a sequence of positive numbers and write

$$s_n = \frac{x_1 + x_2 + x_3 + \ldots + x_n}{n} \text{ and}$$
$$t_n = \frac{\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \ldots + \frac{1}{x_n}}{n}.$$

If $s_n \to S$ and $t_n \to T$, show that $ST \geq 1$.

9.5.21 Exercise. Prove that if $a_1 + a_2 + \ldots$ converges to s, then a $a_2 + a_3 + \ldots$ converges to $s - a_1$.

9.5.22 Exercise. For what values of x does the series $(1-x) + (x - x^2) + (x^2 - x^3) + \ldots$ converge?

9.5.23 Exercise. Prove that the series $(a_1 - a_2) + (a_2 - a_3) + (a_3 - a_4) + \ldots$ converges if and only if the sequence (a_n) converges.

9.5.24 Exercise. Prove that for any $a, b \in \mathbb{R}$ the series $a + (a+b) + (a+2b) + (a+3b) + \ldots$ diverges unless $a = b = 0$.

9.5.25 Exercise. Prove that if $\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty}(a_n + b_n)$ diverges.

9.5.26 Exercise. Let $\sum_{n=1}^{\infty} a_n$ be a convergent series. Let (n_k) be any subsequence of the sequence of positive integers. Finally, let

$$\begin{aligned} b_1 &= a_1 + a_2 + \ldots.. + a_{n_1} \\ b_2 &= a_{n_1+1} + a_{n_1+2} + \ldots.. + a_{n_2} \\ &\vdots \\ b_k &= a_{n_{k-1}+1} + a_{n_{k-1}+2} + \ldots.. + a_{n_k},\ k \in \mathbb{N}. \end{aligned}$$

Prove that $\sum_{k=1}^{\infty} b_k$ converges and has the same sum as $\sum_{n=1}^{\infty} a_n$.

9.5.27 Exercise. Verify that the preceding exercise yields the following important result. If $\sum_{n=1}^{\infty} a_n$ converges, then any series formed from $\sum_{n=1}^{\infty} a_n$ by inserting parentheses [for example, $(a_1 + a_2) + (a_3 + .. + a_7) + (....)...$] converges to the same sum.

9.5.28 Exercise. Give an example of a series $\sum_{n=1}^{\infty} a_n$ such that $(a_1+a_2)+(a_3+a_4)+....$ converges but $a_1 + a_2 + ... + a_{n_1}$ diverges. (This shows that removing parentheses may cause difficulties.)

9.5.29 Exercise. A product $\prod_{n=1}^{\infty}(1 + a_k)$ is said to converge absolutely if the related product $\prod_{n=1}^{\infty}(1 + |a_k|)$ converges.

1. Show that an absolutely convergent product is convergent.
2. Show that an infinite product $\prod_{n=1}^{\infty}(1+a_k)$ converges absolutely if and only if the series of its terms $\sum_{n=1}^{\infty} a_n$ converges absolutely
3. For what values of x does the product $\prod_{n=1}^{\infty}\left(1 + \frac{x}{k}\right)$ converge absolutely?
4. For what values of x does the product $\prod_{n=1}^{\infty}\left(1 + \frac{x}{k^2}\right)$ converge absolutely?
5. For what values of x does the product $\prod_{n=1}^{\infty}\left(1 + x^k\right)$ converge absolutely?
6. Show that $\prod_{n=1}^{\infty}\left(1 + \frac{(-1)^k}{k}\right)$ converges but not absolutely.

9.5.30 Exercise. Let $0 < a \leq b$ and consider the series

$$a + ab + a^2b + a^2b^2 + a^3b^2 + a^3b^3 + ...$$

Show that if $a \geq 1$ then the series diverges while if $b < 1$ the series converges. In general when does the series converge?

9.5.31 Exercise. Prove that the series

$$\frac{a}{b} + \frac{a(a+c)}{b(b+c)} + \frac{a(a+c)(a+2c)}{b(b+c)(b+2c)} + ...a, b, c > 0$$

is convergent if $b > a + c$ and divergent if $b \leq a + c$.

9.5.32 Exercise. Prove that the series

$$1 + \frac{a^2}{1.b} + \frac{a^2(a+1)^2}{1.2.b(b+1)} + \frac{a^2(a+1)^2(a+2)^2}{1.2.3b(b+1)(b+2)} + ...a, b > 0$$

is convergent if $b > 2a$ and divergent if $b \leq 2a$.

9.5.33 Exercise.

1. $\tan\frac{\pi}{4} + \tan\frac{\pi}{8} + \tan\frac{\pi}{12} + ...$ (Use $x < \tan x$ for $0 < x < \frac{\pi}{2}$).
2. $\left(\frac{1}{2}\right)^{\log 1} + \left(\frac{1}{2}\right)^{\log 2} + \left(\frac{1}{2}\right)^{\log 3} + ...$
3. $\left(\frac{1}{3}\right) + \left(\frac{1}{3}\right)^{1+\frac{1}{2}} + \left(\frac{1}{2}\right)^{1+\frac{1}{2}+\frac{1}{3}} + ...$
4. $\left(\frac{1}{4}\right) + \left(\frac{1}{4}\right)^{1+\frac{1}{3}} + \left(\frac{1}{3}\right)^{1+\frac{1}{3}+\frac{1}{5}} + ...$ For 2, 3, 4 Use logarithmic test.

9.5.34 Exercise. Show that the series $\frac{1}{(1+a)^p} - \frac{1}{(2+a)^p} + \frac{1}{(3+a)^p} - ..., a > 0$ is

1. absolutely convergent if $p > 1$,
2. conditionally convergent if $0 < p \leq 1$.

9.5.35 Exercise. Prove that $\frac{\pi}{8} = \frac{1}{1.3} + \frac{1}{5.7} + \frac{1}{9.11} + ...$ (Hint: Use Gregory's series $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - ...$ for $-1 \leq x \leq 1$.)

9.5.36 Exercise. If the sequence (x_n) is monotonically decreasing to zero then the series

$$x_1 - \frac{1}{2}(x_1 + x_2) + \frac{1}{3}(x_1 + x_2 + x_3) - \frac{1}{4}(x_1 + x_2 + x_3 + x_4) + ...$$

converges.

9.5.37 Exercise. Suppose that the series of positive terms $\sum_{n=1}^{\infty} a_n$ diverges. What, if anything, can be concluded about the following series:

1. $\sum_{n=1}^{\infty} \dfrac{a_n}{n}$,
2. $\sum_{n=1}^{\infty} a_n^2$,
3. $\sum_{n=1}^{\infty} \sqrt{a_n}$,
4. $\sum_{n=1}^{\infty} \dfrac{a_n}{a_n + 1}$.

9.5.38 Exercise. Let $a_n \in \mathbb{R}$, such that $\sum_{n=1}^{\infty} |a_n| = \infty$ and $s_m = \sum_{n=1}^{m} a_n \to a \in \mathbb{R}$ as $m \to \infty$. Let $a^+ = \max\{a_n, 0\}$. Show that $\sum_{n=1}^{\infty} a_n^+ = \infty$.

9.5.39 Exercise. Suppose $f(x)$ can be developed in the series $\sum_{n=1}^{\infty} a_n x^n$ on $(-1, 1)$. Show by example that the following statement is false: if $\lim_{x \to 1-} f(x) = A$, then the numerical series $\sum_{n=1}^{\infty} a_n$ converges to A. (Hint: consider $f(x) = \frac{1}{1+x}$.)

9.5.40 Exercise. Show that the root, ratio, Raabe and Bertrand tests fail for the series $\sum_{n=1}^{\infty} 2^{(-1)^n - n}$. However, the simple application of the Comparison test gives the result.

9.5.41 Exercise. Show that the convergence of the series $\sum_{n=2}^{\infty} \frac{1}{(n-\sqrt{n}) \log^2 n}$ cannot be established by the root, ratio and Raabe tests, but it can be proved by the Integral or Bertrand tests.

9.5.42 Exercise. Show that the divergence of the series $\sum_{n=1}^{\infty} \frac{2n!!}{(2n+1)!!}$ cannot be established by the root, ratio and Integral tests, but it can be proved by the Raabe's test.

9.5.43 Exercise. Provide a counterexample to the following statement: if both positive series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are divergent, then the series $\sum_{n=1}^{\infty} \min(a_n, b_n)$ also diverges.

9.5.44 Exercise. Give an example of a divergent series $\sum_{n=1}^{\infty} a_n$ such that $\sum_{n=1}^{\infty} \sqrt{a_n a_{n+1}}$ is convergent.

9.5.45 Exercise. Let $a_n \in \mathbb{R}$, such that $\sum_{n=1}^{\infty} |a_n| = \infty$ and $\sum_{n=1}^{m} a_n \to a \in \mathbb{R}$ as $m \to \infty$. Let $a_n^+ = \max\{a_n, 0\}$. Show that $\sum_{n=1}^{\infty} a_n^+ = \infty$.

9.5.46 Exercise. Discuss the convergence of the series

$$\sum_{n=2}^{\infty} \frac{1}{n \log^p n}$$

depending on $p \geq 0$. These series are known as the **Bertrand series**.

9.5.47 Exercise. Test for the convergence of the series

$$\frac{2}{1^3} - \frac{3}{2^3} + \frac{4}{3^3} - \frac{5}{4^3} + ...; \ \frac{2}{1^2} - \frac{3}{2^2} + \frac{4}{3^2} - \frac{5}{4^2} + ...;$$

9.5.48 Exercise. Show that

$$1 + 2x + 3^2 \frac{x^2}{2!} + 4^3 \frac{x^3}{3!} + 5^4 \frac{x^4}{4!} + ...$$

converges if $x < \frac{1}{e}$.

9.5.49 Exercise. Test for the convergence of the series

$$\frac{a}{b} + \frac{a(a+c)}{b(b+c)} x + \frac{a(a+c)(a+2c)}{b(b+c)(b+2c)} x^2 + ...$$

9.5.50 Exercise. Prove that, when $|x| < 1$,

$$-\frac{\ln(1-x)}{1-x} = x + \left(1 + \frac{1}{2}\right) x^2 + \left(1 + \frac{1}{2} + \frac{1}{3}\right) x^3 + ...$$

9.5.51 Exercise. Prove that the series

$$\left\{ \left(1 + \frac{1}{2}\right)^2 - \left(1 + \frac{1}{3}\right)^2 \right\} + \left\{ \left(1 + \frac{1}{4}\right)^2 - \left(1 + \frac{1}{5}\right)^2 \right\} + ..$$
$$... + \left\{ \left(1 + \frac{1}{2n}\right)^2 - \left(1 + \frac{1}{2n+1}\right)^2 \right\} + ...$$

converges, but that the series obtained by removing brakets oscillates.

9.5.52 Exercise. Prove that if $a_1 + a_2 + a_3 + ...$ is absolutely convergent, then $a_1 + a_2 + a_3 ... = (a_1 + a_4 + a_5 + ...) + (a_2 + a_4 + a_6 + ...)$. Is this true for all conditionally convergent series?

9.5.53 Exercise. What, if anything, is wrong with the following?

$$\begin{aligned}
&1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + ... \\
&= 1 + \left(\frac{1}{2} - 1\right) + \frac{1}{3} + \left(\frac{1}{4} - \frac{1}{2}\right) + \frac{1}{5} + \left(\frac{1}{6} - \frac{1}{3}\right) + ... \\
&= \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + ...\right) - \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + ...\right) = 0.
\end{aligned}$$

9.5.54 Exercise. Show that any conditionally convergent series has a rearrangement that diverges.

9.5.55 Exercise. Show that there exists a rearrangement $\sum_{n=1}^{\infty} a_n$ of $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + ...$ such that, if $t_n = a_1 + a_2 + a_3 + ... + a_n$, then

$$\limsup_{n\to\infty} t_n = 100, \ \liminf_{n\to\infty} t_n = -100.$$

9.5.56 Exercise.

1. Does the ratio test give any information about the series

$$\left(\frac{1}{2}\right)^0 + \left(\frac{1}{4}\right)^1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{4}\right)^3 + \left(\frac{1}{2}\right)^4 + ...?$$

2. Does the series converge?

9.5.57 Exercise. Let $S = \{n_1, n_2, ...\}$ denote the collection of those positive integers that do not involve the digit 0 in their decimal representation. (For example, $7 \in S$ but $101 \notin S$.) Show that $\sum_{k=1}^{\infty} 1/n_k$ converges and has a sum less than 90.

9.5.58 Exercise. Given integers $a_1, a_2, ...$ such that $1 \leq a_n \leq n-1, n = 2, 3, ...$ Show that the sum of the series $\sum_{k=1}^{\infty} \frac{a_n}{n!}$ is rational if, and only if, there exists an integer N such that $a_n = n-1 \; \forall \; n > N$. Hint. For sufficiency, show that $\sum_{n=2}^{\infty}(n-1)/n!$ is a telescoping series with sum 1.

9.5.59 Exercise. Let p and q be fixed integers, $p \geq q \geq 1$, and let

$$x_n = \sum_{k=qn+1}^{pn} \frac{1}{k}, \ y_n = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}.$$

(a) Prove that $\lim_{n\to\infty} x_n = \log \frac{p}{q}$.

(b) When $q = 1, p = 2$, show that $s_{2n} = \sum_{i=n+1}^{2n} x_i = x_n$ and deduce that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \log 2.$$

(c) Rearrange the series in (b), writing alternately p positive terms followed by q negative terms and use (a) to show that this rearrangement has sum $\log 2 + 1/2 \log(p/q)$.

(d) Find the sum of

$$\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{1}{3n-2} - \frac{1}{3n-1}\right).$$

(e) Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be convergent series of non-negative terms. Show that $\sum_{n=1}^{\infty} (a_n b_n)^{1/2}$ is convergent. Give an example to show that the converse implication is false.

(f) Give an example of two divergent series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ of positive terms with the property that $\sum_{n=1}^{\infty} \min\{a_n, b_n\}$ is convergent.

(g) Use the ratio test to show that, for each fixed k and each a such that $0 < a < 1$ the series $\sum_{n=1}^{\infty} n^k a^n$ is convergent. Deduce that $\lim_{n\to\infty} n^k a^n = 0$.

9.5.60 Exercise. Test for convergence (p and q denote fixed real numbers).

1. $\sum_{n=1}^{\infty} n^3 e^{-n}$.
2. $\sum_{n=1}^{\infty} p^n n^p$.
3. $\sum_{n=1}^{\infty} n^{-1-1/n}$.
4. $\sum_{n=1}^{\infty} \frac{1}{n^p - n^q}$.
5. $\sum_{n=3}^{\infty} \left(\frac{1}{\log\log n}\right)^{\log\log n}$.
6. $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{n^2+2}$.
7. $\sum_{n=1}^{\infty} (\sqrt{n+1} - \sqrt{n})$.

9.5.61 Exercise. Consider the alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ...$$

with sum S, and denote its sum to n terms by S_n. Consider also the rearranged series

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + ...$$

and denote its sum to n terms by T_n. For each $n \geq 1$, let

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}.$$

Show that

1. $S_{2n} = H_{2n} - H_n \;\forall\, n \geq 1$.
2. $T_{3n} = H_{4n} - \frac{1}{2}H_{2n} - H_n = S_{4n} + \frac{1}{2}S_{2n}$.

9.5.62 Exercise. Suppose that S is the sum of a convergent series $\sum_{n=1}^{\infty} a_n$. Define $t_n = a_n + a_{n+1} + a_{n+2}$. Which of the following is true?
The series $\sum_{n=1}^{\infty} t_n$

1. diverges.
2. converges to $3S - a_1 - a_2$
3. converges to $3S - a_1 - 2a_2$
4. converges to $3S - 2a_1 - a_2$.

9.5.63 Exercise. Let (a_n) be a sequence of positive real numbers such that

$$a_1 = 1,\; a_{n+1}^2 - 2a_{n+1}a_n - a_n = 0 \;\forall\, n \geq 1.$$

Which of the following is true? The sum of the series $\sum_{n=1}^{\infty} \frac{a_n}{3^n}$ lies in the interval

1. $(1, 2]$
2. $(2, 3]$
3. $(3, 4]$
4. $(4, 5]$.

9.5.64 Exercise. Which one of the following series is divergent?

1. $\sum_{n=1}^{\infty} \frac{1}{n} \sin^2 \frac{1}{n}$
2. $\sum_{n=1}^{\infty} \frac{1}{n^2} \sin \frac{1}{n}$
3. $\sum_{n=1}^{\infty} \frac{1}{n} \tan \frac{1}{n}$
4. $\sum_{n=1}^{\infty} \frac{1}{n} \log n$.

9.5.65 Exercise. Let (a_n) be a sequence of positive real numbers. Which of the following is true? The sum of the series $\sum_{n=1}^{\infty} a_n$ converges if

1. $\sum_{n=1}^{\infty} a_n^2$ converges
2. $\sum_{n=1}^{\infty} \frac{a_{n+1}}{a_n}$ converges
3. $\sum_{n=1}^{\infty} \frac{a_n}{2^n}$ converges
4. $\sum_{n=1}^{\infty} \frac{a_n}{a_{n+1}}$ converges.

9.5.66 Exercise. Let $\phi : \mathbb{N} \to \mathbb{N}$ be a bijective map such that

$$\sum_{n=1}^{\infty} \frac{\phi(n)}{n^2} < \infty.$$

Which of the following is true? The number of such bijective maps is

1. exactly one.
2. finite but more than one.
3. zero.
4. infinite.

Chapter 10

Special Topics

The only way to learn mathematics is to do mathematics.
–Paul R. halmos: A Hilbert Space Problem Book.

10.1 Measure zero

10.1.1 Definition (Sets of measure zero)**.** A set S of real numbers is said to have **measure zero** if, for every $\epsilon > 0$, there is a sequence (I_n) of open intervals, (the length of (I_n) is $\ell(I_n)$), such that

$$S \subseteq \bigcup_{n=1}^{\infty} I_n \text{ and } \sum_{n=1}^{\infty} \ell(I_n) < \epsilon.$$

A property is said to hold **almost everywhere (a.e.)** on $[a, b]$ if it holds at each point of $[a, b] \setminus S$, where S is a set of messure zero.

10.1.1 Problem. Any finite subset of $\mathbb{R}$ is of measuere zero.

10.1.1.1 Solution. Let $F = \{x_1, x_2, ..., x_n\}$ be a finite set, and let $\epsilon > 0$, then

$$\left\{ \left(x_i - \frac{\epsilon}{2(n+1)}, x_i + \frac{\epsilon}{2(n+1)} \right) ;\ i = 1, 2, .., n \right\}$$

is a family of open intervals the sum of whose lengths is $\frac{n}{n+1}\epsilon < \epsilon$. Thus F is of measuere zero. $\square$

10.1.2 Problem. The set $\mathbb{Q}$ is of measure zero.

10.1.2.1 Solution. Let $\mathbb{Q} = \{x_1, x_2, ..., x_n, ...\}$ be an enumeration of $\mathbb{Q}$ and let $\epsilon > 0$. Then the family

$$\left\{ \left(x_i - \frac{\epsilon}{2^{i+2}}, x_i + \frac{\epsilon}{2^{i+2}} \right) ;\ i = 1, 2, .., n... \right\}$$

of open intervals is a covering of $\mathbb{Q}$ and the sum of whose lengths is

$$\sum_{i=1}^{\infty} \frac{\epsilon}{2^{i+1}} = \frac{\epsilon}{2} < \epsilon.$$

Thus the set $\mathbb{Q}$ is of measure zero. However, there exists an uncountable set (Cantor set) which is of measure zero . $\square$

10.2 Notion of nearness, Nets

A real number $x \in \mathbb{R}$ is said to be **ϵ-near to** $a \in D \subseteq \mathbb{R}$ iff $|x-a| < \epsilon$ for $\epsilon > 0$ and $x \in \mathbb{R}$ is said to be **arbitrary-near** to a iff $|x-a| < \epsilon$ for all $\epsilon > 0$, Now one can think it is possible only when a is a limit point of some set. If a is not a limit point, then a is only point near to a.
In the begining we consider the closed interval $I = [a, b]$ and $c \in I$, we define a nearness relation on $I_c = I \setminus \{c\}$ in terms of "near to c" by writing $x_1 \geq x_2$ to mean x_1 is near to c than x_2 is in I_c iff $|x_1 - c| \leq |x_2 - c|$. We shall show that $x_1 \geq x_2$ and $x_2 \geq x_3$ imply $x_1 \geq x_3$
For $x_1 \geq x_2 \Rightarrow |x_1 - c| \leq |x_2 - c|$ and $x_2 \geq x_3 \Rightarrow |x_2 - c| \leq |x_3 - c|$ both imply $|x_1 - c| \leq |x_3 - c| \Rightarrow x_1 \geq x_3$. Again, if $x_1, x_2 \in I_c$, then $\exists\ x_3 \in I_c$ such that $|x_3 - c| \leq \min\{|x_2 - c|, |x_1 - c|\}$. i.e. $x_3 \geq x_1$ and $x_3 \geq x_2$.
Now, we observe that this relation on I_c satisfies

1. $x \geq x, \forall\ x \in I_c$,
2. $x \geq y$ and $y \geq z \Rightarrow x \geq z$
3. if $x, y \in I_c$, then there exists $z \in I_c$ such that $z \geq x, z \geq y$. i.e. z is ϵ-near to c than x, y are in I_c.

Thus, we get the following:

10.2.1 Definition. Let $D(\neq \emptyset)$ be a set and "$\succ$" be a relation on D. The relation $\succ$ is called a **direction** on D or is said to **direct** D, if the relation $\succ$ is

1. **Reflexive**: $x \succ x, \forall\ x \in D$,
2. **Transitive**: $x \succ y$ and $y \succ z \Rightarrow x \succ z, x, y, z \in D$,
3. **Endless**: if $x, y \in D$, then there exists $z \in D$ such that $z \succ x, z \succ y$.

We call $(D, \succ)$ is a **directed** set. If $a \succ b$ we say that a **dominates** b. If $a \succ b$, we sometimes write $b \prec a$.

10.2.2 Example.

1. The set $\mathbb{N}$ of all natural numbers as well as the set $\mathbb{R}$ of all real numbers are directed by $\succ$ in the usual sense, i.e. $x \succ y$ iff $x \geq y$.
2. Let D be the family of all open intervals of $\mathbb{R}$ containing 0, then for $A, B \in D$ define $A \succ B$ iff $A \subseteq B$. Note that if $A, B \in D$, then $A \cap B \subseteq A$ and $A \cap B \subseteq B \Rightarrow C = A \cap B \succ A$ and $C \succ B$.
3. Let E be any nonempty set, let X be the set of finite subsets of E, and define $A \succ B$ if and only if $A \supseteq B$.

10.2.3 Exercise. In each case, show that each relation "$\succ$" is a direction:

1. Let $a \leq c \leq b$ and $D = \{x; a \leq x \leq b; x \neq c\}$. For $x, y \in D$, define $x \succ y$ iff $|x - c| < |y - c|$.
2. Let $I = [a, b]$ by a partition P of I we mean a finite set $P = \{x_0, x_1, .., x_n\}$ such that $a = x_0 < x_1 < .. < x_n = b$. Let D be the set of all partitions of I. For $P, Q \in D$ define $P \succ Q$ iff $Q \subseteq P$, we read P is finer than Q.

3. Let D be the set of all real valued functions on $\mathbb{R}$, define $f \succ g$ iff $f(x) \geq g(x)$, $\forall\, x \in \mathbb{R}$ and for $f, g \in D$.

4. Let $x \in \mathbb{R}$ and $D = \mathcal{U}_x$ (the family of all nbhds of x). For $U, V \in D$ define $U \succ V$ iff $U \subseteq V$.

5. Every linearly ordered set (such as the $\mathbb{N}$ with the usual order) is a directed set.

6. Any collection of sets that is closed under the operation (intersection) is a directed set when ordered by reverse inclusion, i.e. $X \succ Y$ if and only if $Y \subseteq X$.

7. If D and E are directed sets, then their product $D \times E$ is directed by $(d_1, e_1) \succ (d_2, e_2)$ if and only if $d_1 \succ d_2$ in D and $e_1 \succ e_2$ in E.

10.2.4 Definition. Let $(D, \succ)$ be a directed set. A **net** in a set X is a function from S to X. We shall adopt the sequence notation for nets and index by a member of D. Write $(S_n)_{n \in D}$ or simply (S_n) for the net $S : D \to X$.

10.2.5 Definition. A sequence in X is any net $S : \mathbb{N} \to X$.

10.2.6 Definition. Let D be a given directed set and consider an arbitrary subset $A \subseteq D$.

1. The set A is called a **residual subset** of D, iff there exists an element $m \in D$ such that, for each $n \in D$, $n \succ m$ implies $n \in A$.

2. The set A is called a **cofinal subset** of D, iff for every $n \in D$ there exists an element $m \in A$ such that $m \succ n$.

10.2.7 Definition. Let D be a given directed set and consider an arbitrary subset $A \subseteq D$.

1. A net is **in** a set A iff $S_n \in A\ \forall\, n \in D$.

2. A net is **eventually** or **ultimately** in a set A iff $\exists$ a residual subset $E \subseteq D$ such that $S(E) \subseteq A$, i.e $\exists\, m \in D$ such that $n \succ m \Rightarrow S_n \in A$.

3. A net is **frequently** in a set A iff $\exists$ a cofinal subset $E \subseteq D$ such that $S(E) \subseteq A$, i.e for each $n \in D$, $\exists\, m \in E$ such that $m \succ n \Rightarrow S_m \in A$.

10.2.8 Exercise. Show that a cofinal subset of a directed set is also a directed set.

10.2.9 Exercise. The following are equivalent:

1. A net (S_n) is frequently in A.

2. S maps a cofinal subset of D into A.

3. The net (S_n) is not eventually in the complement of A.

10.3 Notion of convergence

10.3.1 Definition. A net (S_n) converges to l iff it is eventually in every nbhd. $U(l)$ of l, i.e. iff $\exists\, m \in D$ such that $\forall\, n \succ m \Rightarrow S_n \in U(l)$ and this situation is described by $(S_n) \to l$ or $S_n \to l$ or $\lim_n S_n = l$.

The familiar limit of calculus "$\lim_{x\to a} f(x) = l$" is an instance of a net convergence. Here D consists of points x near to a and $x \succ y$ means that x is near to a than y is near to a, i.e. $0 < |x-a| \le |y-a|$.
Let $D \subseteq \mathbb{R}$ and $a \in D'$ (the derived set of D), and a function $f\colon D \to \mathbb{R}$. Suppose $l \in \mathbb{R}$. If it is possible that $f(x)$ can be made arbitrary near l while x is sufficiently near a, then we say that f has a limit l when x approaches to a. Note that the point a may or may not belong to D. Now, we show that $\lim_{x\to a} f(x) = l$ is a net convergence.
Let $c \in D'$ and consider the set $D_c = D \setminus \{c\}$. Then for $x, y \in D_c$, define $x \succ y$ iff $|x-c| \le |y-c|$. We show that $\succ$ is a direction on D_c.
It is clear that $x \ge x$. Let $x \succ y$ and $y \succ z$, then $|x-c| \le |y-c|$ and $|y-c| \le |z-c| \Rightarrow |x-c| \le |z-c| \Rightarrow x \succ z$.
Let $x, y \in D_c$ and $\min\{|x-c|, |y-c|\} = \delta$. Since $c \in D'$, so $(c-\delta, c+\delta) \cap D_c \ne \emptyset$. Suppose that $z \in (c-\delta, c+\delta)$ then $|z-c| < \delta$, thus $|z-c| \le |x-c|$ and $|z-c| \le |y-c|$ i.e. $z \succ x$ and $z \succ y$ shows that "$\succ$" is a direction on D_c.

10.3.2 Theorem. Let $c \in D'$. Then $\lim_{x\to c} f(x) = l \Leftrightarrow$ The net $f\colon D_c \to \mathbb{R}$ converges to l.

Proof. $\Leftarrow$ We write the net $(f_x)_{x\in D_c}$ simply as (f_x) where $f_x = f(x)$. Suppose that (f_x) converges to l. Then (f_x) is eventually in every nbhd. of l, which means that $\forall\ \epsilon > 0\ \exists\ y \in D_c$ such that $x \succ y \Rightarrow f_x \in (l-\epsilon, l+\epsilon)$. Now $x \succ y$ means $|x-c| \le |y-c|$ and if $|y-c| < \delta$ then statement becomes $\forall\ \epsilon > 0\ \exists\ \delta > 0$ such that $|f(x)-l| = |f_x - l| < \epsilon$ whenever $|x-c| < \delta$ which is same as $\lim_{x\to c} f(x) = l$.
$\Rightarrow$ Suppose that $\lim_{x\to c} f(x) = l$. Let $U(l)$ be any nbhd. of l then $\exists\ \epsilon > 0$ such that $(l-\epsilon, l+\epsilon) \subseteq U(l)$, and $\exists\ \delta > 0$ such that $y \in (c-\delta, c+\delta) \cap D_c \Rightarrow f(y) \in (l-\epsilon, l+\epsilon)$. Now if $x \succ y$ then $|x-c| \le |y-c| < \delta \Rightarrow f_x = f(x) \in (l-\epsilon, l+\epsilon) \subseteq U(l)$. Hence f is eventually in every nbhd. $U(l)$ of l. The proof is complete. $\square$

10.3.1 Problem. Describe D and the direction '$\succ$' so that these limits are limits of nets.

1. $\lim_{x\to\infty} f(x) = l$.
2. $\lim_{x\to\infty} f(x) = -\infty$.
3. $\lim_{x\to-\infty} f(x) = l$.
4. $\lim_{x\to a+} f(x) = l$.
5. $\lim_{x\to a} f(x) = \infty$.
6. $\sum_{i=1}^{\infty} a_i = a$.

10.3.1.1 Solution.

1. Let D be any subset of $\mathbb{R}$ that is not bounded above. Define a relation $\succ$ on D by $x \succ y$ iff $x \ge y$. Now, let $x, y \in D$, then $\exists\ z \ge \max\{x, y\}$ implies $z \succ x$ and $z \succ y$. Thus $\succ$ is a direction on D.
   We show that $f_x = f(x), x \in D$ is eventually in every nbhd. of l. Let $U(l)$ be a nbhd. of l, then $\exists\ \epsilon > 0$ such that $(l-\epsilon, l+\epsilon) \subseteq U(l)$, and then $\exists\ M > 0$ such that $x_1 > M \Rightarrow f(x_1) \in (l-\epsilon, l+\epsilon)$. Let $x \succ x_1$, then $x \in (x_1, \infty) \subseteq (M, \infty) \Rightarrow f_x = f(x) \in (l-\epsilon, l+\epsilon) \subseteq U(l)$. Thus f_x is eventually in $U(l)$.

2. Left to the reader.

3. Let D be any subset of $\mathbb{R}$ that is not bounded below. Define a relation $\succ$ on D by $x \succ y$ iff $x \leq y$. Now, let $x, y \in D$, then $\exists\, z \leq \min\{x, y\}$ implies $z \succ x$ and $z \succ y$. Thus $\succ$ is a direction on D. Rest is left to the reader.

4. Let D be any subset of $\mathbb{R}$ such that $(a, a+\epsilon) \cap D \neq \emptyset$. Define a relation $\succ$ on D by $x \succ y$ iff $|x-a| \leq |y-a|$. Now, let $x, y \in D$, then $\exists\, z \leq \min\{|x-a|, |y-a|\}$ implies $z \succ x$ and $z \succ y$. Thus $\succ$ is a direction on D. Rest is left to the reader.

5. Left to the reader.

6. The infinite series $\sum_{n=1}^{\infty} a_n$ of real numbers a_n converges to a if and only if given any $\epsilon > 0$ there is an n_0 such that if $n \geq n_0$ then

$$|(a_1 + \ldots + a_n) - a| < \epsilon.$$
$$i.e.\ s_n \in (a-\epsilon, a+\epsilon).$$

This is the same as saying that the sequence $s_1, s_2, \ldots$ of partial sums given by $s_n = a_1 + \ldots + a_n$ converges to a.
Let $D = \mathbb{N}$, and define $\succ$ on $\mathbb{N}$ by $m \succ n$ iff $m \geq n$. Thus $n \succ n_0 \Rightarrow s_n \in (a-\epsilon, a+\epsilon)$. i.e. s_n is eventually in $(a-\epsilon, a+\epsilon)$. □

10.3.3 Exercise. Let (x_α) and (y_α) be real valued nets on the same directed set $(D, \geq)$ with $\lim_\alpha x_\alpha = l$ and $\lim_\alpha y_\alpha = m$, then

1. $\lim_\alpha (x_\alpha + y_\alpha) = l + m$.

2. $\lim_\alpha (x_\alpha . y_\alpha) = lm$.

3. $\lim_\alpha \left(\frac{x_\alpha}{y_\alpha}\right) = \frac{l}{m}$ if $m \neq 0, y_\alpha \neq 0\ \forall\, \alpha \in D$.

10.3.4 Exercise. Let D be the set of all pairs (m, n) of positive integers. Partially order D as follows: $(m, n) \succ (m', n')$ if and only if $m + n \geq m' + n'$.

1. Describe geometrically what $(m, n) \succ (m', n')$ means.

2. Show that if $\lim_{(m,n)} x_{mn} = l$, then $\lim_{n\to\infty} x_{mn} = l\ \forall\, m$, and $\lim_{m\to\infty} x_{mn} = l\ \forall\, n$.

3. Let $x_{mn} = mn/m^2 + n^2$. Show $\lim_{m\to\infty} x_{mn} = 0\ \forall\ n$ and $\lim_{n\to\infty} x_{mn} = 0\ \forall\ m$, but $\lim_{(m,n)} x_{mn}$, fails to exist.

10.3.5 Exercise. Let D be the set of all pairs (m, n) of positive integers. Partially order D as follows: $(m, n) \succ (m', n')$ if and only if $\max\{m, n\} \geq \max\{m', n'\}$.

1. Describe geometrically the set of (m, n) such that $(m, n) \succ (m_0, n_0)$ for a fixed (m_0, n_0).

2. Does $\lim_{(m,n)} x_{mn} = l$, imply $\lim_{n\to\infty} x_{mn} = l\ \forall\, m$?

10.3.6 Exercise. Let D be as above with the ordering $(m, n) > (m', n')$ iff $mn \geq m'n'$. Give examples of nets (x_{mn}) which converge and nets which diverge. What is the connection, if any, between convergence in the ordering of D and the limits $\lim_{m\to\infty} x_{mn}$ and $\lim_{n\to\infty} x_{mn}$?

10.4 Functions of several variables

By a function of several vaiables we generally mean $f : \mathbb{R}^n \to \mathbb{R}^m$. Here, we mainly study the functions from $\mathbb{R}^2 \to \mathbb{R}$. That is with $m = 1, n = 2$.

10.4.1 Functions from $\mathbb{R}^2$ to $\mathbb{R}$. (Limits and Continuity)

Let $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$, i.e. f be a function of two independent variables x, y, then limits of a kind different from those already discussed for the functions from $\mathbb{R} \to \mathbb{R}$, come into consideration. The variables may approach a limiting position either by varying seperately or simultaneously. If (a, b) is a limit point of D, then we have the limits

$$\lim_{x\to a}\lim_{y\to b} f(x, y),\ \lim_{y\to b}\lim_{x\to a} f(x, y),\ \lim_{(x,y)\to(a,b)} f(x, y),$$

the first two types are called **iterated limits** or **repeated imits** and the last as a **simultaneous limit** or **double limit** of $f(x, y)$ as (x, y) approaches to (a, b).
An iterated limit presents nothing essentially new, as it is merely a limit of a limit. However, in double limits we have something quite different from the ordinary limit.

10.4.1 Definition. Let $D \subseteq \mathbb{R}^2$ and $(a, b) \in D'$, then a function $f : D \to \mathbb{R}$ is said to have a **limit** l at (a, b) iff for every $\epsilon > 0$, $\exists\ \delta > 0$ such that $f(\hat{B}((a, b); \delta) \cap D) \subseteq B(l; \epsilon)$. In other words, $f(x, y)$ tends to a limit l as (x, y) tends to (a, b), or $f(x, y)$ converges to l as (x, y) tends to (a, b) iff for each nbhd. V of l we can find a deleted nbhd. $\hat{U} = U \setminus \{(a, b)\}$ of (a, b) such that $f(\hat{U} \cap D) \subseteq V$.

The above is equivalent to the statement

$$\forall\ \epsilon > 0\ \exists\ \delta > 0 \text{ such that } (x, y) \in \hat{U} \cap D \Rightarrow f(x, y) \in B(l; \epsilon).$$

We write the above, symbolically, as $\lim_{(x,y)\to(a,b)} f(x, y) = l$, read as $f(x, y)$ tends to a limit l as (x, y) tends to (a, b), Which is equivalent to the statement

$$\begin{aligned}&\forall\ \epsilon > 0\ \exists\ \delta > 0 \text{ such that } (x, y) \in D \text{ and}\\ &0 < |x - a| < \delta,\ 0 < |y - b| < \delta \Rightarrow |f(x, y) - l| < \epsilon.\end{aligned}$$

The existence of a double limit has a wider range of consequences than is the case with single limits. For example, not only we have the same limiting value if the the variable point (x, y) approaches the limit point (a, b) through any set of values whose limit point is (a, b), but we also have the same limiting value as the variable point approaches its its limting position along any curve whatsoever. Consequently, if we can find two methods of approach to the limit point which give different limiting values, then the double limit does not exist.

10.4.2 Example. Consider the limit $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$. If (x, y) approaches the origin along the line $y = x$, we have,

$$\lim_{x\to 0} \frac{x^2}{x^2 + x^2} = \frac{1}{2}.$$

If, however, we first allow x to approach 0 along a line parallel to the x-axis and then allow y to approach 0 along the y-axis, we have

$$\lim_{y\to 0}\lim_{x\to 0} \frac{xy}{x^2 + y^2} = \lim_{y\to 0} \frac{0}{0 + y^2} = 0.$$

Hence the different results obtained by these two methods of approach to the limit point show that the double limit does not exist.

If a double limit does exist, it follows that the limiting value obtained by assuming any convenient relation between the variables and allowing them to approach the limit point along the corresponding curve gives the value of the limit. However, the converse is not true. For example, the same limiting value of the function may be obtained as the variables approach their limiting values along all straight lines passing through the limit point and yet double limit need not exist.

10.4.3 Example. Consider the limit $\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6}$. If (x, y) approaches the origin along the lines $y = mx$, we have,

$$\lim_{x\to 0} \frac{m^3x^4}{x^2 + m^6x^6} = \lim_{x\to 0} \frac{m^3x^2}{1 + m^6x^4} = 0.$$

However, if (x, y) approaches the origin along the curves $x = my^3, m \neq 0$, then we have,

$$\lim_{y\to 0} \frac{my^6}{m^2y^6 + y^6} = \lim_{y\to 0} \frac{m}{m^2 + 1} = \frac{m}{m^2 + 1}.$$

The double limit, therefore, does not exist; for, that requires that we should obtain same limiting value by all possible method of approach to the limit point.

If the double limit $\lim_{(x,y)\to(a,b)} f(x, y)$ exists, then the single limits $\lim_{x\to a} f(x, b)$, and $\lim_{y\to b} f(a, y)$ exist. It does not follow, however, that the single limits $\lim_{x\to a} f(x, y)$, and $\lim_{y\to b} f(x, y)$ exist for $y \neq b, x \neq a$ respectively.

10.4.4 Example. Consider the function $f : \mathbb{R}^2 \setminus \{(0, 0)\} \to \mathbb{R}$ by $f(x, y) = y \sin \frac{1}{x}$, now for the double limit $\lim_{(x,y)\to(0,0)} y \sin \frac{1}{x}$. We have,

$$\lim_{(x,y)\to(0,0)} y \sin \frac{1}{x} = 0;$$

for, we have

$$\left| y \sin \frac{1}{x} - 0 \right| = \left| y \sin \frac{1}{x} \right| \leq |y| < \epsilon \text{ whenever } 0 < |x| < \epsilon = \delta, 0 < |y| < \epsilon = \delta.$$

However, for any constant value of $y \neq 0$, say $y = c$, we have

$$\lim_{x\to 0} f(x, c) = \lim_{x\to 0} c \sin \frac{1}{x} = c \lim_{x\to 0} \sin \frac{1}{x}$$

does not exist.

Now, we state a result concerning the double limit and iterated limits.

10.4.5 Theorem. If $\lim_{(x,y)\to(a,b)} f(x, y) = A$ and $\lim_{x\to a} f(x, y)$ exists for each constant value of y in the nbhd. of b and likewise $\lim_{y\to b} f(x, y)$ exists for each constant value of x in the nbhd. of a, then

$$A = \lim_{x\to a} \lim_{y\to b} f(x, y) = \lim_{y\to b} \lim_{x\to a} f(x, y) = \lim_{(x,y)\to(a,b)} f(x, y).$$

Proof. By hypothesis the limit $\lim_{x\to a} f(x,y)$ exists for each value of y in the nbhd. of b. The set of limiting values defines a function $F(a,y)$ of y. So, we can write $\lim_{x\to a} f(x,y) = F(a,y)$, where $F(a,y)$ may or may not be identical with $f(a,y)$. That is, for each value of y in a nbhd. $(b-\delta_0, b+\delta_0)$ and for $\epsilon > 0 \ \exists\ \delta_1 > 0$ such that

$$|F(a,y) - f(x,y)| < \epsilon/2 \text{ whenever } |x-a| < \delta_1.$$

Since the double limit exists at (a,b) we have there exists $\delta_2 > 0$ such that

$$|f(x,y) - A| < \epsilon/2 \text{ whenever } |x-a| < \delta_2, |y-b| < \delta_2.$$

Thus

$$|F(a,y) - A| = |F(a,y) - f(x,y)| + |f(x,y) - A| < \epsilon/2 + \epsilon/2 = \epsilon$$

whenever $|y-b| < \delta = \min\{\delta_1, \delta_2\}$. Hence

$$\lim_{y\to b} F(a,y) = A \Rightarrow \lim_{y\to b}\lim_{x\to a} f(x,y) = A.$$

In the same way, we can show that

$$\lim_{x\to a}\lim_{y\to b} f(x,y) = A.$$

and hence we have finally

$$A = \lim_{x\to a}\lim_{y\to b} f(x,y) = \lim_{y\to b}\lim_{x\to a} f(x,y) = \lim_{(x,y)\to(a,b)} f(x,y).$$

□

This theorem states a sufficient condition but not a necessary condition for the interchange the order of iterated limits, but there are cases where the order may be interchanged yet the conditions of the theorem are not satisfied.

10.4.6 Example. Consider the function $f : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ by $f(x,y) = \frac{xy}{x^2+y^2}$, we see that the double limit $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ does not exist and we have

$$\begin{aligned}\lim_{x\to 0}\lim_{y\to 0} \frac{xy}{x^2+y^2} &= \lim_{x\to 0} \frac{x0}{x^2+0} = 0\\ \lim_{y\to 0}\lim_{x\to 0} \frac{xy}{x^2+y^2} &= \lim_{y\to 0} \frac{0y}{0+y^2} = 0,\end{aligned}$$

hence the order of the limits may be interchanged.

Now, we state a necessary and sufficient condition for the interchange of the order of limits.

10.4.7 Theorem. A necessary and sufficient condition that

$$\lim_{x\to a}\lim_{y\to b} f(x,y) = \lim_{y\to b}\lim_{x\to a} f(x,y) \quad \text{is}$$

(a) $\lim_{x\to a} f(x,y) = \phi(y), y \neq b;\quad \lim_{y\to b} f(x,y) = \psi(x), x \neq a.$

(b) $\lim_{y\to b}\phi(y) = A$,

(c) $\exists$ a sequence (y_n) converging to b and $\exists\ m \in \mathbb{N}$ such that for each $y_n, n > m$, there exists $\delta_{y_n} > 0$ and if $\epsilon > 0$, then for every $x \in (a - \delta_{y_n}, a + \delta_{y_n})$ on the line $y = y_n$, we have

$$|f(x, y_n) - \psi(x)| < \epsilon.$$

Proof. Necessary: From the assumed relation,

$$\lim_{x\to a}\lim_{y\to b} f(x,y) = \lim_{y\to b}\lim_{x\to a} f(x,y) = A,$$

the conditions (a) and (b) follow at once. We show that the condition (c) is also satisfied. We have

$$\lim_{x\to a}\lim_{y\to b} f(x,y) = \lim_{x\to a}\psi(x) = A,$$

that is for arrbitrary $\epsilon > 0\ \exists\ \delta_1 > 0$ such that

$$|x - a| < \delta_1 \Rightarrow |\psi(x) - A| < \epsilon/3. \tag{1}$$

We also have $\lim_{y\to b}\lim_{x\to a} f(x,y) = \lim_{y\to b}\phi(y) = A$, and hence for any sequence (y_n) converging to $b\ \exists\ \delta_2 > 0$ such that

$$|y_n - b| < \delta_2 \Rightarrow |A - \phi(y_n)| < \epsilon/3. \tag{2}$$

Moreover, we have $\lim_{x\to a} f(x, y_n) = \phi(y_n)$, hence $\exists\ \delta_3 > 0$ such that

$$|x - a| < \delta_3 \Rightarrow |\phi(y_n) - f(x, y_n)| < \epsilon/3. \tag{3}$$

Let $\delta_{y_n} = \min\{\delta_1, \delta_3\}$. Hence, we have

$$|\psi(x) - f(x, y_n)| \le |\psi(x) - A| + |A - \phi(y_n)| + |\phi(y_n) - f(x, y_n)| < \epsilon$$

whenever $|x - a| < \delta_{y_n}$. Thus (c) is satisfied.

Sufficient: Let $\epsilon > 0$ and (y_n) be any sequence converging to b. Since the first of the conditions (a) holds, so $\lim_{x\to a} f(x, y_n) = \phi(y_n)$ and $\exists\ \delta'_{y_n} > 0$ such that

$$0 < |x - a| < \delta'_{y_n} \Rightarrow |f(x, y_n) - \phi(y_n)| < \epsilon/3. \tag{4}$$

Since (b) holds, then $\exists\ m \in \mathbb{N}$ such that

$$n > m \Rightarrow |\phi(y_n) - A| < \epsilon/3. \tag{5}$$

Now, for each y_n we have from (c)

$$0 < |x - a| < \delta_{y_n} \Rightarrow |f(x, y_n) - \psi(x)| < \epsilon/3. \tag{6}$$

Hence, if $\delta = \min\{\delta_{y_n}, \delta'_{y_n}\}$, so combining (4), (5) and (6) we have. $0 < |x - a| < \delta$ implies

$$\begin{aligned}|\psi(x) - A| =& |\psi(x) - f(x, y_n) + f(x, y_n) - \phi(y_n) + \phi(y_n) - A|\\ \le& |\psi(x) - f(x, y_n)| + |f(x, y_n) - \phi(y_n)| + |\phi(y_n) - A|\\ <& \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.\end{aligned}$$

Thus $\lim_{x\to a}\psi(x) = A$. Hence $\lim_{x\to a}[\lim_{y\to b} f(x,y)] = \lim_{x\to a}\psi(x) = A$, which is equivalent to saying that the order of the two iterated limits may be interchanged. $\square$

In order that $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ shall be continuous in both variables, it must have the same limiting value by all possible approaches to the limit point. A necessary and sufficient condition for continuity therefore involves the condition that the function is not only continuous in each direction but the continuity is uniform for all directions. For, if we put $x = x_0 + r\cos\theta, y = y_0 + r\sin\theta$, we have from the definition of continuity

$$|f(x_0 + r\cos\theta, y_0 + r\sin\theta) - f(x_0, y_0)| < \epsilon$$

holds for all values of r less some number r_0 which is independent of θ. This is equivalent to saying that the transformed function must be uniformly continuous in r for all values of $|\theta| \leq 2\pi$.

10.4.8 Example. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

Let $\epsilon > 0$. Putting $x = r\cos\theta, y = r\sin\theta$, we obtain that

$$f(r\cos\theta, r\sin\theta) = \frac{r^2\cos\theta\sin\theta}{r\sqrt{\sin^2\theta + \cos^2\theta}} = \frac{1}{2}r\sin 2\theta < \epsilon,$$

if we take $r < 2\epsilon\csc\theta$. Since $\csc 2\theta$ is never less than 1, it follows that if we put $r_0 = 2\epsilon$, then however small ϵ may be chosen $f(r\cos\theta, r\sin\theta)$ is always less than ϵ for all values of θ and for $r_0 = 2\epsilon$. The transformed function is uniformly continuous in r for all values of θ.

It must not be inferred that $\lim_{r\to 0} f(x_0 + r\cos\theta, y_0 + r\sin\theta) = f(x_0, y_0)$ for each value of θ implies f is continuous. For example, consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} \frac{xy^3}{x^2+y^6} & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

Here, we see that the double limit does not exist at the origin. This function is, therefore, discontinuous at the origin. However, we have after introducing polar coordinates

$$\lim_{r\to 0} f(r\cos\theta, r\sin\theta) = \lim_{r\to 0} \frac{r^4\cos\theta\sin^3\theta}{r^2\cos^2\theta + r^6\sin^6\theta} = \lim_{r\to 0} r^2 \frac{\cos\theta\sin^3\theta}{\cos^2\theta + r^4\sin^6\theta} = 0,$$

for each constant value of θ.

Again, continuity of f with respect to both variables at a given point (x_0, y_0) implies the continuity at this point with respect to x and to y. It does not imply, however, that f is continuous in each variable seperately in the deleted nbhd. of this point. For example, let

$$f(x,y) = \begin{cases} x\sin\frac{1}{y} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,y) \text{ or } (x,y) = (x,0). \end{cases}$$

Then f is continuous at the origin. However, it is not continuous at $(x_0, 0)$. Since the limit

$$\lim_{y\to 0} f(x_0, y) = \lim_{y\to 0} x_0 \sin\frac{1}{y}, (x_0 \neq 0)$$

does not exist.
If f is known to be merely continuous in x alone and in y alone, then questions arise which involve the relations between the continuity in the two variables and that in each variable seperately. Hence the following theorem results.

10.4.9 Theorem. Let $D = \{(x, y); a \leq x \leq b, c \leq y \leq d\}$ and $f : D \to \mathbb{R}$ be continuous with respect to x and to y. At every point $(x_0, y_0) \in D$, we have

$$\lim_{x \to x_0} \lim_{y \to y_0} f(x, y) = \lim_{y \to y_0} \lim_{x \to x_0} f(x, y) = f(x_0, y_0).$$

Proof. Left to the reader. □

It is often necessary to examine the nature of the limiting function along the boundary of the region where the function is defined. Suppose, for example, that f is continuous in x in the region $D = \{(x, y); a \leq x \leq b, y_0 < y \leq d\}$. For points on the boundary line $y = y_0$, define the function by $\lim_{y \to y_0} f(x, y) = f(x, y_0)$. We shall examine the nature of convergence of $f(x, y)$ to the limiting function $f(x, y_0)$. Thus this problem is a generalization of the corresponding problem which arises in the convergence of an infinite series or a sequence of functions.
Let $x_1 \in [a, b]$. We have then $\lim_{y \to y_0} f(x_1, y) = f(x_1, y_0)$, that is, for $\epsilon > 0\ \exists\ \delta(x_1) > 0$ such that

$$|y - y_0| < \delta(x_1) \Rightarrow |f(x_1, y) - f(x_1, y_0)| < \epsilon. \tag{A}$$

If we keep the value of ϵ fixed while the point x is allowed to change, the value of δ may vary with x. For example, for $x = x_2$ the value of $\delta(x_2)$ may be quite different from $\delta(x_1)$. Then define

$$\delta = \inf\{\delta(x); x \in [a, b],\ |y - y_0| < \delta(x) \Rightarrow |f(x, y) - f(x, y_0)| < \epsilon.\}$$

If $\delta \neq 0$ then $f(x, y)$ is said to **converge uniformly** to the limiting function $f(x, y_0)$. This kind of convergence is quite different from the convergence of $f(x, y)$ for each value of $x \in [a, b]$. The latter means that for any $\epsilon > 0$ there exists $\delta(x)\ \forall\ x \in [a, b]$ which may change change with the point x. On the other hand, uniform convergence means that for every $\epsilon > 0\ \exists\ \delta > 0$ for which (A) holds uniformly $\forall\ x \in [a, b]$, that is, δ is independent of x.
Thus the curves $z = f(x, y_n), y_n \neq 0$ on the planes $y = y_n$ are called the approximation curves of the limiting function $f(x, y_0)$. In the case of uniform convergence, these approximation curves become in the limiting curve $z = f(x, y_0)$, as $y_n \to y_0$. The approximation curves can be conveniently exhibited by projecting them upon zx plane.If the given function converges uniformly, then all the projections of the approximation curves from a certain point on must lie within an ϵ-strip on either side of the curve $z = f(x, y_0)$.

10.4.10 Example. Given $f(x, y) = \frac{xy}{x^2+y^2}; 0 \leq x \leq 1, 0 \leq y \leq 1$, where $f(x, 0) = \lim_{y \to 0} f(x, y)$. Wrte down the approximation curves and show that $f(x, y)$ does not converge uniformly to $f(x, 0)$. We have the value of $f(x, 0)$ is 0 for all values of x, and hence the curve $z = f(x, 0) = 0; y = 0$ is the x-axis itself. For $y = y_n = 1/n$, we have the approximation curves are given by $z = \frac{nx}{1+n^2x^2}$. Now, one can verify easily that for any n, each approximation curve has a peak of height $\frac{1}{2}$ for some value of x. As n increases the peak in the curve goes sharper and approaches the origin, but neither vanishes nor its height changes. Hence, if we take a strip of width ϵ along the x-axis, the approximation curves never lie wholly within it. Consequently, the given function does not converge uniformly to $f(x, 0)$, although the given function is everywhere continuous in x and in y, and we have the following theorem.

10.4.11 Theorem. Let $f(x,y)$ be continuous in $x \in [a,b]$ for $y \neq y_0$ and defined for $y = y_0$ by $f(x,y_0) = \lim_{y\to y_0} f(x,y)$. A necessary and sufficient condition that $f(x,y)$ converges uniformly to $f(x,y_0)$ is that the double limit $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exists at each point $x_0 \in [a,b]$.

Proof. Necessary: Let $f(x,y)$ converge uniformly and $x_0 \in [a,b]$. Now, by the definition of $f(x,y_0)$ we have, for $\epsilon > 0 \; \exists \; \delta > 0$ such that

$$|y - y_0| < \delta \Rightarrow |f(x_0,y) - f(x_0,y_0)| < \epsilon. \tag{B}$$

Because of the continuity of $f(x,y)$ in x, we have on the line $y = y_0+\delta$ an interval $(x_0-\eta, x_0+\eta)$ where η depends on δ, such that $|f(x,y_0+\delta) - f(x_0,y_0+\delta)| < \epsilon/3$. By virtue of the uniform convergence of $f(x,y)$ we can choose $\delta > 0$ independent of x_0 so that for all values $x \in (x_0-\eta, x_0+\eta)$ we have

$$|f(x,y) - f(x,y_0+\delta)| < \epsilon/3.$$

The inequality (B) holds for $y = y_0 + \delta$ and hence we may write

$$|f(x_0,y_0+\delta) - f(x_0,y_0)| < \epsilon/3.$$

Thus

$$\begin{aligned}|f(x,y) - f(x_0,y_0)| &\leq |f(x,y) - f(x,y_0+\delta)| + |f(x,y_0+\delta) - f(x_0,y_0+\delta)| \\ &\quad + |f(x_0,y_0+\delta) - f(x_0,y_0)| < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon,\end{aligned}$$

whenever $|y - y_0| < \delta, |x - x_0| < \eta$. Hence given $\epsilon > 0$ there is a region bounded by the equations $y = y_0, y = y_0 + \delta, x = x \pm \eta$. Hence, we have

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0).$$

Sufficient: Suppose that the double limit $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exists at each point $x_0 \in [a,b]$. We then have about each point (x_0,y_0) of the closed interval $[a,b]$ a region defined by $D = \{(x,y); |x - x_0| < \delta, \; |y - y_0| < \delta\}$ such that for each $(x,y) \in D$, we have for arbitrary $\epsilon > 0$

$$|f(x,y) - f(x_0,y_0)| < \epsilon. \tag{C}$$

Note that, if x_0 is the end point of the given intterval $[a,b]$ then the only that portion of the above region is to be considered which corresponds to values of $x \in [a,b]$. Keeping ϵ fixed, then the value of δ depends on the choice of x_0 and to each value $x \in [a,b], \; \delta(x)$ is defined, Let $\delta_1(x) = \inf\{\delta(x); x \in [a,b]\}$. We claim that $\delta_1 > 0$. If $\delta(x) = 0$ then there must be a point $x' \in [a,b]$ such that in every nbhd, of which the infimum is 0. For $x = x'$, however, the value of δ must be different from 0, since the inequality (C) must hold for $x = x', y = y_0$, Put $\delta(x') = \delta'$. Then in the interval $(x' - \delta'/2, x' + \delta'/2)$ the value of $\delta(x)$ for each point is at least equal to $\delta'/2$. But, as δ' is different from 0, the infimum $\delta(x)$ in this subinterval must be different from 0, which is contrary to our assumption. Thus the condition is sufficient. □

10.4.12 Example. To determine the points of discontinuity of $f(x,y_0)$ in the interval $(0,1)$, where $y_0 = 0$ and where, for $x \neq 0, y \neq 0$, we have

$$f(x,y) = \frac{\left(1 + \sin\frac{\pi}{x}\right)^{\frac{1}{y}} - 1}{\left(1 + \sin\frac{\pi}{x}\right)^{\frac{1}{y}} + 1}.$$

We have

$$f(x,0) = \lim_{y\to 0} \frac{\left(1+\sin\frac{\pi}{x}\right)^{\frac{1}{y}} - 1}{\left(1+\sin\frac{\pi}{x}\right)^{\frac{1}{y}} + 1}, \; x \in (0,1].$$

This limit has the value 0 for $x = 1, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{n}, ..$ However, for $\frac{1}{2n+1} < x < \frac{1}{2n}$, we have $0 < \sin\frac{\pi}{x} \leq 1$ and we get

$$\begin{aligned} f(x,0) &= \lim_{y\to 0} \frac{\left(1+\sin\frac{\pi}{x}\right)^{\frac{1}{y}} - 1}{\left(1+\sin\frac{\pi}{x}\right)^{\frac{1}{y}} + 1} \\ &= \lim_{y\to 0} \frac{1 - \frac{1}{\left(1+\sin\frac{\pi}{x}\right)^{\frac{1}{y}}}}{1 + \frac{1}{\left(1+\sin\frac{\pi}{x}\right)^{\frac{1}{y}}}} = 1. \end{aligned}$$

On the other hand for $\frac{1}{2n} < x < \frac{1}{2n-1}$, we have $0 > \sin\frac{\pi}{x} \geq -1$ and hence

$$f(x,0) = \lim_{y\to 0} \frac{\left(1+\sin\frac{\pi}{x}\right)^{\frac{1}{y}} - 1}{\left(1+\sin\frac{\pi}{x}\right)^{\frac{1}{y}} + 1} = -1.$$

Consequently the function $f(x,0)$ is discontinuous at the points $x = 1, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{n}, ..$

10.5 Differentials

Suppose that f is a function for which $f'(x)$ exists on some interval S. Let $\Delta x \neq 0$ be a number such that $x + \Delta x$ belongs to the nbhd. in the domain of f, and therefore the point $(x + \Delta x, f(x + \Delta x))$ is on the graph of f (figure 10.1).
From the definition of derivative

$$f'(x) = \lim_{\Delta x\to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

and the definition of a limit, it follows that the difference

$$\left| \frac{f(x+\Delta x) - f(x)}{\Delta x} - f'(x) \right|$$

can be made as small as desired for sufficiently small $\Delta x \neq 0$. We may write

$$\frac{f(x+\Delta x) - f(x)}{\Delta x} - f'(x) = k \tag{10.1}$$

where $\lim_{\Delta x\to 0} k = 0$. Now the equation (10.2) can be written as

$$f(x+\Delta x) - f(x) = f'(x)\Delta x + k\Delta x. \tag{10.2}$$

The expression $f'(x)\Delta x$ is of special interest to us. Let $S \subseteq \mathbb{R}$. If $f : S \to \mathbb{R}$ be a function such that f' exists on S and Δx be any non-zero real number, then the **differential** of $f(x)$ with respect to x is equal to $f'(x)$ multiplied by Δx. We denote this differential by $d_x f(x)$, so

$$d_x f(x) = f'(x)\Delta x. \tag{10.3}$$

If $y = f(x)$ then (10.4) becomes

$$d_x y = D_x y \Delta x. \tag{10.4}$$

Where the derivative of $f(x)$ with respect to x is denoted by $D_x f(x) = f'(x)$ i.e.

$$D_x f(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

Now for the identity function $I(x) = x$, i.e. $y = x$, we get

$$D_x x = 1.\Delta x = \Delta x \tag{10.5}$$

Hence (10.5) can be written as

$$d_x f(x) = f'(x) d_x x. \tag{10.6}$$

Hence (10.7) can be written as

$$d_x y = D_x d_x x. \tag{10.7}$$

Suppose that

$$y = U(x),\ x = V(t).$$

Then $y = U(V(t)) = f(t)$ where $f = U \circ V$. We assume that U, V are differentiable. By definition of a differential

$$d_t y = D_t y d_t t$$

and by the chain rule $D_t y = D_x y D_t x$, so

$$d_t y = D_x y D_t x d_t t.$$

In addition, by the definition of differential $d_t x = D_t x d_t t$. Therefore

$$d_t y = D_x y d_t x. \tag{10.8}$$

Let us compare the results

$$d_x y = D_x y d_x x \text{ and } d_t y = D_x y d_t x.$$

This comparison indicates that we can write

$$d_\square y = D_x y d_\square x, \tag{10.9}$$

with the same variable (indepenent or not) to be placed in each of the boxes. For this reason the formula

$$d_x y = D_x y d_x x$$

is usually written in the form

$$dy = D_x y dx. \tag{10.10}$$

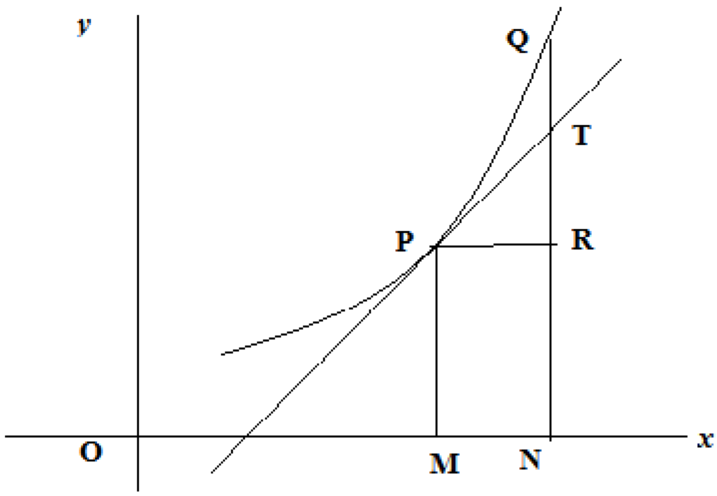

Figure 10.1:

We read it as "the differential of y is equal to the derivative of y with respect to x multiplied by the differential of x," where it is understood that dy and dx are the differentials with respect to the same variable. Similarly, (10.11) can be written as

$$df(x) = f'(x)dx. \tag{10.11}$$

As a consequence of the equality (10.12) we may write

$$D_x y = \frac{dy}{dx}, \tag{10.12}$$

that is, derivative of y with respect to x is equal to the ratio of the differential of y to the differential of x.

10.5.1 Example. If $f(x) = \sin x$, then we write $df(x) = \cos x dx$.

We may give a geometrical interpretation (Figure 10.1) of the differential as follows. Let f be a function which is differentiable on S and let $P(x, y)$ and $Q(x + \Delta x, y + \Delta y)$ be two points on the graph of f for which $x \in S$ and $x + \Delta x \in S$. Let PT be the tangent to the graph of f at P and we have

$$PR = \Delta x = dx, \quad RQ = \Delta y,$$
$$\frac{RT}{PR} = \tan RPT.$$

But we know that $\tan RPT = f'(x)$, therefore,

$$RT = PR.f'(x) = f'(x)dx,$$

and we see that RT represents the differential $df(x)$. If $y = f(x)$, then $RT = dy$.
As we have noted, if $y = f(x)$, then in general $dy \neq \Delta y$. However, if f is alinear function its graph is a straight line and the tangent PT coincides with the graph of f; so if f is linear and if $y = f(x)$, then $dy = \Delta y$.
From (10.3) it follows that

$$|\Delta y - dy| = |k||\Delta x|.$$

Consequently we may take $|\Delta y - dy|$ as small as we please by making $|\Delta x|$ sufficiently small. For this reason dy may be the useful appoximation to Δy when Δx is small.

10.5.2 Example. By the use of the differentials, we find approximately the value of $\sqrt[3]{122}$.

Let $y = x^{\frac{1}{3}}$. Then $dy = \frac{1}{3}x^{-2/3}dx = \frac{1}{3x^{2/3}}dx$. Substitute 125 for x and -3 for dx. Then

$$dy = -\frac{1}{25} = -0.04 \text{ and } y = 5.$$

So an approximation to $y + \Delta y$ for $x = 125$ and $dx = \Delta x = -3$ is

$$y + \Delta y = 5 - 0.04 = 4.96.$$

10.6 Differentiability

In this section we will extend the concept of differentiability of a function of one variable to the case of a function of several variables. We first recall the definitions for a function of one variable.

10.6.1 Linear Transformations

Let n and m be positive integers. We shall denote by $L(\mathbb{R}^n, \mathbb{R}^m)$ the vector space consisting of all linear transformations T from $\mathbb{R}^n$ into $\mathbb{R}^m$. The most basic fact about a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is that T is uniquely determined by its values on any n linearly independent vectors $\mathbf{x}_1, \ldots\ldots, \mathbf{x}_n$. That is, if

$$\mathbf{x} = c_1\mathbf{x}_1 + \ldots\ldots + c_n\mathbf{x}_n \text{ then}$$
$$T(\mathbf{x}) = c_1T(\mathbf{x}_1) + \ldots\ldots + c_nT(\mathbf{x}_n).$$

In particular, T is determined by its values on the standard basis vectors $\mathbf{e}_1 = (1, 0, .., 0, 0), .., \mathbf{e}_n = (0, 0, .., 0, 1)$.

10.6.2 Differentiability of a function of one variable

Let $S \subseteq \mathbb{R}$ and x be an interior point of S i.e. there exists an open set $U; x \in U \subseteq S$. Recall that a function $f : S \to \mathbb{R}$ is **differentiable** at $x \in S$, if for any point $x + h$ in U,

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{A}$$

exists. And we denote it by $D_x f(x)$. The derivative $D_x f(x)$ measures the rate of change of f at x along the x-axis. The process of finding $D_x f(x)$ for a given $f(x)$ is called **differentiation**. Equivalently we can say that f is differentiable at $x \in S$ if there exists a linear map $L : \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{h \to 0} \frac{f(x+h) - f(x) - Lh}{h} = 0. \tag{10.13}$$

In this case L is given by $h \longrightarrow f'(x)h$. Another way of writing (10.14) is

$$f(x+h) = f(x) + Lh + E(h) \text{ with } \lim_{h \to 0} \frac{E(h)}{h} = 0 \text{ i.e. } E(h) = o(h). \tag{10.14}$$

This definition is more suitable for the multivariable case, where h is a vector, so it does not make sense to divide by h.

10.6.3 Affine Transformation:

An **affine transformation** from $\mathbb{R}^n$ into $\mathbb{R}^m$ is a map

$$F : \mathbb{R}^n \to \mathbb{R}^m$$

of the form

$$F(\mathbf{x}) = \mathbf{y_0} + T(\mathbf{x}),$$

where $\mathbf{y_0}$ is a fixed vector in $\mathbb{R}^m$ and $T \in L(\mathbb{R}^n, \mathbb{R}^m)$. Thus, an affine transformation is a linear transformation followed by a translation. The **linear transformations** are the affine transformations which carry $\mathbf{0}$ into $\mathbf{0}$. A real-valued function $f : \mathbb{R} \to \mathbb{R}$ is **affine** means that the graph of f is a straight line. That is, f is linear means that the graph of f is a straight line through the origin.

10.6.4 Differentiability of a vector-valued function of one variable

Analogously we define the derivative of a vector-valued function of one variable. More precisely, if $f : S \subseteq \mathbb{R} \to \mathbb{R}^n; n > 1$, with components $f_1, f_2, , .., f_n$, we say that f is differentiable at $x \in S$ if

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists. It is easily seen that f is differentiable at $x \in S$ if and only if f_i is differentiable at $x \in S$ for all $i = 1, 2, .., n$. Also, f is differentiable $x \in S$ if and only if there exists a linear map $L : \mathbb{R} \to \mathbb{R}^n$ such that

$$\lim_{h \to 0} \frac{f(x+h) - f(x) - Lh}{h} = 0. \tag{10.15}$$

Here and in what follows we write Lh instead of $L(h)$ if L is a linear map.

How can we now generalize the concept of differentiability to functions of several variables, say for a function $f : \mathbb{R}^2 \to \mathbb{R}, z = f(x, y)$. A natural idea is to freeze one variable, say y, define $g(x) = f(x, y)$ and check whether g is differentiable in x. This will lead to the notion of partial derivatives and most of us have seen this already in Calculus courses. However, we will see that the concept of partial derivatives alone is not completely satisfactory. For example we will see that the existence of partial derivatives does not guarantee that the function itself is continuous (as it is the case for a function of one variable).

The notion of the (total) derivative for functions of several variables will not have this deficiency. It is based on a generalisation of the formulation in (A). In order to do that we will need a suitable norm in $\mathbb{R}^n$. We have learnt already, e.g. in topology, that all norms in $\mathbb{R}^n$ are equivalent, and hence properties of sets, such as openness or boundedness, and of functions, such as continuity, do not depend on the choice of the norm.
In the sequel we will always use the Euclidean norm in $\mathbb{R}^n$ and denote it by $|.|$. More precisely, for $\mathbf{x} = (x_1, x_2, .., x_n) \in \mathbb{R}^n$,

$$|\mathbf{x}| = \sqrt{x_1^2 + x_2^2 + .. + x_n^2}.$$

We first consider a few examples of such functions:

10.6.1 Example.

1. 1 Let $f : \mathbb{R}^{n^2} \to \mathbb{R}^{n^2}$ where we identify the vector space of $n \times n$ matrices, and define f by
$$f(A) = A^2,$$
for $A \in \mathbb{R}^{n^2}$.
2. 2 Let $f : \mathbb{R}^n \to \mathbb{R}$ by
$$f(\mathbf{x}) = |\mathbf{x}| = \sqrt{x_1^2 + x_2^2 + .. + x_n^2},$$
for $\mathbf{x} = (x_1, x_2, .., x_n) \in \mathbb{R}^n$.
3. 3 Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ by
$$f(x, y, z) = (x^2 + z, \ln y).$$

10.6.2 Note. If $f\colon \mathbb{R}^n \to \mathbb{R}^m$, then, in Cartesian notation

$$f(x_1, x_2, ..., x_n) = (f_1(x_1, x_2, ..., x_n), f_2(x_1, x_2, ..., x_n), .., f_m(x_1, x_2, ..., x_n)).$$

In vector notation, if

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ . \\ x_n \end{bmatrix}, \text{then } f(\mathbf{x}) = \begin{bmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \\ . \\ f_m(\mathbf{x}) \end{bmatrix}.$$

10.6.5 Differentiability of a vector-valued function of a vector variable

Consider the function $f\colon \mathbb{R}^n \to \mathbb{R}^m$, We say that f is **differentiable** at an interior point point $\mathbf{a} \in \mathbb{R}^n$ if there exists a linear transformation $L\colon \mathbb{R}^n \to \mathbb{R}^m$, called the **total derivative** or the **Jacobi matrix** or the **Jacobian**, of f at $\mathbf{a}$ denoted by $L = Df(\mathbf{a})$ which satisfies the following

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{|f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - Df(\mathbf{a})\mathbf{h}|}{|\mathbf{h}|} = 0. \tag{10.16}$$

Equivalently, we can write f is **differentiable** at a point $\mathbf{a} \in \mathbb{R}^{\mathbf{n}}$ if

$$f(\mathbf{a}+\mathbf{h}) = f(\mathbf{a}) + Df(\mathbf{a})\mathbf{h} + E(\mathbf{h}) \tag{10.17}$$

where

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{|E(\mathbf{h})|}{|\mathbf{h}|} = 0.$$

Equivalently, f must locally be approximated by a linear function, that is

$$f(\mathbf{a}+\mathbf{h}) \approx f(\mathbf{a}) + Df(\mathbf{a})\mathbf{h}$$

where the **error** in the approximation

$$E(\mathbf{h}) = f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - Df(\mathbf{a})\mathbf{h}$$

satisfies

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{|E(\mathbf{h})|}{|\mathbf{h}|} = 0.$$

10.6.3 Remark. The expression $Df(\mathbf{a})\mathbf{h}$ denotes the matrix multiplication. Here $\mathbf{h}$ is an $n \times 1$ column vector:

10.6.4 Remark. Just as in the single variable case, where $h = \Delta x = x - a$, so in this case

$$\mathbf{h} = \Delta\mathbf{x} = \mathbf{x} - \mathbf{a} = \begin{bmatrix} x_1 - a_1 \\ \cdots\cdot \\ \cdots\cdot \\ x_n - a_n \end{bmatrix}.$$

Thus, the idea is that going a distance of $\mathbf{h}$ away from $\mathbf{a}$, to the point $\mathbf{x}$, we may approximate the value of $f(\mathbf{x}) = f(\mathbf{a}+\mathbf{h})$ by the amount $Df(\mathbf{a})\mathbf{h}$. It is literally the tangent line approximation in the real-valued case of $m = 1, n = 1$ i.e. $f\colon \mathbb{R} \to \mathbb{R}$ for then

$$Df(\mathbf{a})\mathbf{h} = f'(x)h.$$

The term $Df(\mathbf{a})\mathbf{h}$ or $f'(x)h$ is also called the **differential** of f at h. And it coincides with the definition that $f : \mathbb{R} \to \mathbb{R}$ is differentiable at an interior point $a \in D$ if there is a constant A such that $f(a+h) = f(a) + Ah + E(h)$ with

$$\lim_{h\to 0} \frac{E(h)}{h} = 0.$$

Again, if $f\colon \mathbb{R} \to \mathbb{R}$, then differential of f at h is denoted by $df(x) = f'(x)h$ and if $f(x) = x$ then differential of x at h is then $dx = 1h$, hence $df(x) = f'(x)dx$.

10.6.5 Remark. Thus to say that a function f is differentiable at a point is equivalent to saying that f has a total derivative there. We shall see that f may be partially differentiable, and to have directional derivatives in all directions, yet not be differentiable. We will explain this further below.

10.6.6 Example.

1. Let $f\colon \mathbb{R}^2 \to \mathbb{R}^3$ defined by $f(x,y) = (x^2, -3xy, x+y)$ (in Cartesian notation). We see that

$$\begin{aligned} f(x+h, y+k) &= ((x+h)^2, -3(x+h)(y+k), x+h+y+k) \\ &= (x^2, -3xy, x+y) + (2x, -3y, 1)h + (0, -3x, 1)k \\ &+ (h^2, -3hk, h+k) \\ &= f(x,y) + \begin{bmatrix} h & k \end{bmatrix} \begin{bmatrix} 2x & -3y & 1 \\ 0 & -3x & 1 \end{bmatrix} + \begin{bmatrix} h^2 & -3hk & h+k \end{bmatrix} \end{aligned}$$

In vector notation, denote $\mathbf{X} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\mathbf{H} = \begin{bmatrix} h \\ k \end{bmatrix}$. Then the above becomes

$$\begin{aligned}
f(\mathbf{X}+\mathbf{H}) &= \begin{bmatrix} (x+h)^2 \\ -3(x+h)(y+k) \\ (x+h)+(y+k) \end{bmatrix} \\
&= \begin{bmatrix} x^2 \\ -3xy \\ x+y \end{bmatrix} + \begin{bmatrix} 2x \\ -3y \\ 1 \end{bmatrix} h + \begin{bmatrix} 0 \\ -3x \\ 1 \end{bmatrix} k + \begin{bmatrix} h^2 \\ -3hk \\ 0 \end{bmatrix} \\
&= f(\mathbf{X}) + \begin{bmatrix} 2x & 0 \\ -3y & -3x \\ 1 & 1 \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix} + E(\mathbf{H}) \\
&= f(\mathbf{X}) + Df(\mathbf{X})\mathbf{H} + E(\mathbf{H})
\end{aligned}$$

2. Let $A = (a_{ij}) \in M_{m\times n}(\mathbb{R})$ and let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear map

$$\mathbf{X} \mapsto A\mathbf{X}.$$

Now

$$\begin{aligned}
f(\mathbf{X}+\mathbf{H}) &= A(\mathbf{X}+\mathbf{H}) \\
&= A\mathbf{X} + A\mathbf{H} + E(\mathbf{H}), \text{where } \mathbf{E}(\mathbf{H}) = \mathbf{0} \\
&= f(\mathbf{X}) + A\mathbf{H} + E(\mathbf{H})
\end{aligned}$$

So in ths case $f(\mathbf{X}+\mathbf{H}) - f(\mathbf{X}) = A\mathbf{H}$ which is linear and the remainder term $E(\mathbf{H})$ is zero. Hence f is differentiable and the linear map $L = Df(\mathbf{X})$ is given by

$$Df(\mathbf{X}) : \mathbf{H} \longrightarrow A\mathbf{H}.$$

3. Let $A = (a_{ij}) \in M_{m\times n}(\mathbb{R})$ be symmetric and let $f\colon \mathbb{R}^n \to \mathbb{R}$ be the quadratic form corresponding to A i.e.

$$f(\mathbf{X}) = \mathbf{X}^t A\mathbf{X} = \langle \mathbf{X}, A\mathbf{X}\rangle.$$

If $\mathbf{H} \in \mathbb{R}^n$, we see

$$\begin{aligned}
f(\mathbf{X}+\mathbf{H}) - f(\mathbf{X}) &= \langle \mathbf{X}+\mathbf{H}, A(\mathbf{X}+\mathbf{H})\rangle - \langle \mathbf{X}, A\mathbf{X}\rangle \\
&= \langle \mathbf{X}, A\mathbf{X}\rangle + \langle \mathbf{H}, A\mathbf{X}\rangle + \langle \mathbf{X}, A\mathbf{H}\rangle + \langle \mathbf{H}, A\mathbf{H}\rangle - \langle \mathbf{X}, A\mathbf{X}\rangle \\
&= 2\langle \mathbf{H}, A\mathbf{X}\rangle + \langle \mathbf{H}, A\mathbf{H}\rangle \\
&= 2\langle A\mathbf{X}, \mathbf{H}\rangle + \langle \mathbf{H}, A\mathbf{H}\rangle
\end{aligned}$$

We see that, $\langle \mathbf{H}, A\mathbf{X}\rangle$ is a scalar and $A^t = A$, so

$$\langle \mathbf{H}, A\mathbf{X}\rangle = \mathbf{H}^t A\mathbf{X} = (\mathbf{X}^t A^t \mathbf{H})^t = \mathbf{X}^t A^t \mathbf{H} = \mathbf{X}^t A\mathbf{H} = \langle \mathbf{X}, A\mathbf{H}\rangle$$

Hence $Df(\mathbf{X}) = 2(A\mathbf{X})^t$ as $2(A\mathbf{X})^t\mathbf{H} = 2\langle A\mathbf{X}, \mathbf{H}\rangle$
Indeed

$$\begin{aligned}
\left|\frac{f(\mathbf{X}+\mathbf{H}) - f(\mathbf{X}) - 2(A\mathbf{X})^t\mathbf{H}}{|\mathbf{H}|}\right| &= \left|\frac{\langle \mathbf{H}, A\mathbf{H}\rangle}{|\mathbf{H}|}\right| \le \frac{|\mathbf{H}||\mathbf{A}\mathbf{H}|}{|\mathbf{H}|} \\
&\le ||A|||\mathbf{H}| \to \mathbf{0} \text{ as } \mathbf{H} \to \mathbf{0},
\end{aligned}$$

where

$$||A|| = \left(\sum_{i,j=1}^{n} a_{ij}^2 \right)^{\frac{1}{2}}.$$

Thus f is differentiable at every $\mathbf{X} \in \mathbb{R}^n$.

4. Let $f : M_{n\times n}(\mathbb{R}) \to M_{n\times n}(\mathbb{R})$ defined by $f(\mathbf{A}) = \mathbf{A}^2$ for $\mathbf{A} \in M_{n\times n}(\mathbb{R})$
Let $\mathbf{H} = (h_{ij}) \in M_{n\times n}(\mathbb{R})$, then

$$\begin{aligned} f(\mathbf{A}+\mathbf{H}) - f(\mathbf{A}) &= (\mathbf{A}+\mathbf{H})(\mathbf{A}+\mathbf{H}) - \mathbf{A}^2 \\ &= \mathbf{AH} + \mathbf{HA} + \mathbf{H}^2. \end{aligned}$$

And the linear term $\mathbf{AH} + \mathbf{HA} = Df(\mathbf{A})\mathbf{H}$. Again

$$\left| \frac{f(\mathbf{X}+\mathbf{H}) - f(\mathbf{X}) - (\mathbf{AH}+\mathbf{HA})}{|\mathbf{H}|} \right| = \frac{|\mathbf{H}^2|}{|\mathbf{H}|} \to 0 \text{ as } \mathbf{H} \to 0.$$

Thus f is differentiable at every $\mathbf{A} \in M_{n\times n}(\mathbb{R})$ with $Df(\mathbf{X})\mathbf{H} = \mathbf{AH} + \mathbf{HA}$.

10.7 Partial derivatives

Now suppose that $f\colon \mathbb{R}^n \to \mathbb{R}$. The **$i$-th partial derivative** of f at $\mathbf{a}$ is the directional derivative of f at $\mathbf{a}$ in a direction $\mathbf{e}_i$ (i-th coordinate axis), that is $\frac{d}{dt}f(\mathbf{a}+t\mathbf{e}_i)$ at $t = 0$ exists, hence the **partial derivatives** of f (if they exist) are

$$(D_1 f)(\mathbf{a}) = \lim_{t\to 0} \frac{f(\mathbf{a}+t\mathbf{e}_i) - f(\mathbf{a})}{t} \tag{10.18}$$

We denote the above limit by $D_1 f, ..., D_n f$ by $\dfrac{\partial f}{\partial x_1}, .., \dfrac{\partial f}{\partial x_n}$.

10.7.1 Theorem. Let $f : \mathbb{R}^2 \to \mathbb{R}$ whose domain is R. If (x_1, y_1) is an interior point of R and if the functions $f_x = \dfrac{\partial f}{\partial x}$ and $f_y = \dfrac{\partial f}{\partial y}$ are continuous on some nbhd. N of (x_1, y_1) then

$$f(x_1+h, y_1+k) - f(x_1, y_1) = hf_x(x_1, y_1) + kf_y(x_1, y_1) + h\theta_1(h,k) + k\theta_2(h,k) \tag{10.19}$$

where h, k are real numbers such that $(x_1+h, y_1+k) \in N$ and θ_1 and θ_2 are the functions such that

$$\lim_{(h,k)\to(0,0)} \theta_1(h,k) = 0, \quad \lim_{(h,k)\to(0,0)} \theta_2(h,k) = 0.$$

10.8 Directional derivative:

Suppose $\mathbf{v}$ is a vector in $\mathbb{R}^n$, then $\mathbf{v}$ determines a line through the origin (the 1-dimensional subspace spanned by $\mathbf{v}$) and $\mathbf{a}$ is a point in $\mathbb{R}^n$, and let $T\colon \mathbb{R} \to \mathbb{R}^n$ be the translation by $t\mathbf{v}$ function

$$T(t) = \mathbf{a} + t\mathbf{v}.$$

In other words, if we translate that line so that it passes through $\mathbf{a}$, we have the line

$$L = \{\mathbf{a} + t\mathbf{v}; \mathbf{t} \in \mathbb{R}\}$$

and we can try to differentiate f at $\mathbf{a}$ along L: We say that $f\colon \mathbb{R}^n \to \mathbb{R}^m$ has a **directional derivative** at $\mathbf{a}$ in the direction of $\mathbf{v}$ if the composition function

$$f \circ T\colon \mathbb{R} \xrightarrow{T} \mathbb{R}^n \xrightarrow{f} \mathbb{R}^m,$$

defined by $(f \circ T)(t) = f(\mathbf{a} + t\mathbf{v})$ is differentiable at $t = 0$, in the sense that the limit

$$\lim_{t\to 0} \frac{f(\mathbf{a} + t\mathbf{v}) - f(\mathbf{a})}{t} \tag{10.20}$$

exists in $\mathbb{R}^m$, and this limit is denoted by $f_{\mathbf{v}}(\mathbf{a})$ or by $D_{\mathbf{v}}f(\mathbf{a})$ or by $f'(\mathbf{a}; \mathbf{v})$.
The derivative $D_{\mathbf{v}}f(\mathbf{a})$ depends not only on the line L but on the vector $\mathbf{v}$ as well, because in (10.19) we used a scale and direction along L which depends on $\mathbf{v}$ as a unit. Sometimes, we may carelessly refer to $D_{\mathbf{v}}f(\mathbf{a})$ as the derivative of f at $\mathbf{v}$ in the direction of the vector $\mathbf{v}$; but, we shall try not to do so, because $D_{\mathbf{v}}f(\mathbf{a})$ depends upon the length of $\mathbf{v}$ as well as its direction. If we let

$$\mathbf{w} = \frac{\mathbf{1}}{|\mathbf{v}|}\mathbf{v}$$

then $\mathbf{w}$ is the vector of unit length which has the same direction as $\mathbf{v}$ and it is $D_{\mathbf{w}}f(\mathbf{a})$ which is properly called the derivative of f at $\mathbf{v}$ in the direction of the vector $\mathbf{v}$. Of course,

$$D_{\mathbf{w}}f(\mathbf{a}) = \frac{1}{|\mathbf{v}|}D_{\mathbf{v}}f(\mathbf{a}).$$

That is a special case of the fact that $D_{\mathbf{v}}f$ is a linear function of $\mathbf{v}$ (if f is, say, a fixed function of class C^1).(Note that, a function f is of class C^1 on an open set $U \subseteq \mathbb{R}^n$, if $\dfrac{\partial f}{\partial x_i}; i = 1, 2, ..., n$ exist and are continuous functions on U.) For a fixed $\mathbf{v}$, and an open set $U \subseteq \mathbb{R}^n$ then $D_{\mathbf{v}}$ is a linear transformation

$$\begin{aligned} D_{\mathbf{v}} &: C^1(U) \to C(U) \text{ and} \\ D_{a\mathbf{v}_1+\mathbf{v}_2} &= aD_{\mathbf{v}_1} + D_{\mathbf{v}_2} \end{aligned}$$

In particular, if $\mathbf{v} = (v_1, v_2, ..., v_n)$, then

$$\begin{aligned} D_{\mathbf{v}} &= v_1D_1 + v_2D_2 + ... + v_nD_n \text{ and} \\ D_{\mathbf{v}}f &= v_1\frac{\partial f}{\partial x_1} + v_2\frac{\partial f}{\partial x_2} + ... + v_n\frac{\partial f}{\partial x_n},\ f \in C^1. \end{aligned}$$

If f is of class C^1 on the open set U, the **gradient** of f is

$$\begin{aligned} f' &= (D_1f, D_2f, .., D_nf) \\ &= \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, .., \frac{\partial f}{\partial x_n}\right). \end{aligned} \tag{10.21}$$

Note that f is a (continuous) map from U into $\mathbb{R}$. Hence the gradient satisfies

$$\begin{aligned} D_{\mathbf{v}}f &= \langle \mathbf{v}, \mathbf{f}' \rangle \\ &= \mathbf{v} \cdot \mathbf{f}'. \end{aligned}$$

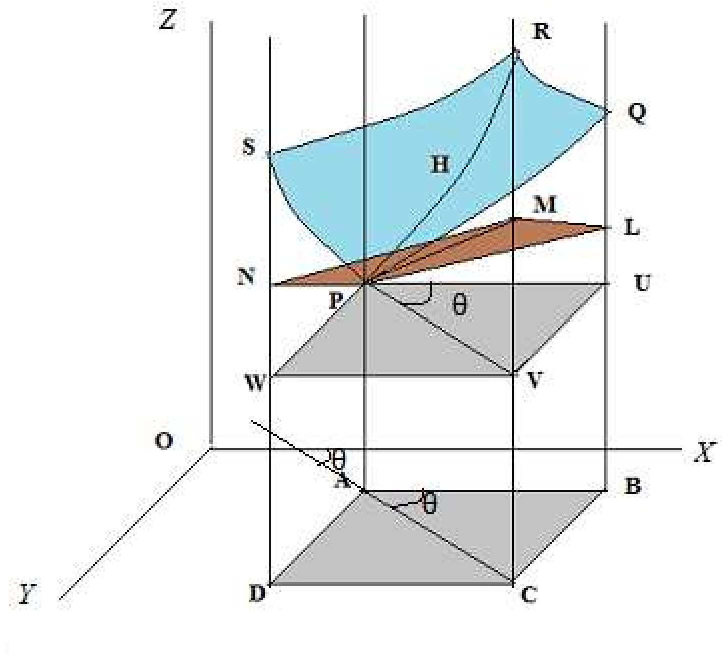

Figure 10.2:

10.9 Geometrical Illustration

In this section we explain (Figure 10.2) the geometric meaning of partial derivatives for the case of two independent variables.
Let $PQRS$ be a portion of the surface $Z = f(X, Y)$ cut off by four planes

$$PABQ\ (Y = y),\ RCDS\ (Y = y + k)$$
$$PADS\ (X = x),\ QBCR\ (X = x + h)$$

Capital letters X, Y, Z represent current co-ordinates of the system, so that the co-ordinates of the corners P, Q, R, S are

$$P \equiv (x, y, f(x, y)),\ Q \equiv (x + h, y, f(x + h, y))$$
$$S \equiv (x, y + k, f(x, y + k)),$$
$$R \equiv (x + h, y + k, y, f(x + h, y + k)).$$

Let $PUVW$ be a plane through P parallel to the XY-plane and cutting the ordinates of P, Q, R, S in P, U, V, W respectively, we have

$$\begin{aligned} UQ &= f(x+h, y) - f(x, y), \\ WS &= f(x, y+k) - f(x, y), \\ VR &= f(x+h, y+k) - f(x, y). \\ \text{Hence } f_x = \frac{\partial z}{\partial x} &= \lim_{h\to 0} \frac{f(x+h, y) - f(x, y)}{h} = \tan UPL \\ \text{and similarly } f_y = \frac{\partial z}{\partial y} &= \lim_{k\to 0} \frac{f(x, y+k) - f(x, y)}{k} = \tan WPN, \end{aligned}$$

where the tangent plane at P to the surface cut UQ, VR, WS at L, M, N respectively. Hence

$$\begin{aligned} UL = PU \tan UPL &= \frac{\partial z}{\partial x} h \\ \text{and } WN = PW \tan WPN &= \frac{\partial z}{\partial y} k. \end{aligned}$$

Also the section made on the tangent plane by the four bounded planes is a parallelogram and the height of the center above the plane $PUVW$ is given by $\frac{1}{2}MV$ and also by $\frac{1}{2}(UL + WN)$. Hence

$$\begin{aligned} MV &= UL + WN \\ &= \frac{\partial z}{\partial x} h + \frac{\partial z}{\partial y} k \end{aligned}$$

is called the **differential** of z and is denoted by dz, so

$$dz = f_x h + f_y k. \tag{10.22}$$

With x, y as independent variables consider the functions $z = I(x, y) = x$ and $z = J(x, y) = y$. Now

$$I_x(x, y) = 1,\ I_y(x, y) = 0;\ J_(x, y) = 0,\ J_y(x, y) = 1$$

Then by using (10.23), we get $dx = h$ and $dy = k$. Therefore, when x, y are independent variables, we can write

$$dz = f_x dx + f_y dy. \tag{10.23}$$

Suppose that $z = f(x, y)$ is an equation of a surface, and let $P(x_1, y_1, z_1)$ be a point on this surface and $PQRS$ of Fig.10.2 represents a portion of this surface. Through the point $A(x_1, y_1, 0)$ in the xy-plane construct a directed line AC which makes a positive angle θ with the positive x-axis $(0 < \theta < 2\pi)$. The set of equations

$$x = x_1 + t\cos\theta, y = y_1 + t\sin\theta, z = 0$$

is a parametric representation of the line AC. The plane $ACRP$, perpendicular to XY-plane through the line AC, intersects the surface $PQRS$ in a curve PHR that has the parametric representation

$$x = x_1 + t\cos\theta, y = y_1 + t\sin\theta, z = f(x + t\cos\theta, y + t\sin\theta) \tag{10.24}$$

For z given in 10.24 we compute

$$\frac{dz}{dt} = f_x(x + t\cos\theta, y + t\sin\theta)\cos\theta + f_y(x + t\cos\theta, y + t\sin\theta)\sin\theta.$$

This result is a formula for the rate of change of the z-coordinate of the point $A(x, y, z)$ with respect t as A moves along the curve PHR. The value of $\frac{dz}{dt}$ at $t = 0$ is called the **directional derivative** of z at (x_1, y_1) in the direction μ and denoted by

$$d_\theta z = f_x(x_1, y_1)\cos\theta + f_y(x_1, y_1)\sin\theta.$$

Note that for $\theta = 0$ we have $d_0 z = f_x(x_1, y_1)$ and for $\theta = \frac{\pi}{2}$ we have $d_{\pi/2} z = f_y(x_1, y_1)$. In other words, the directional derivative of $z = f(x, y)$ at (x_1, y_1) in the direction of x-axis and in the direction of y-axis are $f_x(x_1, y_1)$ and $f_y(x_1, y_1)$ respectively. So the the directional derivatives are the generalizations of partial derivatives.

10.9.1 Theorem. (Implicit Function Theorem) Let $D = \{(x, y) \in \mathbb{R}^2; a \leq x \leq b\}$ and $F : D \to \mathbb{R}$ be a function where its partial derivative with respect to y exists and there exist $m, M > 0$ such that

$$0 < m < \frac{\partial F}{\partial y} \leq M \;\forall\; (x, y) \in D.$$

Then there exists one and only one continuous function $y(x)$ on $[a, b]$ such that

$$F(x, y(x)) = 0.$$

Note*: This means the equation $F(x, y(x)) = 0$ does implicitly define a unique continuous function y in terms of x. To solve this problem consider the vector space $C[a, b]$ of all continuous realvalued functions defined on $[a, b]$ with

$$\| f \| = \max_{a \leq x \leq b} |f(x)|$$

and define a map

$$T : C[a, b] \to C[a, b]$$

by

$$Ty(x) = y(x) - \frac{1}{M} F(x, y(x)).$$

Show that T is a contraction. (Use the Banach Contraction Mapping Theorem.)

10.10 Relationship between the derivatives

10.10.1 Theorem. If $f \colon \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $\mathbf{a}$ in $\mathbb{R}^n$, then all of its directional derivatives at $\mathbf{a}$ exist, and for any choice of vector $\mathbf{v}$ in $\mathbb{R}^n$, we have

$$D_{\mathbf{v}} f(\mathbf{x}) = Df(\mathbf{x})\mathbf{v}. \tag{10.25}$$

The left-hand side is a limit, while the right-hand side is a matrix product, with $\mathbf{v}$ treated as a column vector.

Proof. Since f is differentiable at $\mathbf{a}$, fix $\mathbf{v}$ and consider $\mathbf{h} = t\mathbf{v}$ for some sufficiently small $t \in \mathbb{R}$. Applying the linear approximation and the linearity of the derivative $Df(\mathbf{a})$ we get

$$\begin{aligned}
f(\mathbf{x}+t\mathbf{v}) - f(\mathbf{x}) - tDf(\mathbf{x})\mathbf{v} &= f(\mathbf{x}+t\mathbf{v}) - f(\mathbf{x}) - Df(\mathbf{x})t\mathbf{v} \\
&= f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}) - Df(\mathbf{x})\mathbf{h} \\
&= E(\mathbf{h}) \\
&= E(t\mathbf{v})
\end{aligned}$$

and

$$\lim_{t\to 0}\frac{|E(t\mathbf{v})|}{|t|} = \lim_{t\to 0}\frac{|E(t\mathbf{v})|}{|t||\mathbf{v}|}|\mathbf{v}| = \lim_{t\to 0}\frac{|E(t\mathbf{v})|}{|t\mathbf{v}|}|\mathbf{v}| = \lim_{t\to 0}\frac{|E(\mathbf{h})|}{|\mathbf{h}|}|\mathbf{v}| = 0.|\mathbf{v}| = 0.$$

This means

$$\lim_{t\to 0}\frac{|f(\mathbf{x}+t\mathbf{v}) - f(\mathbf{x}) - tDf(\mathbf{x})\mathbf{v}|}{t} = 0$$

Hence

$$\lim_{t\to 0}\frac{f(\mathbf{x}+t\mathbf{v}) - f(\mathbf{x})}{t} - Df(\mathbf{x})\mathbf{v} = 0$$

i.e.

$$D_{\mathbf{v}}f(\mathbf{x}) = \lim_{t\to 0}\frac{f(\mathbf{x}+t\mathbf{v}) - f(\mathbf{x})}{t} = Df(\mathbf{x})\mathbf{v}.$$

□

10.10.2 Theorem. A function $f\colon \mathbb{R}^n \to \mathbb{R}^m, f = (f_1, f_2, ..., f_n)$, is differentiable at $\mathbf{a}$ if and only if each of its component functions $f_i\colon \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\mathbf{a}$. In that case, we have

$$Df(\mathbf{a}) = D\begin{bmatrix} f_1(\mathbf{a}) \\ f_2(\mathbf{a}) \\ \vdots \\ f_m(\mathbf{a}) \end{bmatrix} = \begin{bmatrix} Df_1(\mathbf{a}) \\ Df_2(\mathbf{a}) \\ \vdots \\ Df_m(\mathbf{a}) \end{bmatrix}. \tag{10.26}$$

10.10.3 Theorem. Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$. If all the partial derivatives $\frac{\partial f_i}{\partial x_i}$ at $\mathbf{x} = \mathbf{a}$ of f exist and continuous at $\mathbf{a}$, then f is differentiable at $\mathbf{a}$.

10.11 Differentiation of composite functions

A function may often be regarded as built up by composition from a number of other functions. If $f(x, y) = xy^3 + y^2, g(x, y) = y^2 \sin x$ and $h(x) = e^{x^2}$, then a function F may be defined by

$$\begin{aligned}
F(x, y) &= f(g(x, y), h(x)) \\
&= y^2 \sin x.e^{3x^2} + e^{2x^2}.
\end{aligned} \tag{A}$$

The introduction of additional variable symbol some times helps to clarify such relations. For example, an equivalent description of the above is obtained by setting $w = F(x, y)$, and writing

$$\begin{aligned}
w &= f(u, v) = uv^3 + v^2 \\
u &= g(x, y) = y^2 \sin x \\
v &= h(x) = e^{x^2}.
\end{aligned}$$

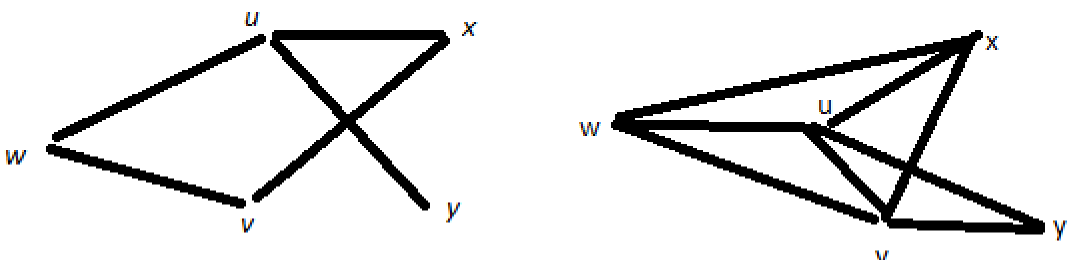

Fig. 3 Fig. 4

Figure 10.3:

These equations express w in terms of x, y indirectly through the intermediate variables u, v. The interdependence involved in this particular example may also be indicated schematically as Fig.3 and by chain rule: If $w = f(u, v)$ where $u = g(x, y), v = h(x, y)$, then we get

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x}$$
$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial y}.$$

Thus we have applying the chain rule to the equations (A) as

$$\frac{\partial w}{\partial x} = v^3 y^2 \cos x + (3uv^2 + 2v)(2xe^{x^2}).$$

The next illustration is somewhat more complicated; it also shows that the quotient notation for partial derivatives is sometimes ambiguous. Let

$$w = f(x, u, v), \ u = g(x, v, y) \ \ v = h(x, y) \tag{B}$$

The corresponding diagram is Fig.4. We see the dependence of w upon x is complicated by the fact that x enters in directly, and also through u and v. Each path in the diagram joining x to w corresponds to a term in the formula for $\frac{\partial w}{\partial x}$, so that we obtain

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial x} + \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial w}{\partial u}\frac{\partial u}{\partial v}\frac{\partial v}{\partial x}. \tag{C}$$

y enters in through v and u, so that

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial u}\frac{\partial u}{\partial v}\frac{\partial v}{\partial y}. \tag{D}$$

In both of these formulas, the partial derivatives must be understood in the correct context of the equations (B). In (C), for example the first occurence of $\frac{\partial w}{\partial x}$ refers to the partial derivative of w regarding it as a function of the independent variables x and y. The second occurence of $\frac{\partial w}{\partial x}$ refers

to the partial derivative of w regarding it as a function of the independent variables x, u and v. The use of numerical subscripts helps to remove such ambiguity. So we may write (C) and (D) in the alternative forms

$$\frac{\partial w}{\partial x} = f_1 + f_2 g_1 + f_3 h_1 + f_2 g_2 h_1$$
$$\frac{\partial w}{\partial y} = f_2 g_3 + f_3 h_2 + f_2 g_2 h_2,$$

where suffixes 1, 2, 3 represent first, second and third variables present in the functions f, g and h.

10.12 Exercises.

10.12.1 Exercise. Evaluate the following limits, or show that the limits do not exist.

1. $\lim_{(x,y)\to(0,0)} (x+y)\frac{y+(x+y)^2}{y-(x+y)^2}$
2. $\lim_{(x,y)\to(0,0)} \log \frac{a(1-e^x)}{a-y^x}$.

10.12.2 Exercise. Let $m : \mathbb{R}^2 \to \mathbb{R}$ be the multiplication function $m(x, y) = xy$. Sketch the preimage of the open interval (1,2) and show that this preimage is open.

10.12.3 Exercise. Let $\mathbb{R}^2 \setminus \mathbb{R}$ be the subset of $\mathbb{R}^2$ consisting of all pairs (x, y) with $y \neq 0$. Define $d : \mathbb{R}^2 \setminus \mathbb{R} \to \mathbb{R}$ to be the division function $d(x, y) = x/y$. Describe the preimage $d^{-1}(a, b)$ of an arbitrary open interval (a, b). Determine whether or not d is continuous.

10.12.4 Exercise. Verify that every point $(w, x, y, z) \in \mathbb{R}^4$ satisfying $w^2 + x^2 + y^2 + z^2 = 1$ can be expressed as $(\cos\theta, x'\sin\theta, y'\sin\theta, z'\sin\theta)$ where $0 \leq \theta < 2\pi$ and $(x', y', z') \in S^2$. $S^2 = \{(x, y, z) \in \mathbb{R}^3; x^2 + y^2 + z^2 = 1\}$.

10.12.5 Exercise. Let

$$f(x,y) = \begin{cases} \frac{xy}{(x^2+y^2)^\alpha} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Then which one of the following is true for f at the point (0,0)?

1. For $\alpha = 1$, is continuous but not differentiable.
2. For $\alpha = \frac{1}{2}$, is continuous and differentiable.
3. For $\alpha = \frac{1}{4}$, is continuous and differentiable.
4. For $\alpha = \frac{3}{4}$, is neither continuous nor differentiable.

10.12.6 Exercise. Let $a, b \in \mathbb{R}$ and let $f : \mathbb{R} \to \mathbb{R}$ be a thrice differentiable function. If $z = e^u f(v)$, where $u = ax + by, v = ax - by$, then which one of the following is true?

1. $b^2 z_{xx} - a^2 z_{yy} = 4a^2b^2 e^u f'(v)$.
2. $b^2 z_{xx} - a^2 z_{yy} = -4e^u f'(v)$.

3. $bz_x + az_y = abz$.
4. $bz_x + az_y = -abz$.

10.12.7 Exercise. Let $g : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function. Define $f : \mathbb{R}^3 \to \mathbb{R}$ by

$$f(x, y, z) = g(x^2 + y^2 - 2z^2).$$

Then $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$ is equal to

1. $4(x^2 + y^2 - 4z^2)g''(x^2 + y^2 - 2z^2)$
2. $4(x^2 + y^2 + 4z^2)g''(x^2 + y^2 - 2z^2)$
3. $4(x^2 + y^2 - 2z^2)g''(x^2 + y^2 - 2z^2)$
4. $4(x^2 + y^2 + 4z^2)g''(x^2 + y^2 - 2z^2) + 8g''(x^2 + y^2 - 2z^2)$

10.12.8 Exercise. Let $g : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function. If $f(x, y) = g(y) + xg'(y)$, then

1. $\frac{\partial f}{\partial x} + y\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial f}{\partial y}$.
2. $\frac{\partial f}{\partial y} + y\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial f}{\partial x}$.
3. $\frac{\partial f}{\partial x} + x\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial f}{\partial y}$.
4. $\frac{\partial f}{\partial y} + x\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial f}{\partial x}$.

10.12.9 Exercise. Let

$$f(x, y) = \begin{cases} x^2 \sin\frac{1}{x} + y^2 \sin\frac{1}{y}, & xy \neq 0 \\ x^2 \sin\frac{1}{x}, & x \neq 0, y = 0 \\ y^2 \sin\frac{1}{y}, & x = 0, y \neq 0 \\ 0 & x = y = 0. \end{cases}$$

Which of the following is true at (0,0)?

1. f is not continuous
2. $\frac{\partial f}{\partial x}$ is continuous but $\frac{\partial f}{\partial y}$ is not continuous
3. f is not differentiable
4. f is differentiable but both $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are not continuous

10.12.10 Exercise. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x, y) = \begin{cases} \frac{x^2 y(x-y)}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

Then find the value of $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$ at the point (0,0).

10.12.11 Exercise. Let $f : \mathbb{R}^2 \to \mathbb{R}$. For $\beta \in \mathbb{R}$, define

$$f(x,y) = \begin{cases} \frac{x^2|x|^\beta y}{x^4+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Which of the following is true at (0,0)?

1. f is continuous for $\beta = 0$
2. f is continuous for $\beta > 0$
3. f is continuous for $\beta < 0$
4. f is not differentiable for any β.

10.12.12 Exercise. Let $f : \mathbb{R}^2 \to \mathbb{R}$. Define

$$f(x,y) = \begin{cases} \frac{|x|}{|x|+|y|}\sqrt{x^4+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Which of the following is true at (0,0)?

1. f is continuous.
2. $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y}$ does not exist.
3. $\frac{\partial f}{\partial y} = 0$ and $\frac{\partial f}{\partial x}$ does not exist.
4. $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$.

10.12.13 Exercise. Let $D \subseteq \mathbb{R}^2$ be defined by $D = \mathbb{R}^2 \setminus \{(x,0); x \in \mathbb{R}\}$. Consider the function $f : D \to \mathbb{R}$ defined by

$$f(x,y) = x \sin \frac{1}{y}.$$

Which one of the following is true?

1. f is discontinuous on D.
2. f is a continuous function on D and cannot be extended continuously to any point outside D.
3. f is a continuous function on D and can be extended continuously to $D \cup \{(0,0)\}$.
4. f is a continuous function on D and can be extended continuously to the whole of $\mathbb{R}^2$.

10.12.14 Exercise. Show that for the following function, the sufficient conditions hold for the differentiability at a point do not hold but the function is differentiable at that point:

$$f(x,y) = \begin{cases} x^2 \sin \frac{1}{x} + y^2 \sin \frac{1}{y}, & xy \neq 0 \\ x^2 \sin \frac{1}{x}, & x \neq 0, y = 0 \\ y^2 \sin \frac{1}{y}, & x = 0, y \neq 0 \\ 0 & x = y = 0. \end{cases}$$

10.12.15 Exercise. Let

$$f(x,y) = \begin{cases} (x^2+y^2)\log(x^2+y^2), & x^2+y^2 \neq 0 \\ 0 & x^2+y^2 = 0. \end{cases}$$

Show that $f_{xy} = f_{yx} = 0$ though neither f_{xy} nor f_{yx} is continuous at (0,0).

10.12.16 Exercise. If

$$f(x,y) = x^2 \tan^{-1}\left(\frac{y}{x}\right) + y^2 \tan^{-1}\left(\frac{x}{y}\right), xy \neq 0, -\frac{\pi}{2} \leq \tan^{-1}\left(\frac{x}{y}\right) \leq \frac{\pi}{2},$$

and $f(x,0) = 0 = f(0,y)$. Show that $f_{xy}(0,0) \neq f_{yx}(0,0)$. Which condition(s) of Schwarz theorem is(are) not satisfied by f?

10.12.17 Exercise. Let

$$f(x,y) = \begin{cases} \frac{x^2y^2}{x^2+y^2}, & x^2+y^2 \neq 0 \\ 0 & x^2+y^2 = 0. \end{cases}$$

Show that $f_{xy} = f_{yx} = 0$, but f does not satisfy the conditions of Schwarz theorem and also f does not satisfy the conditions of Young's theorem.

10.12.18 Exercise. Let

$$f(x,y) = \begin{cases} xy\frac{x^2-y^2}{x^2+y^2}, & x^2+y^2 \neq 0 \\ 0 & x^2+y^2 = 0. \end{cases}$$

1. Show that f is continuous at (0,0).
2. Prove that $\frac{\partial^2 f}{\partial x \partial y} \neq \frac{\partial^2 f}{\partial y \partial x}$ at the origin. Which condition of Schwarz theorem does f violate?

10.12.19 Exercise. Examine the existence of $\lim_{(x,y)\to(0,0)} \frac{xy}{|x|+|y|}$.

10.12.20 Exercise. Let

$$f(x,y) = \begin{cases} x, & \text{if } xy \text{ is rational} \\ y & \text{if } xy \text{ is irrational} \end{cases}$$

Show that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$, but $\lim_{x\to 0} f(x,y)$ does not exist.

10.12.21 Exercise. Let $D \subseteq \mathbb{R}^2$ and $f : D \to \mathbb{R}$. Give examples to show that neither the continuity of f at a point in D implies the existence of the first order partial derivatives at that point nor the existence of the first order partial derivatives at that point ensures the continuity of f at that point.

10.12.22 Exercise. Let

$$f(x,y) = \begin{cases} x, & \text{if } |y| < |x| \\ -x, & \text{if } |y| \geq |x|. \end{cases}$$

Examine whether f is differentiable at (0,0).

10.12.23 Exercise. Let

$$f(x,y) = \begin{cases} \frac{x^6-2y^4}{x^2+y^2}, & x^2+y^2 \neq 0 \\ 0 & x^2+y^2 = 0. \end{cases}$$

Show that f is differentiable at (0,0).

10.12.24 Exercise. Let $f(x,y) = |xy|^p; (x,y) \in \mathbb{R}^2$. Show that f is differentiable at (0,0) only if $p > 1/2$.

10.12.25 Exercise. The equations $u+v = x+y, xu+yv = 1$ define u, v as a functions of x and y. Find $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$.

10.12.26 Exercise.

1. Transform the equation $x^2\frac{\partial z}{\partial x} + y^2\frac{\partial z}{\partial y} = z^2$, taking $u = x, v = \frac{1}{y} - \frac{1}{x}$ for the new independent variables and $w = \frac{1}{z} - \frac{1}{x}$.
2. Transform the equation $y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = (y-x)z$, taking $u = x^2+y^2, v = \frac{1}{x} + \frac{1}{y}$ for the new independent variables and $w = \log z - (x+y)$.
3. Transform the equation $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0$, taking $u = x+y, v = \frac{y}{x}$ for the new independent variables and $w = \frac{z}{x}$.
4. Transform the equation $\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0$, taking $u = x+y, v = x-y$ and $w = xy - z$ where $w = w(u,v)$.

10.12.27 Exercise. The relationships $u = f(x,y), v = F(x,y)$ where f, F are differentiable functions of x and y, specify x and y as a differentiable functions of u and v. Prove that

$$\left(\frac{\partial u}{\partial x}.\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}.\frac{\partial v}{\partial x}\right)\left(\frac{\partial x}{\partial u}.\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}.\frac{\partial y}{\partial u}\right) = 1.$$

10.12.28 Exercise. Let $S \subseteq \mathbb{R}^2$. Suppose (a,b) is an accumulation point of S. Let $f : S \to \mathbb{R}$. If $\lim_{(x,y)\to(a,b)} f(x,y) = L$, then show that

$$\lim_{x\to a} f(x, \phi(x)) = L$$

where ϕ is a real valued function such that $(x, \phi(x)) \in S$ for each $x \in D = dom(\phi)$ and $\lim_{x\to a} \phi(x) = b$.

10.12.29 Exercise. Correct or justify: If we can find two functions ϕ_1 and ϕ_2 such that

$$\lim_{x\to a} f(x, \phi_1(x)) \neq \lim_{x\to a} f(x, \phi_2(x)) \text{ where } \lim_{x\to a} \phi_i(x) = b \ (i = 1, 2),$$

then $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist.

10.12.30 Exercise. Let $D \subseteq \mathbb{R}^2$ and $f : D \to \mathbb{R}$ and f be continuous at $(a,b) \in D$. Show that $f(x,b)$ is continuous at $x = a$ and $f(a,y)$ is continuous at $y = b$. Give an example to show that the converse may not be true.

10.12.31 Exercise. Let

$$f(x,y) = \begin{cases} 1, & \text{if } xy \neq 0 \\ 0, & \text{if } xy = 0. \end{cases}$$

Show that $f(x,0)$ is continuous for all x and $f(0,y)$ is continuous for all y, but f is not continuous at (0,0).

10.12.32 Exercise. Let $a, b \in \mathbb{R}$ and

$$f(x,y) = \begin{cases} (ax+by)\sin\left(\frac{x}{y}\right), & \text{if } y \neq 0 \\ 0, & \text{if } y = 0. \end{cases}$$

Check for the continuity of f at (0,0).

10.12.33 Exercise. Let

$$f(x,y) = \begin{cases} x^2y^2\cos\left(\frac{1}{x}\right), & \text{if } x \neq 0 \text{ and } \forall\, y \in \mathbb{R}. \\ 0, & \text{if } x = 0. \end{cases}$$

Verify the following properties of f.

1. $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}\ \forall\,(x,y) \in \mathbb{R}^2$.
2. Both $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ are continuous together at the origin.
3. Neither $\frac{\partial^2 f}{\partial x \partial y}$ nor $\frac{\partial^2 f}{\partial y \partial x}$ is continuous in x at $x = 0$ if $y \neq 0$.

10.12.34 Exercise. If $f(x,y,\alpha,\beta) = 0, \phi(x,\alpha) = 0, \psi(y,\beta) = 0$; assuming that all concerned first order partial derivatives exist and are continuous, show that

$$\phi_\alpha.\frac{\partial(f,\psi)}{\partial(y,\beta)}.\frac{dy}{dx} + \psi_\alpha.\frac{\partial(f,\phi)}{\partial(x,\alpha)} = 0.$$

10.12.35 Exercise. If $f(x,y,\alpha,\beta) = 0, \alpha = \phi(x,y) = 0, \beta = \psi(x,y) = 0$; assuming that all concerned first order partial derivatives exist and are continuous, show that

$$\frac{dy}{dx} = -\frac{f_x + f_\alpha\phi_x + f_\beta\psi_x}{f_y + f_\alpha\phi_y + f_\beta\psi_y}.$$

10.12.36 Exercise. If u and v are twice differentiable functions of x, y defined by $xy + uv = 1; xu + yv = 1$; find $\frac{\partial^2 u}{\partial x^2}$.

10.12.37 Exercise. If f be twice differentiable functions of x, y and $x = \cosh(u+v); y = \sinh(u-v)$; show that

$$f_{uu} - f_{vv} = 4\sqrt{(x^2-1)(y^2+1)}f_{xy}.$$

10.12.38 Exercise. If f be twice differentiable functions of x, y and $x = (u-v)^2; y = (u+v)^2$; show that

1. $f_{uu} = 2f_x + 2f_y + 4xf_{xx} + 4yf_{yy} + 8\sqrt{xy}f_{xy}$.
2. $f_{vv} = 2f_x + 2f_y + 4xf_{xx} + 4yf_{yy} - 8\sqrt{xy}f_{xy}$.
3. $f_{uv} = f_{vu} = 2f_y - 2f_x - 4xf_{xx} + 4yf_{yy}$.

10.12.39 Exercise. If V be twice differentiable functions of x, y and $u(x^2+y^2) = x, v(x^2+y^2) = y$; show that

$$V_{xx} + V_{yy} = (u^2+v^2)(V_{uu} + V_{vv}).$$

10.12.40 Exercise. If ϕ be twice differentiable functions of x, y and z and if $2x = u^2 - v^2; y = uv, z = w$ show that $\phi_{xx} + \phi_{yy} + \phi_{zz} = 0$ is transformed into $\phi_{uu} + \phi_{vv} + (u^2+v^2)\phi_{ww} = 0$.

10.12.41 Exercise. If z be twice differentiable functions of u, v and if $u = \frac{1}{x} + \frac{1}{y}$, and $v = \frac{1}{x+y}$ find $\frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}$ in terms of $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, x$ and y.

10.12.42 Exercise. If ϕ be a differentiable functions of x, y and z and if $u = x^2 - y^2; v2xy$; show that $y\phi_x - x\phi_y = \frac{(y^2-x^2)^3}{xy}$is transformed into $\frac{\partial \phi}{\partial v} = \frac{u^2-v^2}{v}$.

10.12.43 Exercise. If $f(x, y, z)$ be a homogeneous function of degree $n > 1$ having continuous second order partial derivatives, show that

$$\begin{vmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{vmatrix} = \frac{(n-1)^2}{z^2} \begin{vmatrix} f_{xx} & f_{xy} & f_x \\ f_{yx} & f_{yy} & f_y \\ f_x & f_y & \frac{n}{n-1}f \end{vmatrix}.$$

10.12.44 Exercise. If $x^2 = vw, y^2 = wu, z^2 = uv$ and $f(x, y, z) = \phi(u, v, w)$, show that

$$xf_x + yf_y + zf_z = u\phi_u + v\phi_v + w\phi_w$$

where all partial derivatives are continuous.

10.12.45 Exercise. Show that if $z = f(x, y)$ and $F(x - az, y - bz) = 0$ where F is an arbitrary function of two arguments, satisfies the relation $a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} = 1$, and also that $F\left(\frac{x}{z}, \frac{y}{z}\right) = 0$ yields $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z$.

10.12.46 Exercise. Transform the expression

$$\left\{1 + \left(\frac{\partial z}{\partial x}\right)^2\right\} \frac{\partial^2 z}{\partial y^2} - 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\frac{\partial^2 z}{\partial x \partial y} + \left\{1 + \left(\frac{\partial z}{\partial y}\right)^2\right\} \frac{\partial^2 z}{\partial x^2}$$

by the substitution $x = lu + mv, y = -mu + lv$ where l and m are constants and $l^2 + m^2 = 1$.

10.12.47 Exercise. A set of three variables x, y, z is connected with another set by the equations $x + y + z = u, xy + yz + zx = v, xyz = w$. Prove that

$$\frac{\partial^2 x}{\partial w^2} = -\frac{2(2x-y-z)}{\{(x-y)(x-z)\}^3}.$$

10.12.48 Exercise. If $x = cuv, y = c\sqrt{(1+u^2)(1+v^2)}$ where c is a non-zero constant, show that

$$\frac{1}{y}\left\{y\frac{\partial V}{\partial x} - x\frac{\partial V}{\partial y}\right\} = \frac{v\frac{\partial V}{\partial u} + u\frac{\partial V}{\partial v}}{c(u^2+v^2)}$$

where V is a differentiable function of x and y.

10.12.49 Exercise. A function $f(x, y)$ becomes $g(u, v)$ when $x = \frac{1}{2}(u+v)$ and $y^2 = uv$, prove that

$$\frac{\partial^2 g}{\partial u \partial v} = \frac{1}{4}\left\{\frac{\partial^2 f}{\partial x^2} + 2\frac{x}{y}\frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 f}{\partial y^2} + \frac{1}{y}\frac{\partial f}{\partial y}\right\}$$

10.12.50 Exercise. If $F(u, v)$ be twice differentiable function of u, v and $u = x^2 - y^2, v = xy$ and show that

$$4(u^2+v^2)\frac{\partial^2 F}{\partial u \partial v} + 2u\frac{\partial F}{\partial v} + 2v\frac{\partial F}{\partial u} = xy\left(\frac{\partial^2 F}{\partial x^2} - \frac{\partial^2 F}{\partial y^2}\right) + (x^2-y^2)\frac{\partial^2 F}{\partial x \partial y}.$$

10.12.51 Exercise. If z is a function of two variables x, y and $x = c\cosh u \cos v, y = c \sinh u \sin v (c)$ is a real number, show that

$$\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} = \frac{c^2}{2}(\cosh 2u - \cos 2v)\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right),$$

where all second order partial derivatives are continuous.

10.12.52 Exercise. By the transformation $u = a + cx + dy, v = b - dx + cy$, where a, b, c, d are constants and $c^2 + d^2 = 1$, the function $f(x, y)$ is transformed to $F(u, v)$. Prove that

$$F_{uu}F_{vv} - F_{uv}^2 = f_{xx}f_{yy} - F_{xy}^2,$$

where all second order partial derivatives are continuous.

10.12.53 Exercise. If $F(v^2 - x^2, v^2 - y^2, v^2 - z^2) = 0$ and $v = f(x, y, z)$ having continuous first order partial derivatives, show that

$$\frac{1}{x}\frac{\partial v}{\partial x} + \frac{1}{y}\frac{\partial v}{\partial y} + \frac{1}{z}\frac{\partial v}{\partial z} = \frac{1}{v}$$

10.12.54 Exercise. Let

$$(1)\ \ f(x,y) = \begin{cases} (x^2+y^2)\sin\frac{1}{\sqrt{x^2+y^2}}, & x^2+y^2 \neq 0 \\ 0 & x^2+y^2 = 0 \end{cases}$$

$$(2)\ \ g(x,y) = \begin{cases} (x^2+y^2)\sin\frac{1}{xy}, & xy \neq 0 \\ 0 & xy = 0. \end{cases}$$

Examine the differentiability of f and g at (0,0).

10.12.55 Exercise. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x,y) = \begin{cases} \frac{xy(x-y)}{(x^2+y^2)^{\frac{3}{2}}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Let $f_a : \mathbb{R} \to \mathbb{R}$ be the map $\lambda \mapsto f(a, \lambda)$ and $f_b : \mathbb{R} \to \mathbb{R}$ with $\lambda \mapsto f(\lambda, b)$. Show that for each $a, b \in \mathbb{R}$ the maps f_a and f_b are continuous but that f is not continuous. [Hint: $f(1/n, 1/n) = 0; f(1/n, -1/n) = 1/\sqrt{2}$.]

10.13 Cantor Set

We study in this section a subset $C \subseteq [0,1] = I$, first introduced by G. Cantor, that plays an important role in mathematics. The set C is usually referred to as the **Cantor ternary set**. The term "ternary" comes from the way it is constructed. We shall consider C endowed with the metric induced by the absolute-value metric of $\mathbb{R}$ (the resulting space is usually referred to as the **Cantor space**).

10.13.1 Ternary expansion

The easiest way to understand ternary expansions is to review the fundamentals of decimal expansions of real numbers. Since the decimal system is based on the scale of 10, we have exactly ten digits 0,1,2,...,9 which are the coefficients of integral powers of 10 in the decimal expansion. For example, 19,732.206 represents the sum given by

$$1.10^4 + 9.10^3 + 7.10^2 + +3.!0^1 + 2.10^0 + 2.10^{-1} + 0.10^{-2} + 6.10^{-3}$$
$$\text{i.e. } = 10,000 + 9,000 + 700 + 30 + 2 + \frac{2}{10} + \frac{6}{10^3}$$

In particular, if x is a real number such that $0 \leq x \leq 1$, then the decimal expansion of x has the form

$$0.x_1x_2x_3x_4..... \tag{1}$$

where each $0 \leq b_i \leq 9$. The expression (1) represents the sum

$$S = 0 + x_1 10^{-1} + x_2 10^{-2} + x_3 10^{-3} + 10^{-4} x_4 + ... \tag{2}$$

Note that if each $x_i = 9$, then $S = 1$. Hence

$$0.9999.... = 1.0000...$$

Thus any real number x of the form (1) which eventually ends in 9's, that is $x_i = 9 \;\forall\; i \geq k$, for some $k \in \mathbb{N}$. has equivalent decimal expansion which eventually ends in 0's. Furthermore, it can be shown that a real number of the form (1) has two such equivalent expansion iff $x = \frac{a}{10^k}$ for some non-negative integer k, where a is an integer with $0 \leq a \leq 10^k$. Thus x has either a unique decimal expansion or exactly two equivalent decimal expansions, one ending with 9s and the other ending with 0's.
Since the ternary system is based on a scale of 3, we have exactly three digits 0,1,2 which are the coefficients of integral powers of 3 in the ternary expansion. For example, $(12,110.201)_3$ represents the sum given by

$$1.3^4 + 2.3^3 + 1.3^2 + +1.3^1 + 0.3^0 + 2.3^{-1} + 0.3^{-2} + 1.3^{-3}$$

In particular, if x is a real number such that $0 \leq x \leq 1$, then the ternary expansion of x has the form

$$0.x_1x_2x_3x_4..... \tag{3}$$

where each $0 \leq x_i \leq 2$. The expression (3) represents the sum

$$S = 0 + x_1 3^{-1} + x_2 3^{-2} + x_3 3^{-3} + 3^{-4} x_4 + ... \tag{4}$$

Note that if each $x_i = 2$, then $S = 1$. Hence

$$0.2222.... = 1.0000...$$

Thus any real number x of the form (3) which eventually ends in 2's, that is $x_i = 2\ \forall\ i \geq k$, for some $k \in \mathbb{N}$ has equivalent ternary expansion which eventually ends in 0s. Furthermore, it can be shown that a real number of the form (3) has two such equivalent expansion iff $x = \frac{a}{3^k}$ for some non-negative integer k, where a is an integer with $0 \leq a \leq 3^k$. Thus x has either a unique ternary expansion or exactly two equivalent ternary expansions, one ending with 2's and the other ending with 0's. We begin with ternary expansions of some numbers.

10.13.1 Example. We have

1. Suppose we want to find the ternary expansion of $\frac{11}{26}$, we see that

$$\begin{aligned}
\frac{11}{26} &= \frac{3^2+2}{3^3-1} = \frac{3^2+2}{3^3}\left(1 - \frac{1}{3^3}\right)^{-1} \\
&= \left(\frac{1}{3} + \frac{2}{3^3}\right)\left(1 + \frac{1}{3^3} + \frac{1}{3^6} + \frac{1}{3^9} + ...\right) \\
&= \frac{1}{3} + \frac{0}{3^2} + \frac{2}{3^3} + \frac{1}{3^4} + \frac{0}{3^5} + \frac{2}{3^6} + \frac{1}{3^7} + \frac{0}{3^8} + \frac{2}{3^9} + ... \\
&= 0.102102102...[3].
\end{aligned}$$

2. Consider the number $\frac{11}{27}$. Then we get

$$\frac{11}{27} = \frac{3^2+2}{3^3} = \frac{1}{3} + \frac{2}{3^3} = \frac{1}{3} + \frac{0}{3^2} + \frac{2}{3^3} = 0.102\ [3].$$

3. Consider the number $\frac{1}{5}$.

$$\begin{aligned}
\frac{1}{5} &= \frac{16}{80} = \frac{3^2+2.3+1}{3^4-1} = \frac{3^2+2.3+1}{3^4}\left(1 - \frac{1}{3^4}\right)^{-1} \\
&= \left(\frac{1}{3^2} + \frac{2}{3^3} + \frac{1}{3^4}\right)\left(1 + \frac{1}{3^4} + \frac{1}{3^8} + ...\right) \\
&= \frac{1}{3^2} + \frac{2}{3^3} + \frac{1}{3^4} + \frac{1}{3^6} + \frac{2}{3^7} + \frac{1}{3^8} + ... = 0.121121121..[3].
\end{aligned}$$

10.13.2 Note (An algorithm of ternary expansion of a rational number in (0,1)). Let $\frac{p}{q} \in [0,1], p <$

$q, (p, q) = 1$. Let k be the smallest positive integer such that $q + m = 3^k$ for some $m \in \mathbb{N}$. Then

$$\begin{aligned}
\frac{p}{q} &= \frac{p}{3^k - m} = \frac{p}{3^k} \frac{1}{\left(1 - \frac{m}{3^k}\right)} \\
&= \frac{p}{3^k} \left(1 - \frac{m}{3^k}\right)^{-1} \\
&= \frac{a_0 3^r + a_1 3^{r-1} + .. + a_r}{3^k} \left(1 - \frac{b_0 3^s + b_1 3^{s-1} + .. + b_s}{3^k}\right)^{-1}
\end{aligned}$$

where $a_i, b_j \in \{0, 1, 2\},\ p = a_0 3^r + a_1 3^{r-1} + .. + a_r,\ m = b_0 3^s + b_1 3^{s-1} + .. + b_s$

$$= \left(\frac{a_0}{3^{k-r}} + \frac{a_1}{3^{k-r+1}} + .. + \frac{a_r}{3^k}\right).$$

$$\left\{1 + \left(\frac{b_0}{3^{k-s}} + \frac{b_1}{3^{k-s+1}} + .. + \frac{b_s}{3^k}\right) + \left(\frac{b_0}{3^{k-s}} + \frac{b_1}{3^{k-s+1}} + .. + \frac{b_s}{3^k}\right)^2 + ...\right\}$$

if some numerator is greater than 3, then expand it in ternary expansion.

$$\equiv \sum_{r=1}^{\infty} \frac{c_r}{3^r} \text{ after simplification.}$$

With this background, we now proceed to discuss about the Cantor set. We begin by defining a set $G \subset I$, defined by

$$G = \{x \in [0, 1]; \text{the ternary expansion for } x \text{ contains at least one } 1\}.$$

The set G is well defined, for if $x \in I$, then x has either a unique ternary expansion or exactly two equivalent ternary expansions and if either of these two equivalent expansions of x contains no 1, then $x \notin G$. For example, the real number

$$\frac{1}{3} = 0.1000.....[3] = 0.02222...[3].$$

does not belong to G, since it has an equivalent ternary expansion and one of them contains 1. Similarly, for the real number $\frac{2}{3} = 0.1222... = 0.20000.....[3]$, we conclude that $\frac{2}{3} \notin G$. Now, we begin with a lemma which is the heart of Cantor set.

10.13.3 Lemma. An element $x \in G$ iff $\exists$ positive integers k and m satisfying the following conditions:

1. $0 < m < m + 1 < 3^k$
2. $\dfrac{m}{3^k} < x < \dfrac{m+1}{3^k}$
3. $m = 3t + 1$ for some non-negative integer t.

Proof. Suppose that $x \in G$ and a ternary expansion of x is

$$(0.b_1 b_2 b_3......)_3$$

Let k be the smallest positive integer such that $b_k = p$. Then we have

$$x = (0.b_1 b_2 b_3...b_{k-1} 1 b_{k+1} b_{k+2}.....)_3$$

when $b_i \neq 1$ for $i = 1, 2, ..k-1$. It follows from the definition of G that the digit $b_k = 1$ cannot be followed immediately by all 2's or all by 0's. Thus we get

$$\begin{aligned}
&0.b_1b_2b_3...b_{k-1}1 < x < 0.b_1b_2b_3...b_{k-1}2\\
\Rightarrow &\frac{b_1}{3} + \frac{b_2}{3^2} + .. + \frac{b_{k-1}}{3^{k-1}} + \frac{1}{3^k} < x < \frac{b_1}{3} + \frac{b_2}{3^2} + .. + \frac{b_{k-1}}{3^{k-1}} + \frac{2}{3^k}\\
\Rightarrow &\frac{t}{3^{k-1}} + \frac{1}{3^k} < x < \frac{t}{3^{k-1}} + \frac{2}{3^k}\\
\Rightarrow &\frac{3t+1}{3^k} < x < \frac{3t+2}{3^k}
\end{aligned}$$

Let $3t + 1 = m$, then $\dfrac{m}{3^k} < x < \dfrac{m+1}{3^k}$.

Conversely, suppose that $x \in I$ and k, m are positive integers such that (1), (2) and (3) are satisfied. We show that $x \in G$. From (2) and (3) we have

$$\frac{m}{3^k} = \frac{3t+1}{3^k} = \frac{t}{3^{k-1}} + \frac{1}{3^k}$$

Hence, $\frac{m}{3^k}$ has the ternary expansion of the form

$$\frac{m}{3^k} = 0.a_1a_2....a_{k-1}10000......[3]$$

Similar reasoning shows that $\dfrac{m+1}{3^k}$ has the ternary expansion with the same digits except that $a_k = 2$ instead of 1. However, we use the equivalent ternary expansions ending with 2's, so that

$$\frac{m+1}{3^k} = 0.a_1a_2....a_{k-1}12222...[3]$$

It then follows from (2) that

$$x = 0.a_1a_2a_3...a_{k-1}1a_{k+1}a_{k+2}.....[3]$$

where at least one $a_i \neq 0$ for $i > k$ and at least one $a_i \neq 2$ for $i > k$. Thus the ternary expansion of x must contain at least one 1, and so $x \in G$. □

10.13.4 Remark. Here, we explain how the lemma works: We divide the unit interval [0,1] into three equal parts, i.e. $\left(0, \frac{1}{3}\right), \left(\frac{1}{3}, \frac{2}{3}\right), \left(\frac{2}{3}, 1\right)$. A number $x \in (0, 1)$ whose first place is 0,1,2, are of the forms

$$0.0x_2x_3... \in \left(0, \frac{1}{3}\right), \quad 0.1x_2x_3... \in \left(\frac{1}{3}, \frac{2}{3}\right), \quad 0.2x_2x_3... \in \left(\frac{2}{3}, 1\right).$$

where $x_i \in \{0, 1, 2\}$ according to ternary system, the first one lies in the first one-third of the unit interval, second lies in the middle one-third of the unit interval and third lies in the last one-third of the unit interval. Again, the numbers whose 2nd places are 0,1,2, are of the forms

$$\begin{aligned}
&0.00x_3x_4... \in \left(0, \tfrac{1}{3^2}\right), \quad 0.01x_3x_4... \in \left(\tfrac{1}{3^2}, \tfrac{2}{3^2}\right), \quad 0.02x_3x_4.. \in \left(\tfrac{2}{3^2}, \tfrac{3}{3^2}\right),\\
&0.10x_3x_4... \in \left(\tfrac{3}{3^2}, \tfrac{4}{3^2}\right), \quad 0.11x_3x_4... \in \left(\tfrac{4}{3^2}, \tfrac{5}{3^2}\right), \quad 0.12x_3x_4... \in \left(\tfrac{5}{3^2}, \tfrac{6}{3^2}\right),\\
&0.20x_3x_4... \in \left(\tfrac{6}{3^2}, \tfrac{7}{3^2}\right), \quad 0.21x_3x_4... \in \left(\tfrac{7}{3^2}, \tfrac{8}{3^2}\right), \quad 0.22x_3x_4... \in \left(\tfrac{8}{3^2}, \tfrac{9}{3^2}\right).
\end{aligned}$$

We observe that in the first step, the number whose first place is 1, lies in $\left(\frac{1}{3}, \frac{2}{3}\right)$. Here $k = 1$ and with $t = 0$ implies $m = 1$ so that $0 < 1 < 2 < 3^1$. And in the 2nd step, the number whose 2nd place is 1, lies in $\left(\frac{1}{3^2}, \frac{2}{3^2}\right), \left(\frac{4}{3^2}, \frac{5}{3^2}\right) \left(\frac{7}{3^2}, \frac{8}{3^2}\right)$. Here $k = 2$ and $t = 0, 1, 2$ implies $m = 1, 4, 7$ so that $0 < 1 < 2 < 4 < 5 < 7 < 8 < 3^2$. Thus, we see that if an expansion contains 1 in the 1st or 2nd place, then it lies within the intervals

$$\underbrace{\left(\frac{1}{3^2}, \frac{2}{3^2}\right)}; \underbrace{\left(\frac{3}{3^2}, \frac{4}{3^2}\right), \left(\frac{4}{3^2}, \frac{5}{3^2}\right), \left(\frac{5}{3^2}, \frac{6}{3^2}\right)}; \underbrace{\left(\frac{7}{3^2}, \frac{8}{3^2}\right)}.$$

As an immediate application of the above lemma, we have the following

10.13.1 Problem.

1. The set G is open.

2. If $a, b \in [0, 1], a \neq b$, then $\exists\, x \in G$ such that $a < x < b$.

3. $\overline{G} = I$, i.e. G is dense in I.

10.13.1.1 Solution.

1. Let $x \in G$. Then by the lemma, there exist positive integers k and m satisfying the three conditions, but by (2) of the lemma, we have

$$x \in \left(\frac{m}{3^k}, \frac{m+1}{3^k}\right) \subseteq G,$$

shows that x is an interior point of G. Thus G is open.

2. Suppose that $a < b$ and $r = b - a$. Since $\lim_n \frac{1}{3^n} = 0$, $\exists\, k \in \mathbb{N}$ such that $\frac{1}{3^k} < \frac{r}{4}$. Let y be the smallest positive integer such that $\frac{y}{3^k} > a$. Then $\frac{y-1}{3^k} \leq a$ and hence

$$b = a + r > a + \frac{4}{3^k} = \frac{a.3^k + 4}{3^k} \geq \frac{y - 1 + 4}{3^k} = \frac{y + 3}{3^k}.$$

Thus the four points $\frac{y}{3^k}$, $\frac{y+1}{3^k}$, $\frac{y+2}{3^k}$, $\frac{y+3}{3^k}$ are contained in (a, b). Since every third successive positive is divisible by 3, it follows that one of the three integers $y, y + 1, y + 2$ must have the form $3t + 1$ for some non-negative integer t.

3. It suffices to show that $I \subseteq \overline{G}$; so let $p \in I$. We show that $p \in \overline{G} = G \cup G'$. Let $p \notin G$ and (r, s) be any interval containing p then $(r, s) \cap I \setminus \{p\} \neq \emptyset$, so let q be any point in this intersection. Then p and q are distinct points of I, hence by (2) $\exists\, x \in G$ between p and q, clearly $x \in (r, s) \cap I \setminus \{p\}$. Hence $p \in G'$. Since $G \subseteq I \Rightarrow \overline{G} \subseteq I$, hence $\overline{G} = I$. □

10.13.5 Definition. Cantor set C is the subset of $[0, 1] = I$ consisting of precisely the real numbers which have ternary expansions containing no 1's. Thus Cantor set is the set

$$C = I \setminus G.$$

Since the Cantor set is the set $[0, 1] \setminus G$, so, what remains after the removal of G from the closed interval [0,1]. Clearly, the endpoints of all the removed open intervals must then be the members of the Cantor set, as well as any cluster points of the set of end points.

10.13.6 Remark. To understand about the remaining points after removal of middle thirds in each step we observe that, in the first place, what has been removed is a union of open sets (indeed, of open intervals), and so is open; what remains is its complement (with respect to [0,1]), and so is a closed set. The endpoints of the various middle thirds were not removed, so they remain; and since the remaining set is closed, every limit point of endpoints remains. For example, if we start from $\frac{1}{3}$ and take the closest endpoint in the second step $\left(\frac{1}{3}-\frac{1}{9}=\frac{2}{9}\right)$, then the closest endpoint in the third step $\left(\frac{1}{3}-\frac{1}{9}+\frac{1}{27}\right)$ and so on, the (only) limit point of this set of points is $\left(\frac{1}{3}-\frac{1}{9}+\frac{1}{27}-..=\frac{1}{4}\right)$. Thus there are, In fact, limit points of endpoints which are not endpoints. The Cantor set is the set that remains after we have removed all the middle thirds: it consists of all the endpoints and of their limit points.

10.13.7 Remark. To study the properties of the Cantor set C, we need a simple characterization of its elements. Since C was obtained by deleting open middle thirds, it is not surprising that the desired characterization of its elements is provided by their ternary expansions. Thus dividing the closed interval [0,1] into 3 equal subintervals and remove the central open interval $I_1=(\frac{1}{3},\frac{2}{3})$, so that $C_1=[0,1]\setminus I_1=\left[0,\frac{1}{3}\right]\cup\left[\frac{2}{3},1\right]$. Subdivide each of these intervals in 3 equal parts and remove their central open intervals. If I_2 is the set that has been removed

$$I_2=\left(\frac{1}{3^2},\frac{2}{3^2}\right)\bigcup\left(\frac{7}{3^2},\frac{8}{3^2}\right)$$
$$C_2=[0,1]\setminus I_1\cup I_2=\left[0,\frac{1}{3^2}\right]\bigcup\left[\frac{2}{3^2},\frac{3}{3^2}\right]\bigcup\left[\frac{6}{3^2},\frac{7}{3^2}\right]\bigcup\left[\frac{8}{3^2},1\right].$$

We subdivide each of the closed intervals making up $[0,1]\setminus I_1\cup I_2$, into 3 equal subintervals and remove their central open intervals. If I_3 is the set that has been removed

$$\begin{aligned}&I_3=\left(\frac{1}{3^3},\frac{2}{3^3}\right)\bigcup\left(\frac{7}{3^3},\frac{8}{3^3}\right)\bigcup\left(\frac{19}{3^2},\frac{20}{3^2}\right)\bigcup\left(\frac{25}{3^2},\frac{26}{3^2}\right)\\&C_3=[0,1]\setminus I_1\cup I_2\cup I_3=\left[0,\frac{1}{3^3}\right]\bigcup\left[\frac{2}{3^3},\frac{3}{3^3}\right]\bigcup\left[\frac{6}{3^3},\frac{7}{3^3}\right]\bigcup\left[\frac{8}{3^3},\frac{9}{3^3}\right]\\&\bigcup\left[\frac{18}{3^3},\frac{19}{3^3}\right]\bigcup\left[\frac{20}{3^3},\frac{21}{3^3}\right]\bigcup\left[\frac{24}{3^3},\frac{25}{3^3}\right]\bigcup\left[\frac{26}{3^3},1\right].\end{aligned}\tag{5}$$

Proceeding in this manner, it defines a sequence of disjoint open sets I_n, each being the finite, disjoint union of open intervals and satisfying

$$\ell(I_n)=\frac{2^{n-1}}{3^n}\text{ and }\sum_{n=1}^{\infty}\ell(I_n)=1.$$

The Cantor set $C=[0,1]\setminus\bigcup_{n=1}^{\infty}I_n$ is the set that remains after removing, the union of the I_n out of $[0,1]$. The Cantor set C is closed and each of its point is an accumulation point of the extremes of the intervals I_n. Thus C coincides with the set of all its accumulation points.
In general, C_k consists of 2^k disjoint closed intervals and, having C_k, C_{k+1} is obtained by removing middle thirds from each of the intervals that make up C_k. Then it is easy to see that

$$C=\bigcap_{k=0}^{\infty}C_k.$$

Let

$$\mathcal{E} = \{(e_n);\ \text{where } e_n \text{ is either 0 or 1}\}.$$

Every element $x \in C$ can be represented as $x = \frac{2}{3^i}e_i$ for some sequence $(e_n) \in \mathcal{E}$. Every element of C is associated to one and only one sequence $(e_n) \in \mathcal{E}$ by the representation formula above. For example

$$\begin{aligned}
&\frac{1}{3} \leftrightarrow (0,1,1,1,..,1,..),\ \frac{2}{3} \leftrightarrow (1,0,0,0,..,0,..),\\
&\frac{1}{9} \leftrightarrow (0,0,1,1,..,1,..),\ \frac{2}{9} \leftrightarrow (0,1,0,0,...,0,.),\\
&\frac{7}{9} \leftrightarrow (1,0,1,1,..,1,..),\ \frac{8}{9} \leftrightarrow (1,1,0,0,...,0,.),...
\end{aligned} \tag{6}$$

Vice versa any such sequence identifies one and only one element of C by (6). The set of all sequences in $\mathcal{E}$ has the cardinality of the real numbers in $[0,1]$, being their binary representation. Thus C has the cardinality of $\mathbb{R}$ and therefore is uncountable. It also follows from (6) that the Cantor set could be defined alternatively as the set of those numbers in [0,1] whose ternary expansion has only the digits 0 and 2. The two definitions are equivalent.

10.13.2 Problem. Show that $\frac{1}{4} \in C$, but is not an endpoint of any of the intervals in any of the sets C_k for $k \in \mathbb{N}$.

10.13.2.1 Solution. The ternary expansion of $\frac{1}{4}$ is as follows:

$$\begin{aligned}
\frac{1}{4} = \frac{2}{8} &= \frac{2}{3^2-1}\\
&= \frac{2}{3^2}\left(1-\frac{1}{3^2}\right)^{-1}\\
&= \frac{2}{3^2}\left(1+\frac{1}{3^2}+\frac{1}{3^4}+\frac{1}{3^6}+...\right)\\
&= \frac{2}{3^2}+\frac{2}{3^4}+\frac{2}{3^6}+\frac{2}{3^8}+...\\
&= .020202...[3].
\end{aligned}$$

Hence $\frac{1}{4} \in C$. Notice that $x \in C_k$ is an endpoint if $x = 0, x = 1$, or if $x = 3^{-k}$ for some $k \in \mathbb{N}$. Clearly $1/4 \neq 0, 1$, and $\frac{1}{4} \neq 3^{-k}$ for all $k \in \mathbb{N}$. Therefore $1/4$ is not an endpoint. □

10.13.3 Problem. The Cantor set C can also be described in terms of ternary expansions. Every number in [0,1] has a ternary expansion

$$\sum_{n=1}^{\infty} \frac{a_n}{3^n} \text{ where } a_k \in \{0,1,2\}.$$

Prove that $x \in C$ if and only if x has a representation as above where every a_k is either 0 or 2.

10.13.3.1 Solution. ($\Rightarrow$) Let $x \in C$. We build a ternary expansion for x of the desired form as follows. Consider $C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$. It must be that x belongs to one of $[0, \frac{1}{3}]$ (in which case let

first digit of the ternary expansion for x be 0) or $[\frac{2}{3}, 1]$ (in which case let first digit of the ternary expansion for x be 2). Next, consider $C_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$. The interval of C_1 to which x currently belongs will be divided into three subintervals, and so we append a 0 to the ternary expansion of x if it belongs to the leftmost subinterval or a 2 if it belongs to the rightmost subinterval. Continuing in this way, we see that x has an associated ternary expansion containing only the digits 0 and 2.
($\Leftarrow$) Let

$$x = \sum_{n=1}^{\infty} \frac{a_n}{3^n} \text{ where } a_k \in \{0, 2\}.$$

We can locate x on the real line as follows. If $a_1 = 0$, we choose the left subinterval of C_1. If $a_1 = 2$, we choose the rightmost subinterval of C_1. When we form C_2, the interval we have just chosen will be subdivided into three subintervals. If $a_2 = 0$, we select the leftmost subinterval. If $a_2 = 2$, we select the rightmost subinterval. Continue in this way. Since the length of these intervals can be made arbitrarily small, we see that the ternary expansion of x uniquely specifies its location on the real line. □

10.13.4 Problem. The Cantor-Lebesgue function is defined on C by

$$F(x) = \sum_{k=1}^{\infty} \frac{b_k}{2^k} \text{ if } x = \sum_{k=1}^{\infty} \frac{a_k}{3^k}, \text{ where } b_k = \frac{a_k}{2}; a_k \in \{0, 2\}.$$

Show that F is well defined and continuous on C, and moreover $F(0) = 0$ as well as $F(1) = 1$.

10.13.4.1 Solution. Let $x, x' \in C$ with $x = x'$. Denote the k-th digit of the ternery expansion of x and x' by a_k and a'_k, respectively. Claim $a_k = a'_k \ \forall\ k$.
Proof of claim: Suppose not. Then, $a_N \neq a'_N$ for some N. From the construction of C_k, we see that x and x' must belong to different subintervals in C_N, and so $x \neq x'$, which is a contradiction. Now, let $b_k = \frac{a_k}{2}$ and $b'_k = \frac{a'_k}{2}$. Then $b_k = b'_k \ \forall\ k$. Hence

$$F(x) = \sum_{k=1}^{\infty} \frac{b_k}{2^k} = \sum_{k=1}^{\infty} \frac{b'_k}{2^k} = F(x'),$$

and so F is well defined.
To show that F is continuous, let $\epsilon > 0$. Consider the binary expansion of ϵ (denote the k-th digit of ϵ by ϵ_k). Construct $\delta > 0$ such that $\delta_k = 2\epsilon_k$ for all k. Let N be the first nonzero digit of δ and ϵ. Then, $|x - x'| < \delta$ implies that the first $N-1$ digits of x and x' agree. Hence, the first $N-1$ digits of $F(x)$ and $F(x')$ agree, and so $|F(x) - F(x')| < \epsilon$. Therefore, F is continuous.
Since 0 is represented in ternary form by choosing the leftmost subinterval, and so for $x = 0, b_k = 0/2 = 0$ for all k. Similarly, 1 is represented in ternary form by chosing the rightmost subinterval, and so for $x = 1, b_k = 2/2 = 1$ for all k. Hence

$$F(0) = \sum_{k=1}^{\infty} \frac{0}{2^k} = 0$$

$$F(1) = \sum_{k=1}^{\infty} \frac{1}{2^k} = \sum_{k=1}^{\infty} \frac{b_k}{2^k} = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1.$$

10.13.5 Problem. Let C be the Cantor set. Prove that $C + C = \{x + y; x, y \in C\} = [0, 2]$.

10.13.5.1 Solution. Note that for any subsets A, B, C, D of $\mathbb{R}$, we have

$$(A \cup B) + (C \cup D) = (A + C) \cup (A + D) \cup (B + C) \cup (B + D). \tag{7}$$

Call $C_0 = [0, 1], C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right], C_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$ and so on. It is easy to check, say by induction, that $C_{n+1} = \frac{1}{3}C_n \cup \left(\frac{2}{3} + \frac{1}{3}C_n\right)$. Now, let C be the Cantor set and let C_n be the n-th approximation to C. Note that $C \subset [0, 1]$. Thus, by the second to last remark of the previous paragraph we have $C + C \subset [0, 1] + [0, 1] = [0, 2]$. Thus $C + C \subset [0, 2]$. The opposite inclusion is more delicate and we turn to it now. Fix $s \in [0, 2]$. We first must find, for all n, a pair of elements $x_n, y_n \in C_n$ so that $x_n + y_n = s$. It is clear that $C_0 + C_0 = [0, 2]$. Suppose now that $C_n + C_n = [0, 2]$. Note that

$$C_{n+1} + C_{n+1} = \left(\frac{1}{3}C_n \cup \left(\frac{2}{3} + \frac{1}{3}C_n\right)\right) + \left(\frac{1}{3}C_n \cup \left(\frac{2}{3} + \frac{1}{3}C_n\right)\right).$$

The sum of the first terms is $\left(\frac{1}{3}C_n + \frac{1}{3}C_n\right) = \frac{1}{3}[0, 2] = \left[0, \frac{2}{3}\right]$, by induction. Similarly the sum of the outer terms is $\frac{1}{3}C_n + \left(\frac{2}{3} + \frac{1}{3}C_n\right) = \frac{2}{3} + \left[0, \frac{2}{3}\right] = \left[\frac{2}{3}, \frac{4}{3}\right]$. The sum of the inner terms is $\left(\frac{2}{3} + \frac{1}{3}C_n\right) + \frac{1}{3}C_n = \left[\frac{2}{3}, \frac{4}{3}\right]$, giving the same set as the outer terms. Finally the sum of the last terms is $\left(\frac{2}{3} + \frac{1}{3}C_n\right) + \left(\frac{2}{3} + \frac{1}{3}C_n\right) = \left[\frac{4}{3}, 2\right]$ By the above identity (7) instructs us to take the union of these four sets. This gives $C_{n+1} + C_{n+1} = [0, 2]$, as desired. It follows that for all n there exists $x_n, y_n \in C_n$ so that $x_n + y_n = s \in [0, 2]$.
Recall that $C_n \subset [0, 1]$ for all n. It follows that the sequence (x_n) is bounded. Thus by Bolzano-Weierstrass the sequence (x_n) has a convergent subsequence (x_{n_i}). Suppose that (x_{n_i}) converges to $x \in [0, 1]$. Thus we have

$$\lim_{i\to\infty} y_{n_i} = \lim_{i\to\infty} (s - x_{n_i}) = s - x$$

Thus the subsequence (y_{n_i}) also converges and converges to $y = s - x$. Now, we show that $x, y \in C$, the Cantor set, and the proof will be complete. Recall that $C_{n+1} \subset C_n$ for all $n \in \mathbb{N}$. So, for all $M \in \mathbb{N}$ and for all $k \geq M$ we have that (x_{n_k}) is contained in C_{n_k}. As C_{n_k} is closed (it is a finite union of closed intervals) we deduce that the limit x is contained in C_{n_k}. Finally, it follows that $x \in \bigcap_k C_{n_k}$. But this last intersection is exactly equal to the Cantor set. (Check this!) A similar argument shows that $y \in C$ and the proof is complete. □

10.13.6 Problem.

1. Cantor set is the set $[0, 1] \setminus G$.
2. Cantor set is closed.
3. Cantor set contains no connected subset consisting of more than one point.
4. Cantor set is uncountable.

10.13.6.1 Solution.

1. It follows from the definition of G and definition (11.10.3).
2. Since the complement of the Cantor set C is $(-\infty, 0) \cup G \cup (1, \infty)$, the union of three open sets and hence is open.

3. Let $(a, b) \subseteq I \setminus G$. Then $(a, b) \subseteq I \cap G^C$ implies $(a, b) \subseteq G^C$ implies $(a, b) \cap G = \emptyset$ which is a cotradiction, for any two points $a, b \in I \ \exists \ x \in G$ such that $a < x < b$ implies $(a, b) \cap G \neq \emptyset$.

4. Let $f : C \to [0, 1]$. A point $x \in C$ has a ternary expansion of the form $x = 0.x_1x_2x_3x_4.....[3]$ where x_i is either 0 or 2. For each $i \in \mathbb{N}$, define $y_i = x_i/2$ so that each y_i is either 0 or 1 and let $y = 0.y_1y_2y_3y_4.....[2]$. Finally, for each $x \in C$, define $f(x) = y$. Since every real number in [0,1] has a binary representation of the form $0.y_1y_2y_3y_4.....[2]$, where each y_i is either 0 or 1. Now it is clear that f is surjective. Hence C is uncountable. □

10.13.7 Problem. The set G is the union of countable collection of disjoint open intervals the sum of whose lengths is 1.

10.13.7.1 Solution. From the remark (11.10.4), it is easy to establish that $G = \cup_{k=1}^{\infty} I_k$. Thus G is the union of countable collection of disjoint open intervals. According to the lemma, G contains the open interval $\left(\frac{1}{3}, \frac{2}{3}\right)$ of length $\frac{1}{3}$, and for each integer $k \geq 2, G$ contains 2^{k-1} open intervals each of length $\frac{1}{3^k}$. If S denote the sum of the lengths of all these open intervals, then

$$\begin{aligned} S &= \frac{1}{3} + 2\frac{1}{3^2} + 4\frac{1}{3^3} + 8\frac{1}{3^4} + ... \qquad (7) \\ &= \frac{1}{3} + \frac{2}{3^2} + \frac{4}{3^3} + \frac{8}{3^4} + ... = \frac{\frac{1}{3}}{1 - \frac{2}{3}} = 1. \quad \square \end{aligned}$$

10.13.8 Problem. The Cantor set C is

1. nowhere dense.

2. perfect.

3. of measure zero.

10.13.8.1 Solution.

1. Recall that a set is nowhere dense iff it contains no open intervals and if possible let $(a, b) \subseteq I \setminus G$. Then $(a, b) \subseteq I \cap G^C$ implies $(a, b) \subseteq G^C$ implies $(a, b) \cap G = \emptyset$ which is a cotradiction, for any two points $a, b \in I \ \exists \ x \in G$ such that $a < x < b$ implies $(a, b) \cap G \neq \emptyset$. Thus no open interval (a, b) is contained in $I \setminus G$. Hence C is nowhere dense.

2. Recall that a set S is perfect iff it $S = S'$. Let $p \in C$. Then p has a ternary expansion of the form $0.x_1x_2x_3x_4.....[3]$ with $x_i \in \{0, 2\}$. Let $\epsilon > 0$ then $\exists \ k \in \mathbb{N}$ such that $\frac{1}{3^k} < \epsilon$. If $x_{k+1} = 0$, then define $q = 0.x_1x_2x_3x_4...x_ky_{k+1}x_{k+2}..$ with $y_{k+1} = 2$ and if $x_{k+1} = 2$, then define $q = 0.x_1x_2x_3x_4...x_ky_{k+1}x_{k+2}..$ with $y_{k+1} = 0$. Clearly $q \in C, q \neq p$, and

$$0 < |p - q| = \frac{a_{k+1} - b_{k+1}}{3^{k+1}} = \frac{2}{3^{k+1}} < \frac{1}{3^k} < \epsilon.$$

Thus $q \in \hat{B}(p; \epsilon)$. Hence p is a cluster point of C and we have $C \subseteq C' \subseteq \overline{C} = C$ implies $C = C'$. Thus C is perfect.

3. Let $\epsilon > 0$. Since G is the countable union of disjoint open intervals I_n the sum of whose lengths is 1, so for $n \in \mathbb{N}$, let S_n be the n-th partial sum of the series (7). Since (S_n) is an increasing sequence converging to 1, so $\exists \ k \in \mathbb{N}$ such that $S_k > 1 - \frac{\epsilon}{3}$. Thus there are finite number of

disjoint open intervals $I_1, I_2, .., I_N$ in G, the sum of whose lengths is greater than $1 - \frac{\epsilon}{3}$. Let $I_i = (a_i, b_i); 1 \leq i \leq N$ where $0 < a_1 < b_1 < a_2 < b_2 < .. < a_N < b_N < 1$. Then

$$C \subseteq [0, a_1] \cup [b_1, a_2] \cup [b_2, a_3] \cup ... \cup [b_N, 1]$$

and the sum of the lengths of these intervals is less than $\frac{\epsilon}{3}$. Let $b_0 = 0, a_{N+1} = 1$, and for each $i = 1, 2, ..., N + 1$, define the open interval G_i by

$$G_i = \left(b_{i-1} - \frac{\epsilon}{6N}, a_i + \frac{\epsilon}{6N}\right).$$

Then $\{G_i; i = 1, 2, .., N+1\}$ is a countable collection of open intervals covering the Cantor set. Furthermore, the sum of the lengths of $G_i, i = 1, 2, .., N + 1$ is

$$\sum_{i=1}^{N+1} |G_i| = S + (N + 1).\frac{\epsilon}{3N} \leq S + (N + N)\frac{\epsilon}{3N} < \frac{\epsilon}{3} + \frac{2\epsilon}{3} = \epsilon.$$

Hence C is of measure zero. □

10.13.8 Remark. We see that each point of C is a limit point. Since the limit points of endpoints are naturally limit points of the set, it is merely a question of showing that the endpoints are limit points. In particular, consider the point $\frac{1}{3}$. To the left of it there is an interval of length $\frac{1}{3}$ from which we remove the middle third, leaving an interval of length $\frac{1}{9}$ adjacent to the point $\frac{1}{3}$; then we remove the interval $\left(\frac{7}{27}, \frac{8}{27}\right)$ leaving an interval of length $\frac{1}{27}$ adjacent to $\frac{1}{3}$; and so on. In any neighborhood of $\frac{1}{3}$ there will always be a short interval that is not removed at some step, and this interval will contain an endpoint belonging to a subsequent step. Hence $\frac{1}{3}$ is a limit point of endpoints.

10.13.9 Problem. Every non-empty perfect set $E \subseteq \mathbb{R}$ is uncountable.

10.13.9.1 Solution. Suppose false, so E may be enumerated as a sequence (x_n). Form the sequence (y_n) in E inductively as follows. Let $y_1 = x_1, y_2 = x_2$, and choose $\epsilon_1, 0 < \epsilon_1 < |x_1 - x_2|$. Since E is perfect we may choose $y_3 \in E, y_3 \in N(x_2, \epsilon_1), y_3 \neq x_3$, and with a neighbourhood $N(y_3, \epsilon_2) \subseteq N(x_2, \epsilon_1)$. We may suppose that $0 < \epsilon_2 < \epsilon_1/2$ and that x_1, x_2, x_3 are not in $N(y_3, \epsilon_2)$. Now choose $y_4 \in E, y_4 \in N(y_3, \epsilon_2)$, with a neighbourhood $N(y_4, \epsilon_3) \subseteq N(y_3, \epsilon_2)$, such that $0 < \epsilon_3 < \epsilon_2/2$ and that $x_1, .., x_4 \notin N(y_4, \epsilon_3)$, etc., by induction. Then (y_n) is a Cauchy sequence in E with limit y_0 and $y_0 \in E$ as any perfect set is closed. But for each $n, N(y_n, \epsilon_{n-1})$ contains y_0 but does not contain x_n. So $y_0 \neq x_n$ for any n, and so no such enumeration of E exists. □

10.13.10 Problem. The Cantor set C is compact, nowhere dense, totally disconnected, and perfect. Moreover, there is a surjective map from C onto $[0, 1]$ and hence C has the cardinality of the continuum: $|C| = |[0, 1] = |\mathbb{R}| = \mathfrak{c}$.

10.13.10.1 Solution. First, C is compact because it is a closed subset of the compact set $[0, 1]$. Next, to show that C is nowhere dense i.e., that $\overline{C}^\circ = \emptyset$. Note that if $(\alpha, \beta) \subseteq C$ for some $\alpha \leq \beta$ in $\mathbb{R}$, then $C_n \ \forall \ n \in \mathbb{N}$. Since C_n is a union of 2^n pairwise disjoint intervals of equal length $1/3^n$, we must have $\beta - \alpha \leq \left(\frac{2}{3}\right)^n \ \forall \ n \in \mathbb{N}$, which is absurd. This also shows that C is totally disconnected (why?). To prove that C is perfect, let $x \in C$ be arbitrary. Given any $\epsilon > 0$, $\exists \ n \in \mathbb{N}$ such that $1/3^n < \epsilon$. Let I_n denote the subinterval in C_n containing x. If $x_n \neq x$ is an endpoint of I_n, then we

have $x_n \in C$ and $0 < |x - x_n| < \epsilon$, which proves that x is a limit point of C. Finally, if we define the map $f : C \to [0,1]$ by

$$f\left(\sum_{n=1}^{\infty} \frac{x_n}{3^n}\right) = \sum_{n=1}^{\infty} \frac{x_n/2}{2^n}$$

then $f(0.x_1x_2x_3...)_3 = (0.y_1y_2y_3....)_2$ where $y_n = x_n/2$. Since each has a binary expansion, the map ϕ is onto and we get $|[0,1]| \leq |C|$. But $C \subseteq [0,1]$ implies $|C| \leq |[0,1]|$ thus we have $|C| = |[0,1]|$ and the proof is complete. □

10.13.9 Remark. We shall see later that the map $f : C \to [0,1]$ defined above is actually continuous and can be extended to a continuous, monotone map $\phi : [0,1] \to [0,1]$ called the Cantor (ternary) function.

Here, we point out, that the ternary (or base 3) expansion of a number $x \in [0,1]$ is unique except when $x = m/3^n$ for some positive integers m, n, where we may assume that 3 does not divide m. In these exceptional cases, we have two ternary expansions, one ending with a string of 0's and, the other with a string of 2's.

$$x = (0.x_1x_2...x_n1000..)_3 = (0.x_1x_2..x_n0222...)_3;$$

for some n. For example, $\frac{1}{3} = (0.1000..)_3 = (0.0222...)_3$, since

$$\sum_{n=2}^{\infty} \frac{2}{3^n} = \frac{2}{3^2} \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n = \frac{1}{3}.$$

It is therefore possible to write all the exceptional x's i.e. those with a terminating ternary expansion so that their expansions end with strings of 2's and, as we shall see presently, have no 1's. First, we observe that if in the expansion $x = (0 : x_1x_2x_3...)_3$ we have $x_k = 1$ for some $k \in \mathbb{N}$, then x must belong to (the closure of) one of the middle thirds deleted in the construction of the Cantor set. This can be seen inductively. For example, if $x_1 = 1$, then $x \in \left[\frac{1}{3}, \frac{2}{3}\right]$. Indeed, we have $x = \frac{x_1}{3} + t$, where the "tail" t satisfies

$$t = \sum_{n=2}^{\infty} \frac{x_n}{3^n} \leq \sum_{n=2}^{\infty} \frac{2}{3^n} = \frac{1}{3}.$$

Also, $x_2 = 1$ implies that $x \in \left[\frac{1}{9}, \frac{2}{9}\right]$, if $x_1 = 0; x \in \left[\frac{4}{9}, \frac{5}{9}\right]$, if $x_1 = 1$; and $x \in \left[\frac{7}{9}, \frac{8}{9}\right]$, if $x_1 = 2$. Thus, deleting the middle third $\left[\frac{1}{3}, \frac{2}{3}\right]$ removes all numbers x whose unique ternary expansions satisfy $x_1 = 1$. Similarly, deleting the middle third $x \in \left[\frac{1}{9}, \frac{2}{9}\right]$ of $\left[0, \frac{1}{3}\right]$ removes all x's whose unique expansions satisfy $x_1 = 0$ and $x_2 = 1$, while deleting the middle third $\left[\frac{7}{9}, \frac{9}{9}\right]$ of $\left[\frac{2}{3}, 1\right]$ removes the x's whose unique expansions satisfy $x_1 = 2$ and $x_2 = 1$. If all middle thirds are deleted up to the n-th stage, then deleting the middle thirds of the remaining subintervals will remove all numbers x whose unique expansions satisfy $x_j = 0$ or 2 for $1 \leq j \leq n$ and $x_{n+1} = 1$. Also, if x is the left endpoint of one of the deleted middle thirds, then its expansion has the form $x = (0.x_1x_2...x_n1000..)_3$, with $x_k = 0$ or 2 for $1 \leq k \leq n$. But then we also have $x = (0.x_1x_2..x_n0222...)_3$. Summarizing these observations, we have: the Cantor set C is the set of all $x \in [0,1]$ such that $x = (0.x_1x_2x_3...)_3$; where each x_k is either 0 or 2.

10.13.2 Cantor Function:

In this section we explore the Cantor ternary function. This function provides an interesting example of a function that is uniformly continuous on a closed interval and of bounded variation on the closed interval but is not absolutely continuous. The closed, bounded interval that we work on is [0,1]. Let $f : C \to [0,1] = I$. A point $x \in C$ has a ternary expansion of the form $x = 0.x_1x_2x_3x_4.....[3]$ where x_i is either 0 or 2. For each $i \in \mathbb{N}$, define $y_i = x_i/2$ so that each y_i is either 0 or 1 and let $y = 0.y_1y_2y_3y_4.....[2]$. Finally, for each $x \in C$, define $f(x) = y$. In other words,

$$f(x) = f(0.x_1x_2x_3x_4....[3]) = y_1y_2y_3y_4.....[2] = .\frac{x_1}{2}\frac{x_2}{2}\frac{x_3}{2}\frac{x_4}{2}....[2].$$

Now we begin with the mapping $f : C \to I$ defined above. The idea is to extend the domain of f to all of [0,1]$= I$ and this can be done by the following way. We see that the points $\frac{1}{3}, \frac{2}{3} \in C$ with $f\left(\frac{1}{3}\right) = f\left(\frac{2}{3}\right)$ and $f(\frac{2}{3}) = f(0.2...[3]) = 0.1...[2] = \frac{1}{2}$ according to above construction, whereas the open interval $\left(\frac{1}{3}, \frac{2}{3}\right) \subset G$. Define

$$\phi(x) = \frac{1}{2} \text{ for } \frac{1}{3} \le x \le \frac{2}{3}.$$

Now consider the points $\frac{1}{9}, \frac{2}{9} \in C$ with $f\left(\frac{1}{9}\right) = f\left(\frac{2}{9}\right) = f(0.02..[3]) = 0.01..[2] = \frac{1}{4}$, and $\left(\frac{1}{9}, \frac{2}{9}\right) \subset G$. Define

$$\phi(x) = \frac{1}{4} \text{ for } \frac{1}{9} \le x \le \frac{2}{9}.$$

For the points $\frac{7}{9}, \frac{8}{9} \in C$ with $f\left(\frac{7}{9}\right) = f\left(\frac{8}{9}\right) = f(0.22..[3]) = 0.11..[2] = \frac{3}{4}$, and $\left(\frac{7}{9}, \frac{8}{9}\right) \subset G$. Define

$$\phi(x) = \frac{3}{4} \text{ for } \frac{7}{9} \le x \le \frac{8}{9}.$$

Clearly we can continue this process, definimg ϕ on each open middle third (a, b) to be the common value of f at the end points $a, b \in C$. Thus ϕ is defined on all of I, and $\phi(x) = f(x)$ for $x \in C$. The mapping ϕ is called the **Cantor function**.

10.13.11 Problem. Let $f : C \to [0,1]$ by the above paragraph. Verify the following: $f\left(\frac{1}{27}\right) = f\left(\frac{2}{27}\right) = \frac{1}{8},\ f\left(\frac{7}{27}\right) = f\left(\frac{8}{27}\right) = \frac{3}{8}, f\left(\frac{19}{27}\right) = f\left(\frac{20}{27}\right) = \frac{5}{8}$ and $f\left(\frac{25}{27}\right) = f\left(\frac{26}{27}\right) = \frac{7}{8}$.

10.13.11.1 Solution. Since $\frac{2}{27} = \frac{2}{3^3} = .002_3$, so $f\left(\frac{1}{27}\right) = f\left(\frac{2}{27}\right) = .001_2 = \frac{1}{2^3} = \frac{1}{8}$, and $\frac{8}{27} = \frac{3.2+2}{3^3} = \frac{2}{3^2} + \frac{2}{3^3} = .022_3$ implies $f\left(\frac{7}{27}\right) = f\left(\frac{8}{27}\right) = .011_2 = \frac{1}{2^2} + \frac{1}{2^3} = \frac{3}{8}$. Similarly for others. And finally we have the following

10.13.10 Definition. (Cantor's Ternary Function or Cantor Function). Given a point $x \in [0,1]$ with $x = (0.x_1x_2x_3...)_3$, i.e. $x = \sum_{n=1}^{\infty} \dfrac{x_n}{3^n}, x_n \in \{0,1,2\}$. Then the Cantor function $\phi : [0,1] \to \mathbb{R}$ can be defined as

$$\phi(x) = y = (0.y_1y_2y_3...)_2, \text{ where}$$
$$y_i = \begin{cases} 0, & \text{if } \exists\ k < i \text{ such that } x_k = 1 \\ 1, & \text{if } x_i = 1 \text{ and } \nexists\ k < i \text{ such that } x_k = 1 \\ \frac{x_i}{2}, & \text{if } x_i = 0, 2 \text{ and } \nexists\ k < i \text{ such that } x_k = 1. \end{cases}$$

Applying the above definition we can verify the following examples.

10.13.11 Example. Let $x = \frac{8}{27} = .022$, then

$$\phi(x) = f(.022) = .011...[2] = \frac{1}{2^2} + \frac{1}{2^3} = \frac{3}{8}.$$

10.13.12 Example. Let $x = \frac{1}{2} = 0.1111.....[3]$, then

$$\phi(x) = 0.1.....[2] = \frac{1}{2}.$$

10.13.13 Example. Let $x = \frac{10}{27} = 0.101.....[3]$, then

$$\begin{aligned} 0.101.....[3] &= 0.1.......[2] \\ \Rightarrow \ \phi(x) &= 0.1.....[2] = \frac{1}{2}. \end{aligned}$$

Again, as $\frac{1}{3} \leq \frac{10}{27} < \frac{1}{2} \leq \frac{2}{3}$ so $\phi(1/2) = \phi(10/27) = 1/2$.

10.13.12 Problem. The Cantor function ϕ has the following properties:

1. ϕ is well-defined on $[0,1]$. Moreover $\phi(0) = 0$ as well as $\phi(1) = 1$.
2. $\phi : [0,1] \to [0,1]$ is surjective.
3. ϕ is non-decreasing on $[0,1]$.
4. ϕ is continuous on $[0,1]$.
5. ϕ is constant on each interval contained in the complement of the Cantor set C. i.e. ϕ is constant on each interval in G.
6. ϕ is differentiable on G and $\phi'(x) = 0$ for each $x \in G$.

10.13.12.1 Solution.

1. Note that x has two ternary expansions if and only if it is an endpoint of a removed middle third. In this case, we either have

   $$x = (0.x_1x_2....x_{N-1}1\overline{0})_3 = (0.x_1x_2....x_{N-1}0\overline{2})_3$$

   where $x_i = 0$ or 2 for $1 \leq i \leq N-1$ and $\overline{0}$ and $\overline{2}$ indicate (infinite) strings of 0's and 2's, respectively, in which case we obtain the same value

   $$\phi(x) = \left(0.\frac{x_1}{2}\frac{x_2}{2}....\frac{x_{N-1}}{2}1\overline{0}\right)_2 = \left(0.\frac{x_1}{2}\frac{x_2}{2}....\frac{x_{N-1}}{2}0\overline{1}\right)_2$$

   or we have

   $$x = (0.x_1x_2....x_{N-1}2\overline{0})_3 = (0.x_1x_2....x_{N-1}1\overline{2})_3$$

   in which case we have the obviously unique value

   $$\phi(x) = \left(0.\frac{x_1}{2}\frac{x_2}{2}....\frac{x_{N-1}}{2}1\overline{0}\right)_2 = \left(0.\frac{x_1}{2}\frac{x_2}{2}....\frac{x_{N-1}}{2}0\overline{1}\right)_2$$

2. Let

$$x = (0.x_1x_2....)_3 \text{ and } x' = (0.x'_1x'_2....)_3$$

be two points in $[0,1]$ with $x < x'$ and let $N_x \leq \infty$ and $N_{x'} \leq \infty$ be as above. If m is the smallest index with $x_m \neq x'_m$ then $x_m < x'_m$ and hence $y_m < y'_m$ so that

$$\phi(x) = \sum_{k=1}^{N_x} \frac{y_k}{2^k} \leq \sum_{k=1}^{N_{x'}} \frac{y'_k}{2^k} = \phi(x').$$

Also, by Theorem (12.3.2) ϕ is onto and hence so is ϕ.

3. Left to the reader.

4. To prove the continuity, let $\epsilon > 0$ be given and pick m so that $1/2^m \leq \epsilon$. If $x, x' \in [0,1]$ satisfy $|x - x'| < 1/3^m$, then we can pick ternary expansions

$$x = (0.x_1x_2....)_3 \text{ and } x' = (0.x'_1x'_2....)_3$$

with $x_k = x'_k$ for $1 \leq k \leq m$. It follows that the first m digits of the binary expansions of $\phi(x)$ and $\phi(x')$ are equal and hence $|\phi(x) - \phi(x')| \leq 1/2^{m+1} < \epsilon$. In fact, since ϕ is increasing, it can only have jump discontinuities. However, being onto, satisfies the Intermediate Value Property and hence cannot have jump discontinuities and must be continuous.

5. Note that if

$$x = (0.x_1x_2....)_3 \text{ and } x' = (0.x'_1x'_2....)_3$$

and belong to the same middle third in the complement $I \setminus C$; then the smallest index m with $x_m = 1$ is also the smallest with $x'_m = 1$ and hence $\phi(x) = \phi(x')$, i.e. ϕ is constant on all such middle thirds.

6. Let $x \in G$. Since G is open, $\exists$ an open interval $(x-\delta, x+\delta)$ such that $N(x;\delta) = (x-\delta, x+\delta) \subseteq G$. It follows from the definition of ϕ that ϕ is constant on N Hence ϕ is differentiable at x, and $\phi(x) = 0$. Thus the Cantor function is an example of a continuous non-constant monotone function whose derivative vanishes almost everywhere. □

10.13.14 Exercise. Does the Cantor set contain the point $\sqrt{\pi} - 1 = 0.77245...$?

10.13.15 Exercise. Does the Cantor set contain any irrational points? If so, find one explicitly.

10.13.16 Exercise. Find a limit point that is not an endpoint.

10.13.17 Remark (A surprising reaction)**.** When I was a sophomore, an older student showed me the Cantor set, but admitted that he had never been able to find a limit point that was not an endpoint. He later left mathematics for biology. (A PRIMER OF REAL FUNCTIONS: RALPH P. BOAS)

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